Discussion of Phase Space and Emittances

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Introduction

- Figure of merit for accelerator performance
  - Luminosity for colliders
  - Brightness and coherence for light sources
  - Close relation with beam intensity in phase space
- Emittance and beam brightness in phase space
  - Important and fundamental quantities
  - Characterize beam quality
- Past and recent development
  - Path toward higher brightness of charged beams
  - Recent development
    - Magnetized beam
    - Emittance exchange/transfer in phase space manipulation
- Goal of this tutorial
  - Review the theoretical foundation
  - Introduce recent developments

Outline of the tutorial

- Phase space concept for single particle dynamics
- Emittance for a bunch of particles
- Emittance dilution and mitigation
- Electron cooling and magnetized beam
- Phase space manipulation
- Discussion and Summary

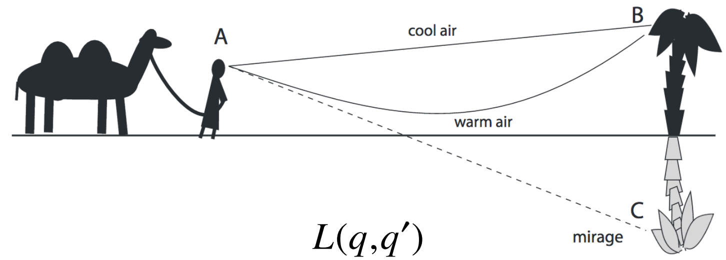
II. Phase Space Concept for the Single Particle Dynamics

- Hamiltonian mechanics and its origin in geometrical optics
- Flow in canonical phase space
  - Linear and nonlinear harmonic oscillator
  - Benefits of phase space description
  - Symplectic map and its properties
    - Invariant: Liouville theorem
- Single particle dynamics in accelerators
  - Lorentz force and equation of motion
  - Hamiltonian (t or s as independent variable)
  - Transverse and longitudinal equations of motion

- Courant-Snyder theory
  - Hill's equation
  - Invariant for linear map: action amplitude
  - Phase space ellipse and Poincare section
- Particle dynamics in a solenoid
- Various phase spaces
  - Coupling of subspaces in 6D phase space
  - Trace space, nominal trace space, canonical phase space

Origin of Phase Space Concept

- Geometric optics for light rays



Fermat's Principle (1662)

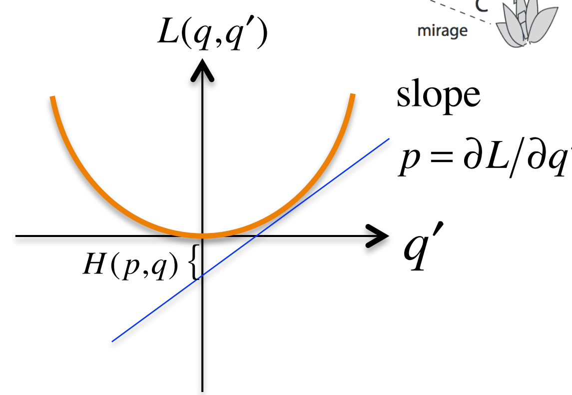
$$\delta S = \delta \int_A^B n(x,y,z)\,ds = 0$$

↓

Lagrangian Equation

$$L(q,q',z) = n(x,y,z)\sqrt{1+x'^2+y'^2}$$

$$\frac{d}{dz}\frac{\partial L}{\partial q'} - \frac{\partial L}{\partial q} = 0 \quad \text{for } q=(x,y)$$



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Eikonal Equation

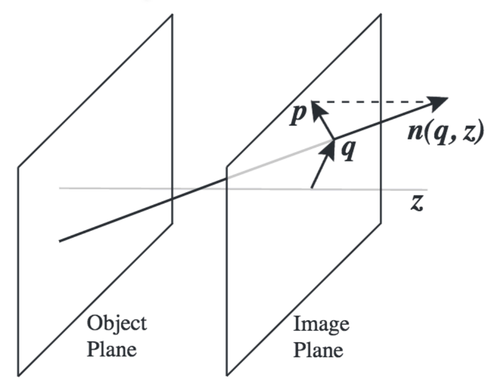
$$\frac{d}{ds}\left(n\frac{d\vec{r}}{ds}\right) = \nabla n$$

↓

Hamilton's Equation (1828)

$$H(p,q,s) = p\dot{q} - L(q,\dot{q},s)$$

$$\begin{cases} dq/ds = \partial H/\partial p \\ dp/ds = -\partial H/\partial q \end{cases}$$



Propagation of light is described by the flow in phase space

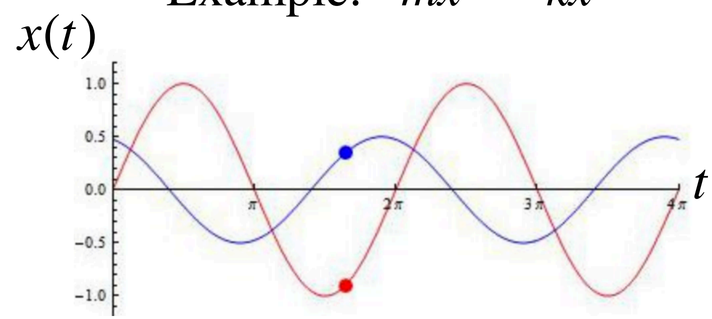
Phase Space Concepts in Particle Dynamics

- Classical Mechanics

Newton's Equation (1686)

$$m\frac{d^2\vec{x}}{dt^2} = F$$

Example: $m\ddot{x} = -kx$



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Lagrangian Equation

$$L = L(\vec{q},\dot{\vec{q}},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad \text{for } \vec{q} = (x,y,z)$$

↓

Extremal Action Principle

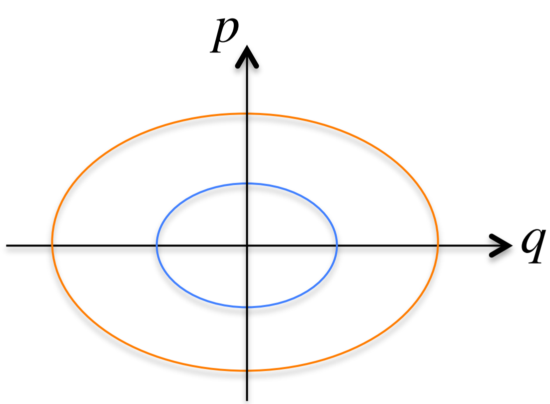
$$\delta S = \delta\int_A^B L(\vec{q},\dot{\vec{q}},t)dt = 0$$

↓

Hamilton's Equation

$$H(p,q,s) = p\dot{q} - L(q,\dot{q},s) = T + V$$

$$\begin{cases} dq/ds = \partial H/\partial p \\ dp/ds = -\partial H/\partial q \end{cases}$$



Simple Harmonic Oscillator

- Newton's equation

$$m\ddot{x} = -kx, \quad \text{or} \;\; \ddot{x} + \omega^2 x = 0 \quad (\text{for } \omega = \sqrt{k/m})$$

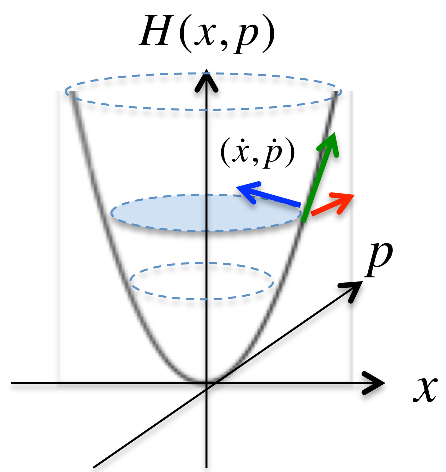
- Hamiltonian dynamics in phase space

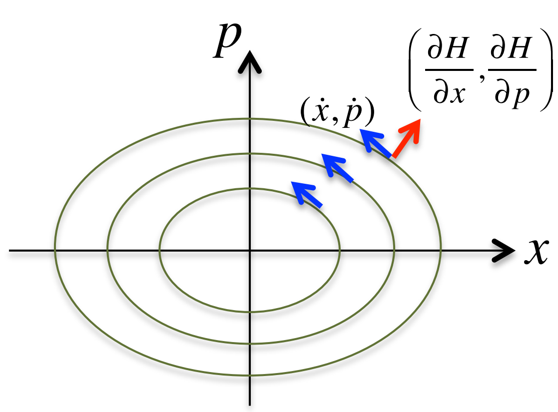
$$H(x,p) = T + V = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \quad \text{(total energy)}$$

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -\partial H/\partial p \\ \partial H/\partial x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial x \\ \partial H/\partial p \end{pmatrix}$$

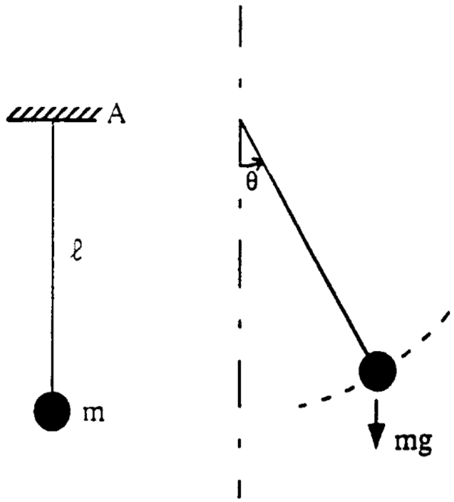
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0, \quad \Rightarrow H = \text{constant}$$

  - (x,p) are on equal footing in equation of motion, or called conjugate pair
  - Solutions are phase space orbits (x(t), p(t))
  - Phase space velocity vector is equal and perpendicular to the gradient vector of the Hamiltonian
  - Hamiltonian or energy is conserved





Nonharmonic Pendulum



Equation of motion:

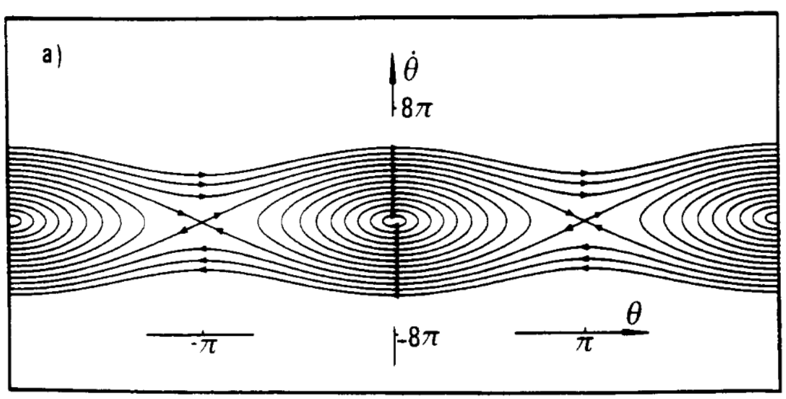
$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0 \qquad \left(\omega = \sqrt{\frac{g}{l}}\right)$$

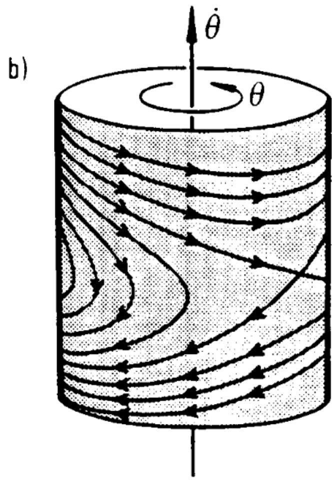
Hamiltonian (allows general coordinates):

$$H(\theta, p_\theta) = T + V = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

Trajectories in phase space are curves of constant energy E:

$$H(\theta, p_\theta) = E$$





Hamilton's Equation

- Hamilton's equation (for motion in n-dim space)

Vector In 2n-dim phase space

$$\begin{pmatrix} \dot{q}_1 \\ \dot{p}_1 \\ \dot{q}_2 \\ \dot{p}_2 \\ \vdots \end{pmatrix} = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{bmatrix} \cdot \begin{pmatrix} \partial H/\partial q_1 \\ \partial H/\partial p_1 \\ \partial H/\partial q_2 \\ \partial H/\partial p_2 \\ \vdots \end{pmatrix},$$

J: 2nx2n

$$z = \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \\ \vdots \end{pmatrix}, \qquad \frac{dz}{dt} = J \cdot \nabla_z H$$

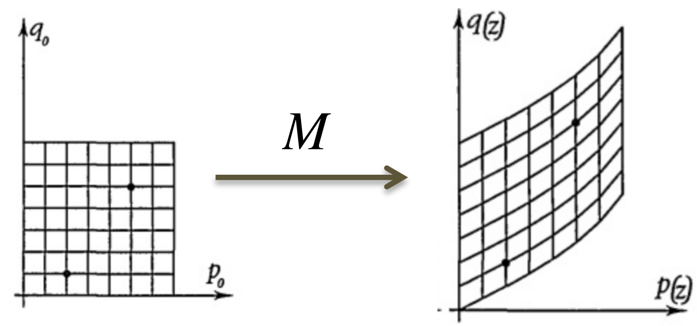
  - Phase space vector at time t depends on its previous state vector
  - At any time its flow is governed by the Hamiltonian equation

- Benefits of phase space description
  - Phase space vector is a complete representation of state of motion
  - Particle motion is described by the trajectories in phase space, these trajectories do not cross each other
  - Stability of dynamics are indicated by stable regions in the phase space structure

Symplectic Transformation

- Map for phase space flow (canonical transformation)

$$z_1 = z(t_1) \xrightarrow{M = \left(\frac{\partial z_{2i}(z_1)}{\partial z_{1j}}\right)_{2n\times 2n}} z_2 = z(t_2)$$

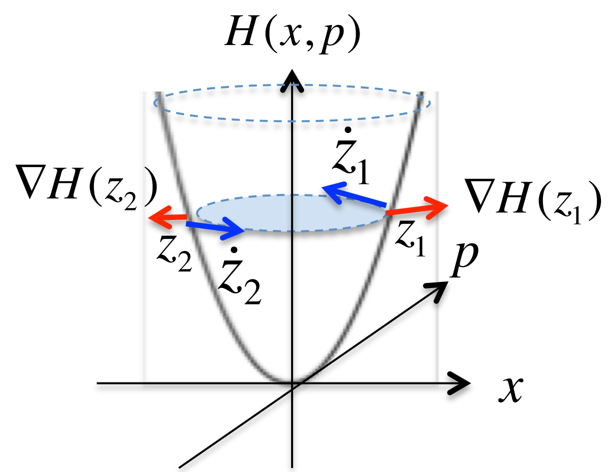


- Symplectic condition

$t = t_1$: $$\frac{dz_1}{dt} = J\ \nabla H(z_1, t_1)$$

$\dot{z}_2 = M\dot{z}_1$ $\qquad$ $\nabla H(z_1) = \tilde{M}\nabla H(z_2)$

$t = t_2$: $$\frac{dz_2}{dt} = J\ \nabla H(z_2, t_2)$$



M must satisfy the symplectic condition:

$$MJ\tilde{M} = J$$

Symplectic Constraints

- Symplectic condition

For n-dim space or 2n-dim phase space,

$$M J \tilde{M} = J$$

for

$$J = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{bmatrix}$$

- Number of constraints

Example: n=2, M is a map in 4-dim phase space

J is a 4×4 matrix wiht antisymmetry

$M J \tilde{M} = J$ sets 6 constraints for the 16 elements of M

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$$

In general, for 2n-dim phase space, symplectic condition sets K=n(2n-1) constraints for 2nx2n matrix elements of M.

n=1, K=1;   n=2, K=6;   n=3, K=15

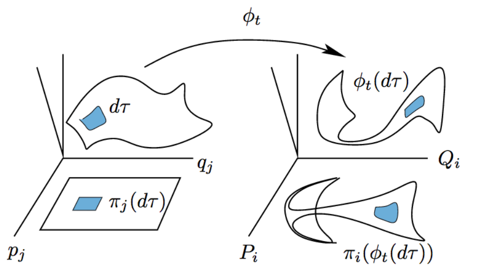
Features of Symplectic Transformation

- Liouville's Theorem

From $M J \tilde{M} = J$, we have

$$\det M = 1$$

or $\int_{\Omega'} d^{2n} z' = \int_{\Omega} \det M \; d^{2n} z = \text{constant}$



In 2D phase space, $\int_{\Omega} dp \wedge dq = \oint_{S} p\, dq = \text{constant}$

Fundamental invariant: Phase space volume of any enclosed boundary is conserved under symplectic map

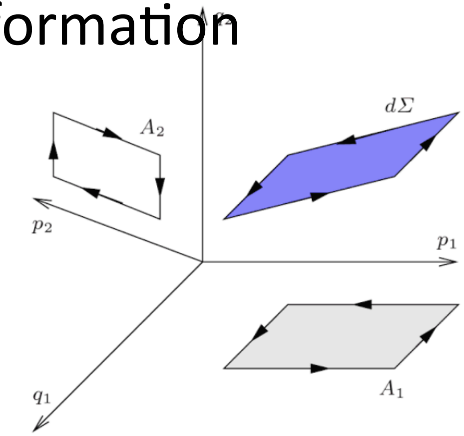
Locally, a parallelogram is transformed into a parallelogram.
Globally, the shape of the volume may change by the symplectic map,
but the volume stays constant.

Poincare-Cartan Invariants

- More conserved quantities: sum of projected subspace volumes are invariants under symplectic transformation

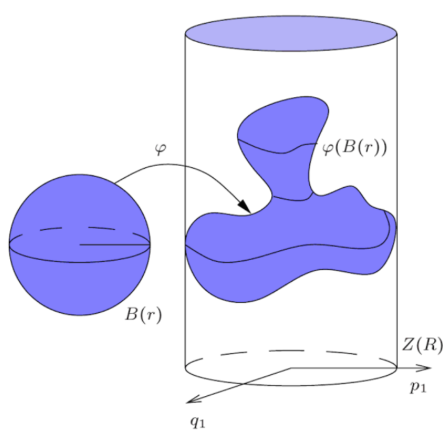
$$\Sigma_{V_2} = \int_{\mathrm{proj}(V_2)} dp_x dx + \int_{\mathrm{proj}(V_2)} dp_y dy + \int_{\mathrm{proj}(V_2)} dp_z dz$$

$$\Sigma_{V_4} = \int_{\mathrm{proj}(V_4)} dp_x dp_y dxdy + \int_{\mathrm{proj}(V_4)} dp_y dp_z dydz + \int_{\mathrm{proj}(V_4)} dp_z dp_x dzdx$$



- Gromov's nonsqueezing theorem

Starting from a ball with projected shadow $\Pi_2 = \pi R^2$
After canonical transform, the shadow on any plane will never decrease below its original value πR^2



Single Particle Dynamics In Accelerators

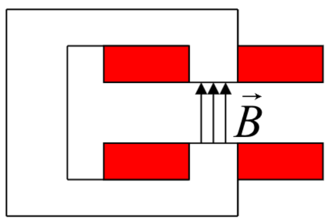
- Equation of motion for charged particles

$$\frac{d\gamma m\vec{v}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right), \quad \text{with} \quad \begin{cases} \vec{B} = \nabla\times\vec{A} \\ \vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \end{cases}$$

- EM fields in typical elements

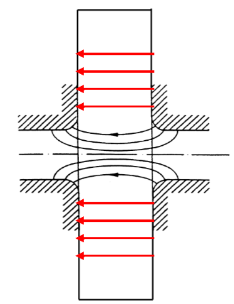
Dipole field:

$$B_y = B_0$$



Acceleration field:

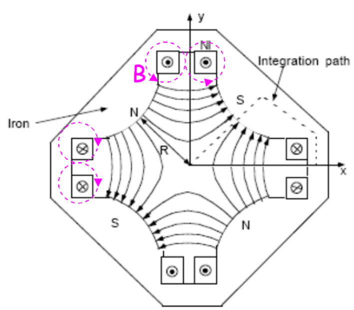
$$E_z = E_0(r,z)\cos(\omega t + \varphi)$$



Quadruple field:

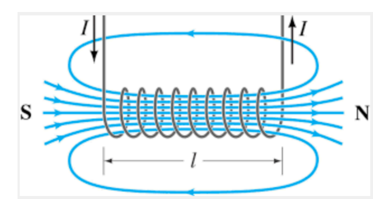
$$\vec{B} = B_1(x\vec{e}_x + y\vec{e}_y)$$

$$\nabla\times\vec{B} = 0: \quad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

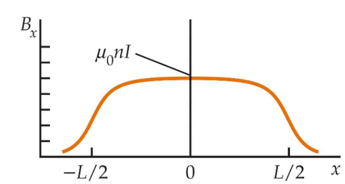


Solenoidal field:

$$B_0(z) = \frac{\mu_0 R^2}{2}\int_{-\infty}^{\infty} dz' \frac{I_\theta(z')\Delta r}{[(z-z')^2 + R^2]^{3/2}}$$



$$\begin{cases} B_z = B(z) - \frac{r^2}{4}B''(z) + \cdots \\ B_r = -\frac{r}{2}B'(z) - \frac{r^3}{16}B'''(z) + \cdots \end{cases}$$



Hamiltonian for Charged Particles

- Hamiltonian for a free particle

$$H = c\sqrt{p^2 + m^2c^2} \qquad (\text{relativistic},\ \vec{p} = \gamma m\vec{v})$$

$$H \simeq mc^2 + \frac{p^2}{2m} \qquad (\text{nonrelativistic},\ p \ll mc)$$

- Hamiltonian for a charged particle in EM fields

minimal coupling: $\vec{p} \to \vec{P} - e\vec{A}/c,\ \ H \to H - e\Phi$

$$H = \mathrm{c}\sqrt{\left(\vec{P} - \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 + m^2c^2} + e\Phi(\vec{x},t)$$

Independent variable: t

canonical conjugate pairs:

$(x,P_x),(y,P_y),(z,P_z)$

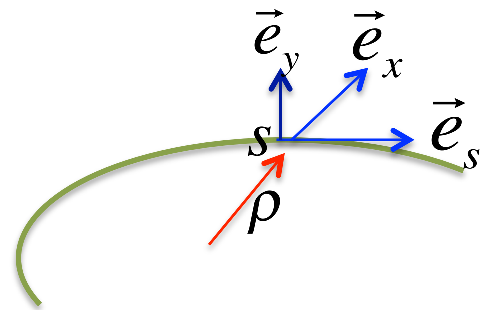
for $\vec{P} = \vec{p} + \dfrac{e}{c}\vec{A}$

Hamiltonian for Charged Particles

- Hamiltonian for charged particle in accelerators

(1) From lab frame to the curvillinear coordinates:

$$z \to s,\ P_z \to (\vec{P}\cdot\vec{e}_s)\cdot(1+x/\rho),\ A_z \to (\vec{A}\cdot\vec{e}_s)\cdot(1+x/\rho)$$



(2) Let s be independent variable: $(t,-H) \to (s,-P_s)$

New Hamiltonian:

$$H = -P_s = -eA_s/c - (1+x/\rho)\sqrt{(H-e\Phi)^2/c^2 - m^2c^2 - (P_x - eA_x/c)^2 - (P_y - eA_y/c)^2}$$

(3) For canonical pair: $z \equiv s - \beta_0 ct,\ \ \delta = \dfrac{1}{\beta_0^2}\dfrac{E-E_0}{E_0} \approx \dfrac{\Delta p}{p_0}$

$$H = -\left(1+\frac{x}{\rho}\right)\frac{qA_s}{p_0} - \frac{x(1+\delta)}{\rho} + \frac{1}{2(1+\delta)p_0}\left[\left(P_x - \frac{eA_x}{p_0}\right)^2 + \left(P_y - \frac{eA_y}{p_0}\right)^2\right] + \frac{1}{2\gamma_0^2}\delta^2 + \cdots$$

Canonical conjugate pairs: $(x,P_x),(y,P_y),(z,\delta)$

- Independent variable: s
- The new Hamiltonian is no longer total energy

Transverse Equation of Motion

- Vector potentials

The magnetic field for dipoles and quadruples is described by the vector potential

$$\begin{aligned} B_y &= -B_0(s) + B_1(s)x + \cdots \\ B_x &= B_1(s)y + \cdots \end{aligned}$$

$$B_0(s) = \frac{p_0 c}{e\rho(s)}$$

$$K_1(s) = \frac{eB_1(s)}{p_0 c}$$

$$\frac{qA_s}{cp_0} = -\left(\frac{x}{\rho} + \left(\frac{1}{\rho^2} - K_1\right)\frac{x^2}{2} + K_1\frac{y^2}{2}\right) + \cdots$$

- Hamiltonian to the 2nd order

$$H = \frac{p_x^2 + p_y^2}{2p} + \frac{1}{2}\left(\frac{1}{\rho^2} - K_1\right)x^2 + \frac{1}{2}K_1 y^2 - \frac{x\delta}{\rho}\cdots$$

when δ=0, the motion in x,y,z are decoupled

$$H = H_1(x, p_x) + H_2(y, p_y)$$

- Linear equation for transverse motion (for δ=0)

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{\partial H_1}{\partial p_x} \\ \dfrac{dp_x}{ds} = -\dfrac{\partial H_1}{\partial x} \end{cases}$$

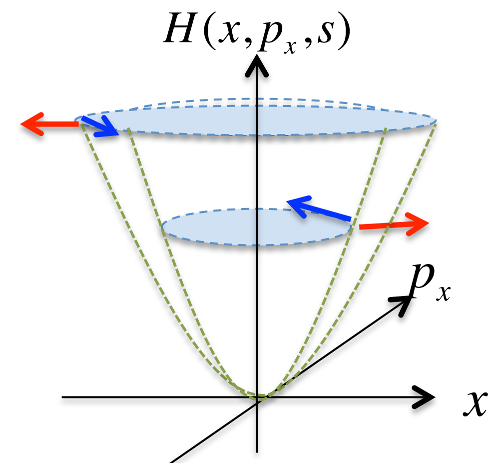
$$\begin{cases} x'' + \left(\dfrac{1}{\rho^2} - K_1\right)x = 0 \\ y'' + \dfrac{p_0}{p}K_1 y = 0 \end{cases}$$

If $d\delta/ds = -\partial H/\partial z = 0 \rightarrow \delta = \delta_0$

Hill's Equation

- Hill's equation for a time dependent, periodic harmonic oscillator

$$\frac{d^2x}{ds^2} + K(s)x = 0$$



for periodic lattice focusing: $K(s+L) = K(s)$

Hamiltonian: $H(x,p_x,s) = \frac{p_x^2}{2p_0} + \frac{K(s)x^2}{2}$

(H=-p_s is not energy)

Hamiltonian Equation:
$$\begin{cases} x' = \partial H/\partial p_x = p_x/p_0 \\ p_x' = -\partial H/\partial x = -K(s)x \end{cases}$$

(energy is conserved)

Canonical conjugate pair : (x, p_x) for $p_x = p_0 x'$

For constant energy, we can consider particle motion in (x,x')

Courant-Snyder Theory of Linear Beam Optics

- Courant-Snyder parameterization of transport map (periodic lattice)

Linear map: $\begin{pmatrix} x \\ x' \end{pmatrix}_{s+L} = M(s+L|s)\begin{pmatrix} x \\ x' \end{pmatrix}_s$,

$MJ\tilde{M} = J, \quad \det(\mathrm{M})=1,$

M has 3 free parameters

C-S parametrization:

$$M(s) = M(s+L|s) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}\cos\mu + \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{S}\sin\mu = I\cos\mu + S\sin\mu,$$

for $\beta\gamma\text{-}\alpha^2 = 1, \quad S^2 = -I$ $\quad (\alpha,\beta,\gamma)$: Twiss parameters

Key features:

$M^k = I\cos(k\mu) + S\sin(k\mu)$ similar to Euler's identity: $e^{ik\theta} = \cos k\theta + i\sin k\theta$

Hill's equation requires the Twiss parameters $(\beta,\alpha,\gamma,\mu)(s)$ to satisfy:

$$\alpha = -\beta'/2, \; \mu(s) = \int_0^s ds/\beta(s) + \varphi_0, \; \beta\beta''/2 - \beta'^2/4 + K(s)\beta^2 = 0$$

Solution for Hill's Equation

- General solution

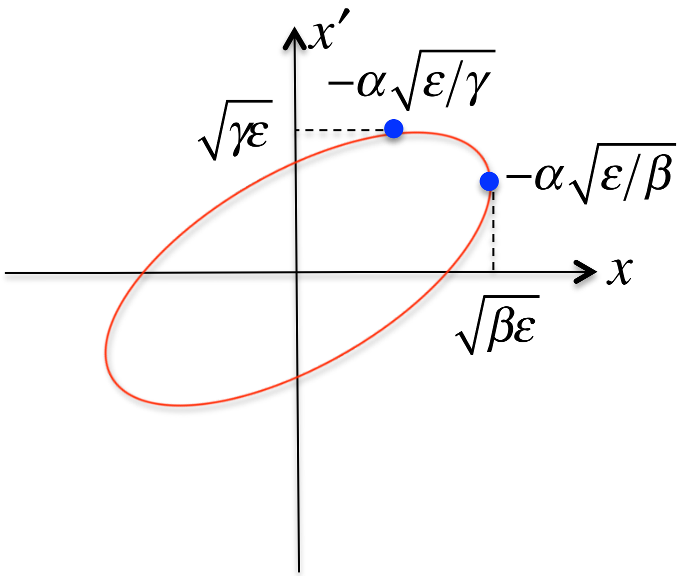
$$\begin{cases} x(s) = \sqrt{2J\beta(s)}\cos\varphi \\ x'(s) = -\sqrt{2J/\beta(s)}(\sin\varphi + \alpha(s)\cos\varphi) \end{cases}$$

for $\varphi(s) = \varphi(0) + \int_0^s \frac{ds_1}{\beta(s_1)}$

- Courant-Snyder Invariant

From $\sin^2\varphi + \cos^2\varphi = 1$,

$$J = \frac{1}{2\beta}\left[x^2 + (\beta x' + \alpha x)^2\right]$$

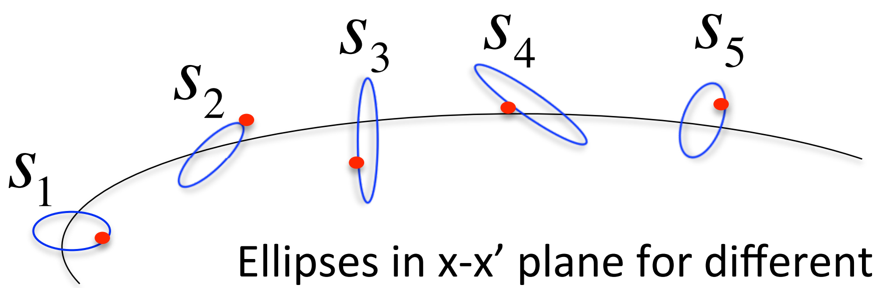


Phase Space Ellipse and Courant-Snyder Invariant

- Phase space ellipse for ring optics (Poincare section)

$$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = 2J$$

Area of ellipse=2πJ



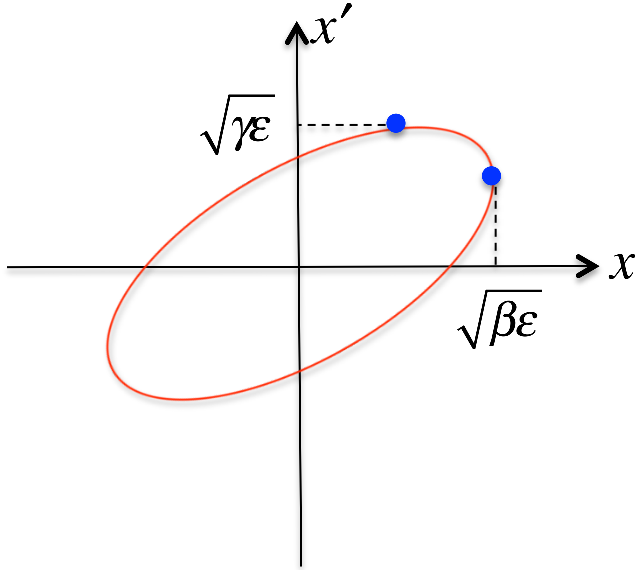
Ellipses in x-x' plane for different s

$$M(s_5|s_1) = M(s_5|s_4)\cdots M(s_3|s_2)M(s_2|s_1)$$

Unlike the harmonic oscillator, here the phase space ellipses changes with s. But because of detM=1, the area is conserved.

- Poincare invariant

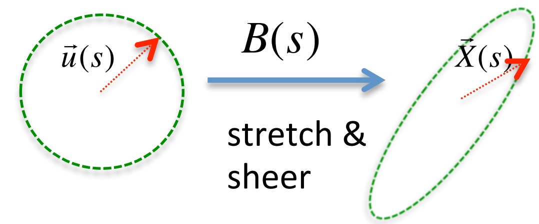
$$\text{Area enclosed}=\oint x'\,dx = 2\pi J$$



General Beam Transport

- General solution in terms of Twiss parameters

$$\underbrace{\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}}_{X(s)} = \sqrt{2J} \underbrace{\begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\alpha(s)/\sqrt{\beta(s)} & 1/\sqrt{\beta(s)} \end{pmatrix}}_{B(s)} \cdot \underbrace{\begin{pmatrix} \cos(\varphi(s)+\varphi_0) \\ -\sin(\varphi(s)+\varphi_0) \end{pmatrix}}_{u(s)}$$

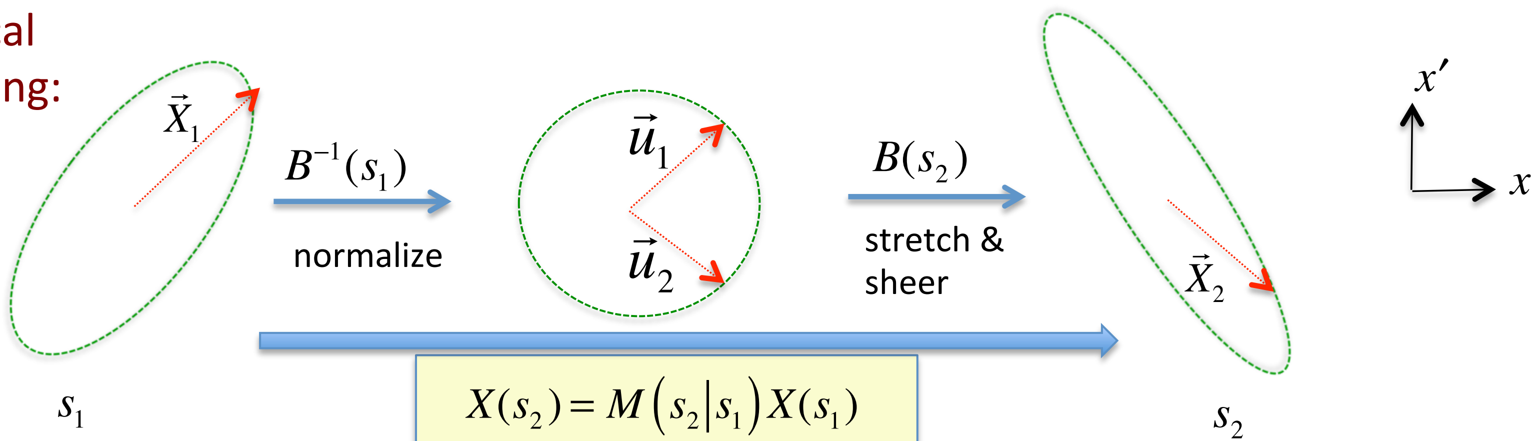


$\det B(s) = 1,\quad$ area preserved

- General expression of transport matrix (general lattice)

$$M(s_2|s_1) = B(s_2)\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} B^{-1}(s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi + \alpha_1 \sin\psi) & \sqrt{\beta_1\beta_2}\sin\psi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\psi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\psi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi - \alpha_2\sin\psi) \end{pmatrix}$$

Physical meaning:



Motion in a Solenoid

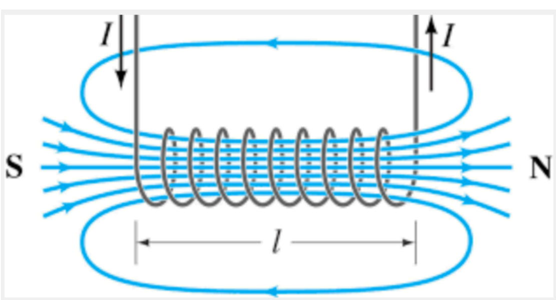
- Special features about solenoid
  - It plays an important role in magentized beam
  - x-y coupling
  - Constant of motion from cylindrical symmetry
  - Canonical momentum includes the vector potential

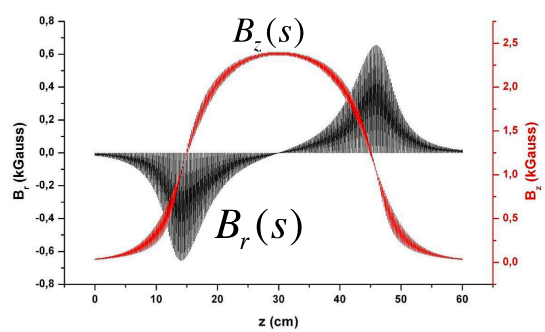
- The magnetic field in solenoid is cylindrically symmetric

$$\begin{cases} \nabla \times \vec{B} = 0 \\ \nabla \cdot \vec{B} = 0 \end{cases} \quad \begin{cases} B_z = B_z(s) + \cdots \\ B_r = -\frac{r}{2} B_z'(s) + \cdots \end{cases}$$

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{A} = \frac{B_z(s)}{2}(-y\vec{e}_x + x\vec{e}_y)$$

$$\text{or} \quad 2\pi r A_\theta = \pi r^2 B_z, \qquad A_\theta = \frac{B_z(s) r}{2} + \cdots$$





Motion in a Solenoid

- Hamiltonian

In Cartesian coordinates:

$$H = \frac{(P_x - eA_x/c)^2}{2p_0} + \frac{(P_y - eA_y/c)^2}{2p_0}$$

$$= \underbrace{\frac{P_x^2}{2p_0} + \frac{1}{8p_0}\left(\frac{eB_z}{c}\right)^2 x^2}_{H_x(s)} + \underbrace{\frac{P_y^2}{2p_0} + \frac{1}{8p_0}\left(\frac{eB_z}{c}\right)^2 y^2}_{H_y(s)} + \underbrace{\frac{eB_z}{2p_0 c}(P_x y - P_y x)}_{\text{coupling term}}$$

Canonical conjugate pairs: $(x, P_x), (y, P_y)$

Canonical momentum:

$$\begin{cases} P_x = p_0 x' + eA_x/c \\ P_y = p_0 y' + eA_y/c \end{cases} \quad \text{for} \quad \begin{cases} A_x = -B_z y/2 \\ A_y = B_z x/2 \end{cases}$$

The properties of symplectic transformation apply to the canonical conjugate pairs

Motion in a Solenoid

- Hamiltonian

In cylindrical coordinates:

$$H = \frac{1}{2p_0}\left(P_r^2 + \frac{e^2 B_z^{\,2}}{4c^2} r^2\right) + \frac{1}{2p_0}\left(\frac{P_\theta^2}{r^2} - \frac{eB_z P_\theta}{c}\right)$$

Canonical conjugate pairs: $(r, P_r), (\theta, P_\theta)$

- Cylindrical symmetry

$$\frac{dP_\theta}{ds} = -\frac{\partial H}{\partial \theta} = 0$$

(Busch's Theorem)

Canonical Angular Momentum (CAM)

$$P_\theta = \gamma\beta m r^2 \theta' + er A_\theta / c = \text{constant}$$

$$\text{i.e.} \quad \vec{L} = \vec{r} \times \vec{P} = \vec{r} \times (\vec{p} + e\vec{A}/c)$$

- Radial motion

$$\frac{dr}{ds} = -\frac{\partial H}{\partial P_r}, \quad \frac{dP_r}{ds} = -\frac{\partial H}{\partial r}$$

$$r'' + \left(\frac{eB_z}{2p_0 c}\right)^2 r - \left(\frac{P_\theta}{p_0}\right)^2 \frac{1}{r^3} = 0$$

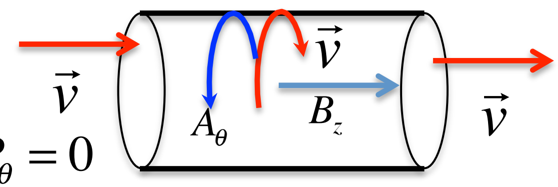
focusing   repulsive centrifugal force

Focusing in a Solenoid

- Example: quiet beam entering a solenoid

Outside the solenoid, the beam is quiet:

$$v_r = v_\theta = 0, \;\; B_z(s) = 0 \qquad \vec{v} \quad P_\theta = 0$$



Canonical angular momentum is conserved: $P_\theta = \gamma\beta m r^2 \theta' + erA_\theta / c = 0$

Inside the solenoid body, the charge acquires angular frequency

$$A_\theta = \frac{r}{2} B_z(s), \;\; \text{so} \;\; \gamma m r^2 \dot{\theta} = -\frac{r^2}{2c} eB_z(s), \quad \omega_L = \dot{\theta} = -\frac{eB_z}{2p_0} = -\frac{\omega_c}{2}$$

Radial focusing inside the solenoid:

$$P_\theta = 0$$

$$r'' + \left(\frac{eB_z}{2p_0 c}\right)^2 r - \left(\frac{P_\theta}{p_0}\right)^2 \frac{1}{r^3} = 0$$

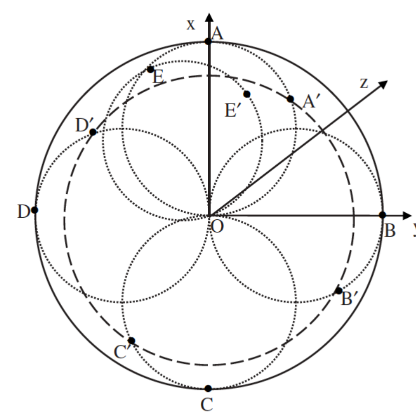
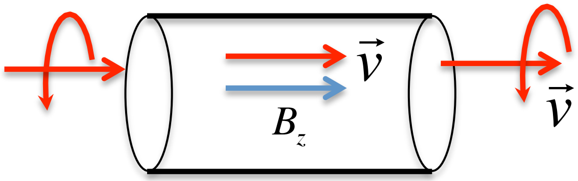


Illustration of solenoid focusing

Motion in a Solenoid

- Example: rotating beam entering a solenoid

At entrance, the particle has transverse rotation



$$v_r = 0, \quad v_\theta = \frac{\omega_c}{2}, \quad B_z(s) = 0$$

$A_\theta = 0$

Canonical angular momentum is conserved: $P_\theta = \gamma\beta m r^2\theta' + erA_\theta/c = er^2\frac{B_z}{2c}$

Inside the solenoid body, particle transverse motion is quiet

$$P_\theta = \gamma\beta m r^2\theta' + erA_\theta/c = er^2\frac{B_z}{2c},$$

(=)

with $A_\theta = \frac{r}{2}B_z(s)$, so $\theta' = 0$

Radial focusing inside the solenoid:

$$r'' + \left(\frac{eB_z}{2p_0c}\right)^2 r - \left(\frac{P_\theta}{p_0}\right)^2\frac{1}{r^3} = 0$$

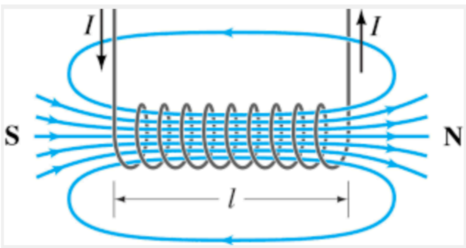
cancelled

$\Rightarrow$ $r = \text{constant}$ if $r'(0) = 0$

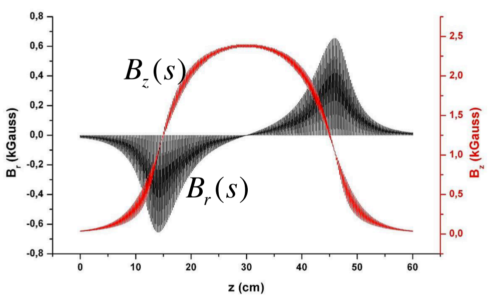
No focusing in r

Transport Matrix of a Solenoid

Transport for (x, x', y, y')



$$F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & K & 0 \\ 0 & 0 & 1 & 0 \\ -K & 0 & 0 & 1 \end{pmatrix} \qquad F_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -K & 0 \\ 0 & 0 & 1 & 0 \\ K & 0 & 0 & 1 \end{pmatrix}$$



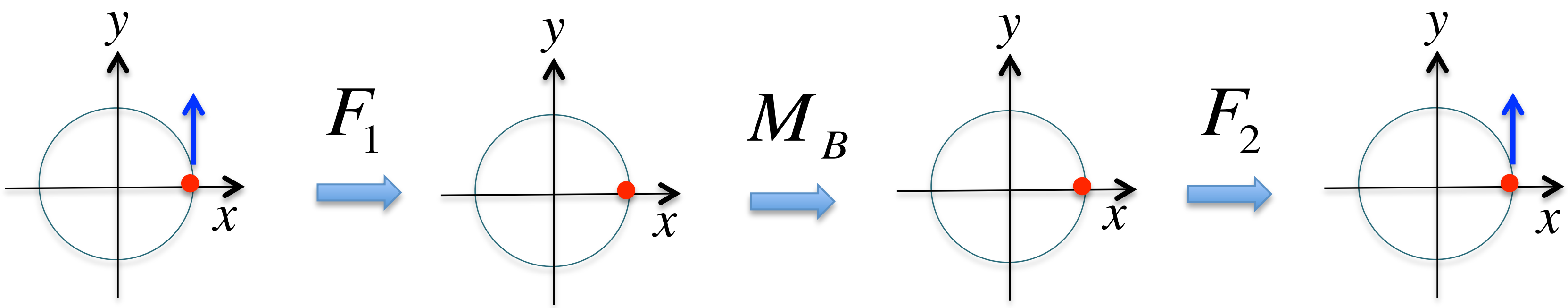
$$M_B = \begin{pmatrix} 1 & \sin\theta/2K & 0 & (1-\cos\theta)/2K \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & -(1-\cos\theta)/2K & 1 & \sin\theta/2K \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$K = eB/2pc$$

$$M = F_2 M_B F_1 = \begin{pmatrix} C^2 & CS/K & CS & S^2/K \\ -KCS & C^2 & -KS^2 & CS \\ -CS & -S^2/K & C^2 & CS/K \\ KS^2 & -CS & -KCS & C^2 \end{pmatrix}$$

Role of Solenoid Fringe Fields

$$K = eB/2pc$$



Incoming particle

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ K \end{pmatrix}$$

Rotation stopped after entrance fringe

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Rotation resumed after exit fringe

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ K \end{pmatrix}$$

Quiet beam good for Electron cooling

Coupled and Decoupled Motion

- Phase space transport for a general form of Hamiltonian

$$H = H(x, P_x, y, P_y, z, \delta)$$

  - symplectic map depends on phase space coordinates (nonlinear map)
  - coupling among subspaces $(x, p_x), (y, p_y)$ and (z, δ)

- Decoupled motion

$$H = H_1(x, P_x) + H_2(y, P_y) + H_3(z, \delta)$$

Motions in each subspace are transported separately

$$\begin{pmatrix} x(s_2) \\ P_x(s_2) \end{pmatrix} = M_1 \begin{pmatrix} x(s_1) \\ P_x(s_1) \end{pmatrix} \qquad M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}$$

etc.

Notions of Phase Spaces

- Canonical phase space

$$(x, P_x) \quad \text{for} \quad P_x = p_x + eA_x \qquad \text{(conjugate pair when } A_x \neq 0\text{)}$$

- Normalized phase space

$$(x, p_x) \quad \text{for} \quad p_x / mc = (\gamma\beta) \cdot x' \qquad \text{(when } A_x = 0\text{)}$$

- Trace space

$$(x, x') \qquad \text{(when } A_x = 0, \text{ and } p_0 = \gamma\beta mc = \text{const.)}$$

II. Emittance for a Bunch of Particles

- Liouville's Theorem for a Bunch of Particles
  - Beam as a vector in the 6N-dim phase space
  - Beam as an ensemble in the 6-dim phase space
  - Liouville's Theorem and incompressible flow

- Statistical description of emittance: moment matrix

- Role of Emittance in machine performance
  - Colliders: luminosity
  - Synchrotron radiation source: photon brilliance
  - FEL: gain performance and transverse/longitudinal coherence

- Notions of emittance
  - Canonical, normalized and geometric emittances
  - Machine and beam ellipses
  - Slice and projected emittance
  - Beam core and halo
  - Rms, 95%, etc
- Invariance of emittance under linear transport
  - Invariant for 2D linear transport
  - 6D emittance preservation
  - Decoupled system: emittance for each subspace is preserved
  - Eigen emittances and invariants of moments

Beam Transport in 6N-dimensional Phase Space

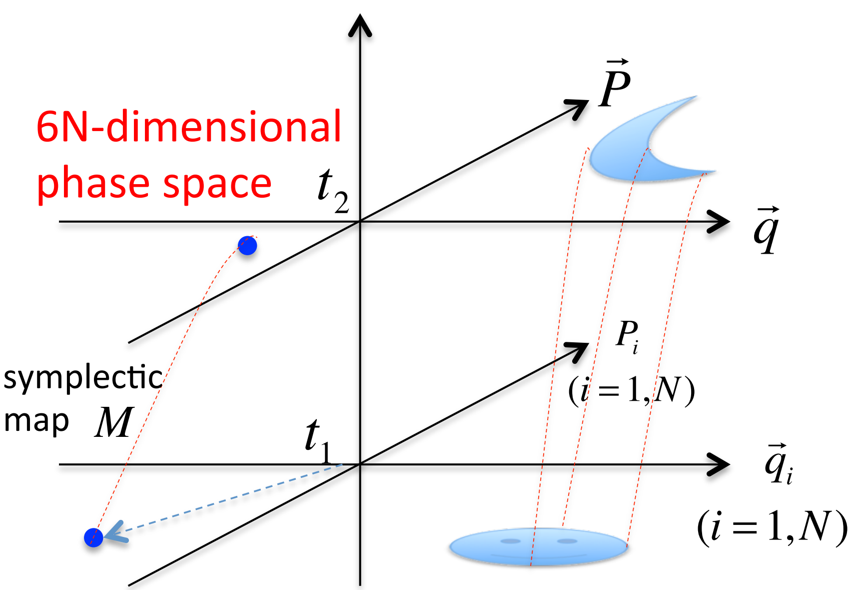
- A charged bunch in accelerators consists of an ensemble of *N* charged particles

A bunch of N particles:

described by $(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N)$ in configuration space $\longrightarrow$ $3N$ degree of freedom

canonical phase space $z = \left(\vec{r}_1, \vec{P}_1, \vec{r}_2, \vec{P}_2, \cdots \vec{r}_N, \vec{P}_N\right) \longrightarrow 6N$ degree of freedom

- The state of the beam can be represented by one state vector in the 6N dimensional phase space



For Hamiltonian $H = H(q_1, p_1, \cdots q_{6N}, p_{6N})$

$$\frac{dz}{dt} = J \cdot \nabla_z H, \quad z_t = M(t, t_0) z_0, \quad \text{with } MJM^T = J$$

$$\det M_{6N\times 6N} = 1, \quad \longrightarrow \int_{\Omega(t_1)} d^{6N} z_0 = \int_{\Omega(t_1)} d^{6N} z_t$$

Liouville Theorem: (a property of the mapping)

The volume enclosed by any hypersurface in the 6N phase space is conserved for system governed by Hamiltonian dynamics

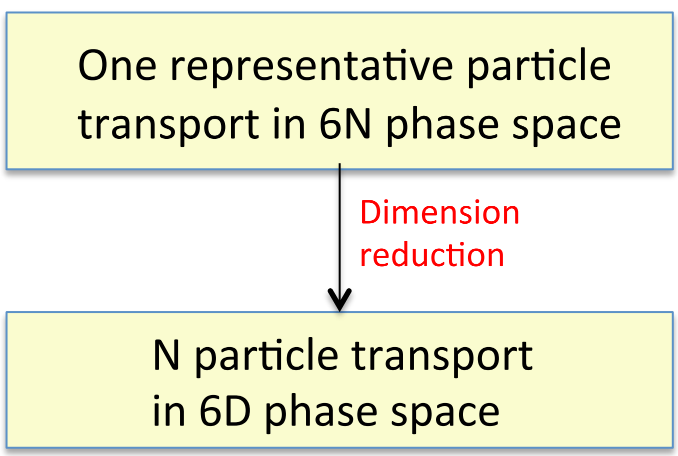
Beam Transport in 6D Phase Space

- Motion for an ensemble of non-interacting particles in a beam

When N particles are identical and non-interacting with each other,

$$H_{tot} = H(\vec{q}_1,\vec{p}_1) + H(\vec{q}_2,\vec{p}_2) + \cdots H(\vec{q}_N,\vec{p}_N)$$

we can consider only one particle motion in 6D phase space governed by $H(\vec{q},\vec{p})$



Beam Transport in 6D Phase Space

- Motion for an ensemble of mutually interacting particles

If the particle collective interaction can be described by potentials $\left(\Phi_{col}(\vec{r},t),\vec{A}_{col}(\vec{r},t)\right)$, as smooth functions of $\vec{r}$,

$$\begin{cases} \Phi = \Phi_{ext} + \Phi_{col} \\ \vec{A} = \vec{A}_{ext} + \vec{A}_{col} \end{cases}, \quad \text{with} \quad \begin{cases} \vec{E}_{col} = -\nabla\Phi_{col} - \dfrac{\partial \vec{A}_{col}}{c\,\partial t} \\ \vec{B}_{col} = \nabla \times \vec{A}_{col} \end{cases}$$

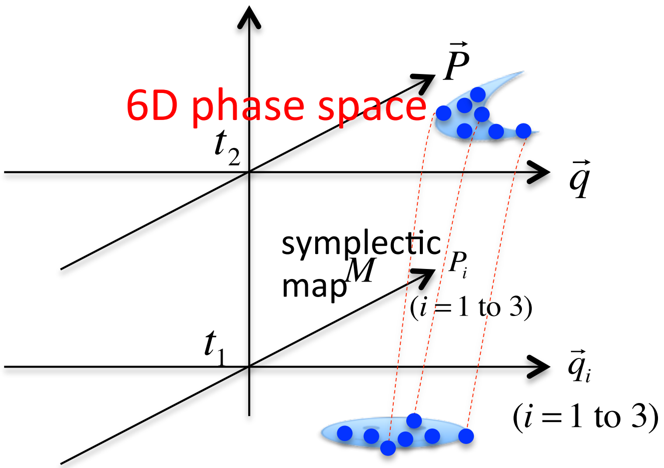
Then motion of each particle in 6D phase space is governed by the single particle Hamiltonian

$$H = c\sqrt{\left(\vec{P} - \frac{q}{c}\vec{A}(\vec{x},t)\right)^2 + m^2c^2} + q\Phi(\vec{x},t)$$

The smooth function requirement applies to the mean field of collective interaction, and it does not apply to Coulomb collision.

Incompressible Flow in 6D Phase Space

- Time evolution governed by reduced Hamiltonian is considered as symplectic mapping



- For large *N* particles closely clustered in small phase space volume, the beam state can be represented by phase space density
  $f(\vec{q},\vec{p},t)$

The number of particles in a small volume $d^6z = d^3\vec{q}d^3\vec{p}$ is

$$dN = f(\vec{q},\vec{p},t)d^3\vec{q}d^3\vec{p}$$

During transport

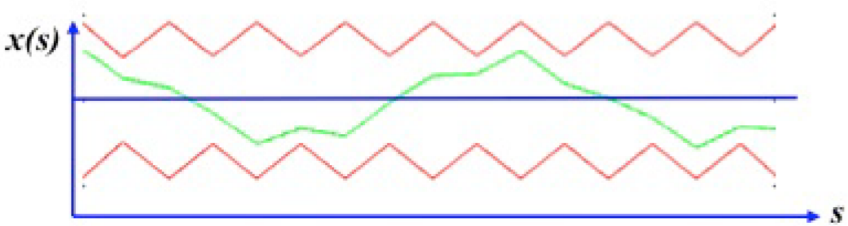
- dN=invariant (particle number conservation)
- $d^3\vec{q}\ d^3\vec{p} = \text{invariant}$ (Liouville theorem)

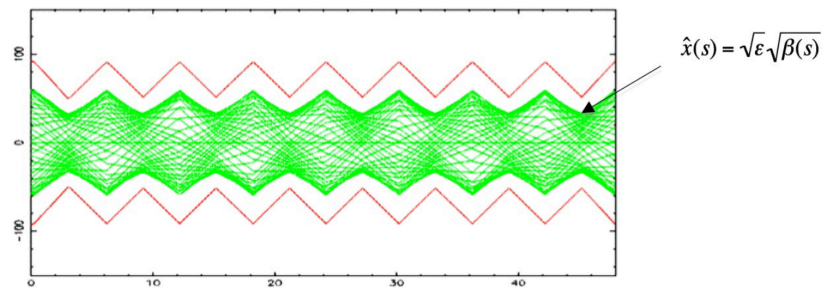
$\Rightarrow$ phase space density $f(q,p,t) = \text{constant}$

Beam motioncan be viewed as an incompressible flow in 6D phase space

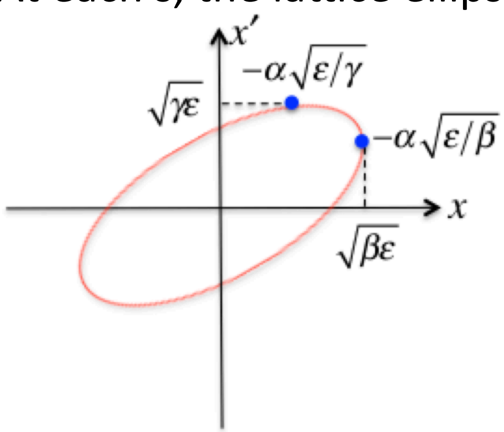
Phase Space Flow for a Beam of Particles

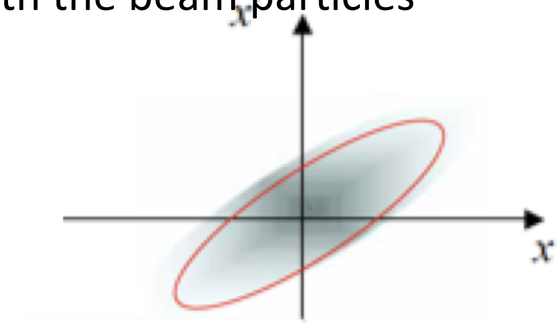
- A beam usually consists of N=10^9-10^{10} electron or ions.
- The particles in the beam can be viewed an ensemble in phase-space evolving in time following symplectic map, including the constraint of Liouville's theorem.





- At each s, the lattice ellipse can be filled with the beam particles

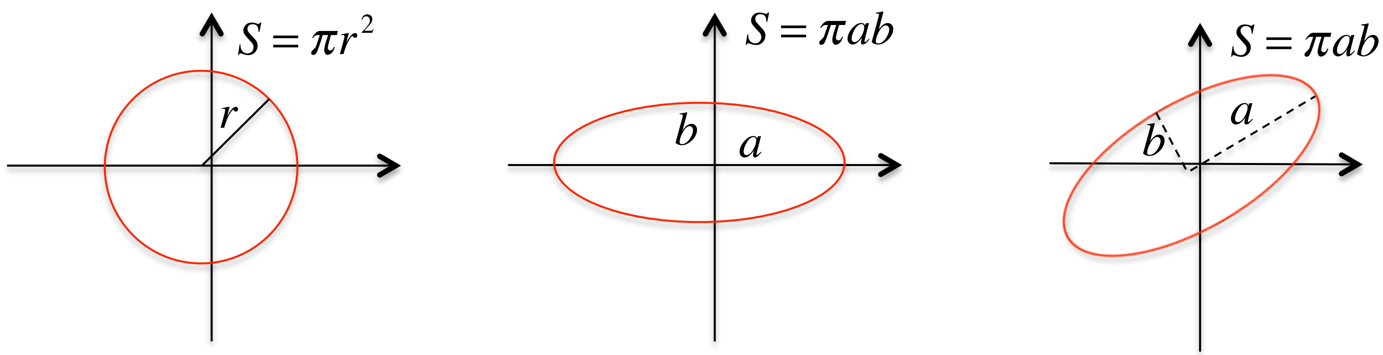




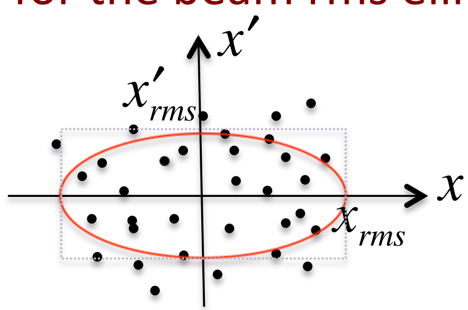
Statistical Description of Emittance

- For a beam in 2D x-x' phase space, we need to characterize the spread of beam particles in phase space.
- Emittance is a measure of the phase space area occupied by a beam.

- Area for an ellipse



- Area for the beam rms ellipse in the (x,x') space



$$\varepsilon_x = \frac{S}{\pi} = x_{rms} x'_{rms}$$

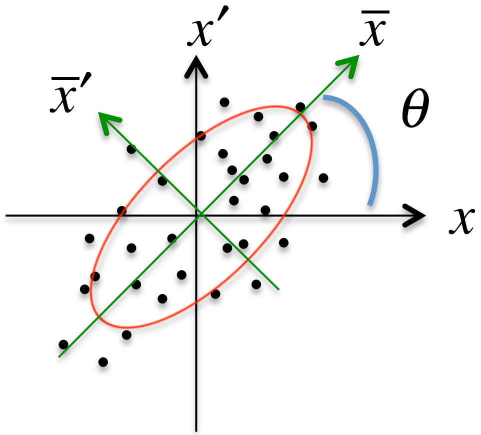
(Unit: mm-mrad)

for
$$\begin{cases} x_{rms}^2 = \langle x^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i^2 \\ x'^2_{rms} = \langle x'^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} x'^2_i \end{cases}$$

General RMS Ellipse and Emittance

Choosing θ for least square condition:

$$\frac{d}{d\theta}\sum_i \bar{x}_i^2 = \frac{d}{d\theta}\sum_i \left(x_i'\cos\theta - x_i\sin\theta\right)^2 = 0$$



The solution:

$$\begin{cases} \tan 2\theta = \dfrac{2\langle xx'\rangle}{\langle x^2\rangle - \langle x'^2\rangle}, \\ \langle \bar{x}^2\rangle = \dfrac{1}{2}\left(\langle x^2\rangle + \langle x'^2\rangle + \dfrac{2\langle xx'\rangle}{\sin 2\theta}\right) \\ \langle \bar{x}'^2\rangle = \dfrac{1}{2}\left(\langle x^2\rangle + \langle x'^2\rangle - \dfrac{2\langle xx'\rangle}{\sin 2\theta}\right) \end{cases}$$

$$\varepsilon_x = \frac{S}{\pi} = \bar{x}_{rms}\bar{x}'_{rms}$$
$$= \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

Beam Matrix

- Beam Matrix in terms of 2nd-order moments

Define beam matrix:

$$\text{For } X = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad \Sigma = \langle X\tilde{X} \rangle = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

$$\begin{cases} x_{rms}^2 = \langle x^2 \rangle = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\left(x_i - \langle x \rangle\right)^2 \\ x_{rms}'^2 = \langle x'^2 \rangle = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\left(x_i - \langle x \rangle\right)^2, \\ \langle xx' \rangle = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\left(x_i - \langle x \rangle\right)\left(x_i' - \langle x' \rangle\right) \end{cases}$$

The emittance is related to beam matrix by:

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} = \sqrt{\det \Sigma}$$

Emittance and Courant-Snyder Invariant

- Particle orbit in terms of lattice Twiss parameters

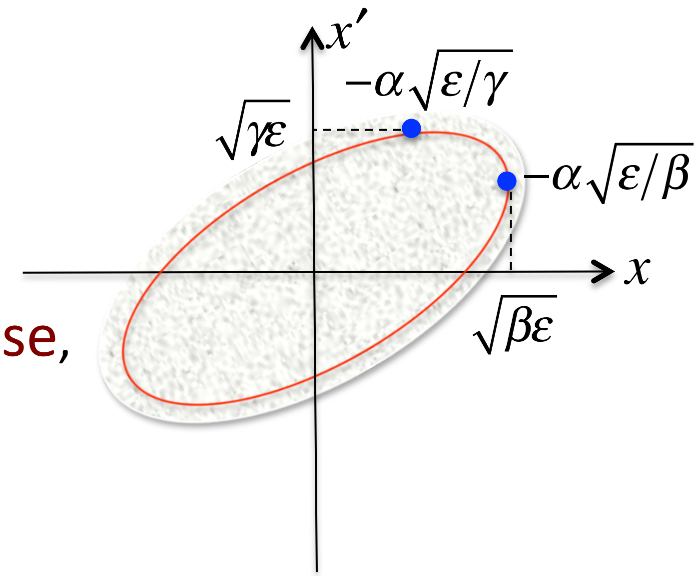
$$(i=1 \text{ to } N) \qquad \begin{cases} x_i(s) = \sqrt{2J_i\beta(s)}\cos(\psi(s)+\varphi_i) \\ x_i'(s) = -\sqrt{2J_i/\beta(s)}\left[\alpha(s)\cos(\psi(s)+\varphi_i)+\sin(\psi(s)+\varphi_i)\right] \end{cases}$$

For matched beam, for each J_i, particles are uniformly distributed around the ellipse,

$$\langle x^2\rangle = \frac{1}{N}\sum_{i=1}^{N} 2J_i\beta(s)\cos^2(\psi(s)+\varphi_i) = \beta(s)\langle J\rangle,$$

$$\langle x'^2\rangle = \gamma(s)\langle J\rangle, \text{ and } \langle xx'\rangle = \alpha(s)\langle J\rangle$$

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N} J_i = \langle J\rangle$$



- Beam ellipse matched with lattice ellipse,

$$\Sigma_{2D} = \varepsilon\begin{pmatrix} \beta & \alpha \\ \alpha & \gamma \end{pmatrix} \quad \text{with} \quad \beta\gamma-\alpha^2=1$$

Invariance of Emittance Under Linear Symplectic Transport

- Invariance of emittance for matched beam

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N} J_i = \langle J \rangle$$

Average of area for ellipse of each particle, conserved during symplectic transport

- Invariance for general beam distribution

$$\text{For} \quad X(s_2) = \begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = MX(s_1), \quad \Sigma(s_2) = \langle X(s_2)\tilde{X}(s_2) \rangle = M\Sigma(s_1)\tilde{M}$$

$$\Sigma(s_2) = M\Sigma(s_1)\tilde{M},$$

$$\det\Sigma(s_2) = \left(\det M\right)^2 \det\Sigma(s_1)$$

$= 1$ (symplectic)

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} = \sqrt{\det \Sigma} = \text{const.}$$

Emittance for Various Phase Spaces

- **Emittance in trace space** (x, x')

  (geometric emittance)
  invariant for constant energy

$$\varepsilon_x^2 = \det \Sigma_{xx'} = \det \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

- **Emittance in normalized trace space** $(x, \overline{p}_x)$

  for the kinetic momentum

  $\overline{p}_x = p_x / mc = \gamma_0 \beta_0 x'$

$$\varepsilon_{nx}^2 = \det \Sigma_{x\overline{p}_x} = \det \begin{pmatrix} \langle x^2 \rangle & \langle x\overline{p}_x \rangle \\ \langle x\overline{p}_x \rangle & \langle \overline{p}_x^2 \rangle \end{pmatrix}$$

  Adiabatic damping:
  reduction of geometric emittance during acceleration $\varepsilon_{nx} = \gamma_0 \beta_0 \varepsilon_x$

- **Emittance in canonical phase space** $(x, \overline{P}_x)$

  for the canonical momentum

  $\overline{P}_x = P_x / mc, \;\; P_x = p_x + e A_x / c$

$$\varepsilon_{cx}^2 = \det \Sigma_{x\overline{P}_x} = \det \begin{pmatrix} \langle x^2 \rangle & \langle x\overline{P}_x \rangle \\ \langle x\overline{P}_x \rangle & \langle \overline{P}_x^2 \rangle \end{pmatrix}$$

Emittance for 6D Phase Space

- Particles in 6D phase space $(x, P_x, y, P_y, z, \delta)$ form a 6D RMS ellipsoid

Beam matrix

$$\Sigma_{6D} = \langle X\tilde{X} \rangle = \begin{pmatrix} \langle x^2 \rangle & \langle xP_x \rangle & \langle xy \rangle & \langle xP_y \rangle & \langle xz \rangle & \langle xP_z \rangle \\ \langle P_x x \rangle & \langle P_x^2 \rangle & \langle P_x y \rangle & \langle P_x P_y \rangle & \langle P_x z \rangle & \langle P_x P_z \rangle \\ \langle yx \rangle & \langle yP_x \rangle & \langle y^2 \rangle & \langle yP_y \rangle & \langle yz \rangle & \langle yP_z \rangle \\ \langle P_y x \rangle & \langle P_y x' \rangle & \langle P_y y \rangle & \langle P_y^2 \rangle & \langle P_y z \rangle & \langle P_y P_z \rangle \\ \langle zx \rangle & \langle zP_x \rangle & \langle zy \rangle & \langle zP_y \rangle & \langle z^2 \rangle & \langle zP_z \rangle \\ \langle P_z x \rangle & \langle P_z P_x \rangle & \langle P_z y \rangle & \langle P_z P_y \rangle & \langle P_z z \rangle & \langle P_z^2 \rangle \end{pmatrix}$$

Beam 6D rms emittance:

$$\varepsilon_{6D} = \sqrt{\det \Sigma_{6D}}$$

Invariant for symplectic transport

Projected emittance in (x, P_x), or in (y, P_y), (z, δ)

$$\varepsilon_{cx}^2 = \det \Sigma_{2D} = \det \begin{pmatrix} \langle x^2 \rangle & \langle xP_x \rangle \\ \langle xP_x \rangle & \langle P_x^2 \rangle \end{pmatrix}$$

Eigen Emittance

- Diaganolization of the beam 2nd moment matrix

At each s, there exists similarity transformation:

$$X(s) = R(s)U(s), \quad \text{or} \quad \begin{pmatrix} x \\ P_x \\ y \\ P_y \\ z \\ P_z \end{pmatrix} = R \begin{pmatrix} u_x \\ u_{P_x} \\ u_y \\ u_{P_y} \\ u_z \\ u_{P_z} \end{pmatrix},$$

such that

$$\Sigma(s) = \left\langle X\tilde{X} \right\rangle = R(s)D(s)R^{-1}(s)$$

for

$$D = \left\langle U\tilde{U} \right\rangle = \begin{pmatrix} \langle u_x^2 \rangle & & & & & \\ & \langle u_{P_x}^2 \rangle & & & 0 & \\ & & \langle u_y^2 \rangle & & & \\ & & & \langle u_{P_y}^2 \rangle & & \\ & 0 & & & \langle u_z^2 \rangle & \\ & & & & & \langle u_{P_z}^2 \rangle \end{pmatrix}$$

- Eigen emittances:

$$\varepsilon_x^2(s) = \left\langle u_x^2 \right\rangle \left\langle u_{P_x}^2 \right\rangle$$

$$\varepsilon_y^2(s) = \left\langle u_y^2 \right\rangle \left\langle u_{P_y}^2 \right\rangle$$

$$\varepsilon_z^2(s) = \left\langle u_z^2 \right\rangle \left\langle u_{P_z}^2 \right\rangle$$

Invariants of Eigen Emittance

- Invariants of eigen emittance under linear symplectic transport

$$I_0 = \varepsilon_x^2 \cdot \varepsilon_y^2 \cdot \varepsilon_z^2$$

(volume invariance)

$$I_2 = \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2$$

(sum of projected area for all subspaces)

$$I_4 = \varepsilon_x^4 + \varepsilon_y^4 + \varepsilon_z^4$$

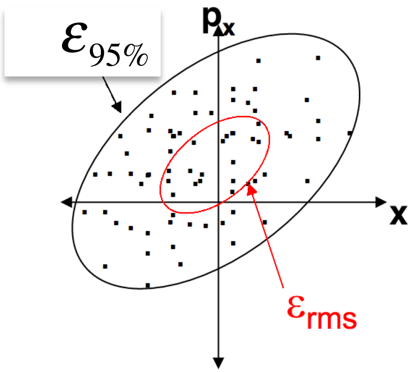
$$I_4 = \varepsilon_x^6 + \varepsilon_y^6 + \varepsilon_z^6$$

Notions of Emittance

- Different definitions of emittance.

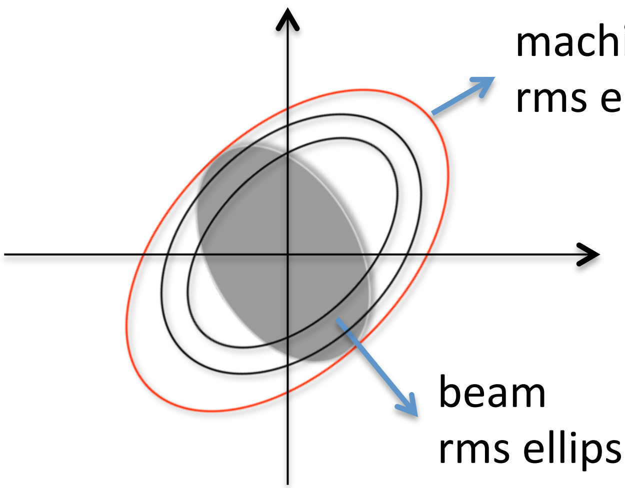
$\varepsilon_{95\%}$ is the area of ellipse that contains 95% of the beam, with the same (β, α, γ) for the ellipse as the rms one.

$\varepsilon_{100\%}$ is the area of ellipse that contains 100% of the beam.



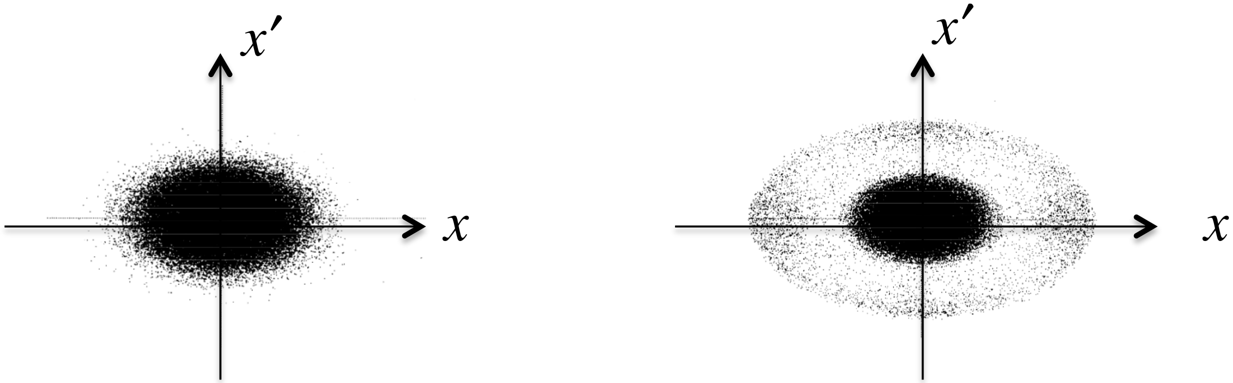
- Beam and machine ellipse

A beam injected into a lattice may not have its rms ellipse matched to the machine ellipse.



Notions of Emittance

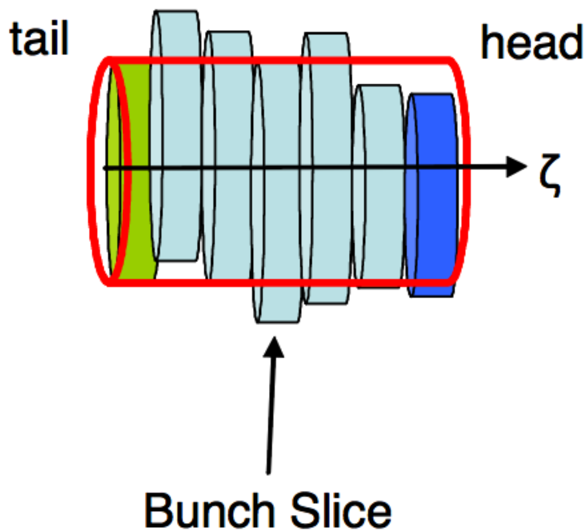
- Beam core and halo

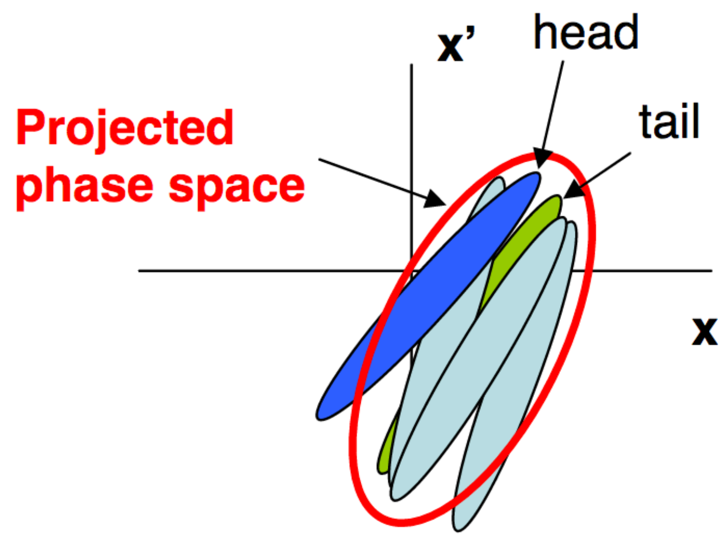


- Halo may have different ellipse as the core distribution
- The rms emittance measures the spread of particles for the core of the beam. It is a good description of beam distribution if the beam is dominated by distribution in the core
- Each tail particles has contribution to the emittance than a particle in the beam core. Emittance is not a good measure of beam-core spread if there are many tail particles

Notions of Emittance

- Slice and projected emittance





- Each longitudinal slice of the bunch may have its own phase space ellipse and slice emittance in the x-x' phase space

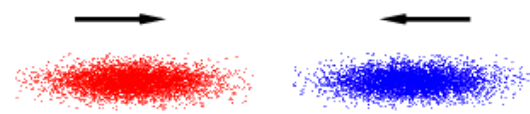
- The projection of the whole beam on the x-x' phase space will have larger rms spread, or larger projected emittance

Role of Emittance in Colliders

- Luminosity
  - The performance of a hadron or lepton collider is characterized by the beam energy and the luminosity.
  - Luminosity is the event rate for unit cross-section area [cm^{-2}s^{-1}]

$$L = \frac{dR}{\sigma_{\text{int}} dt} \quad \left[\text{cm}^{-2}\text{s}^{-1}\right]$$

$$= 2cf_{rep} \int \rho_2(x,y,z-ct)\rho_1(x,y,z+ct)\ d^3\vec{x}dt$$



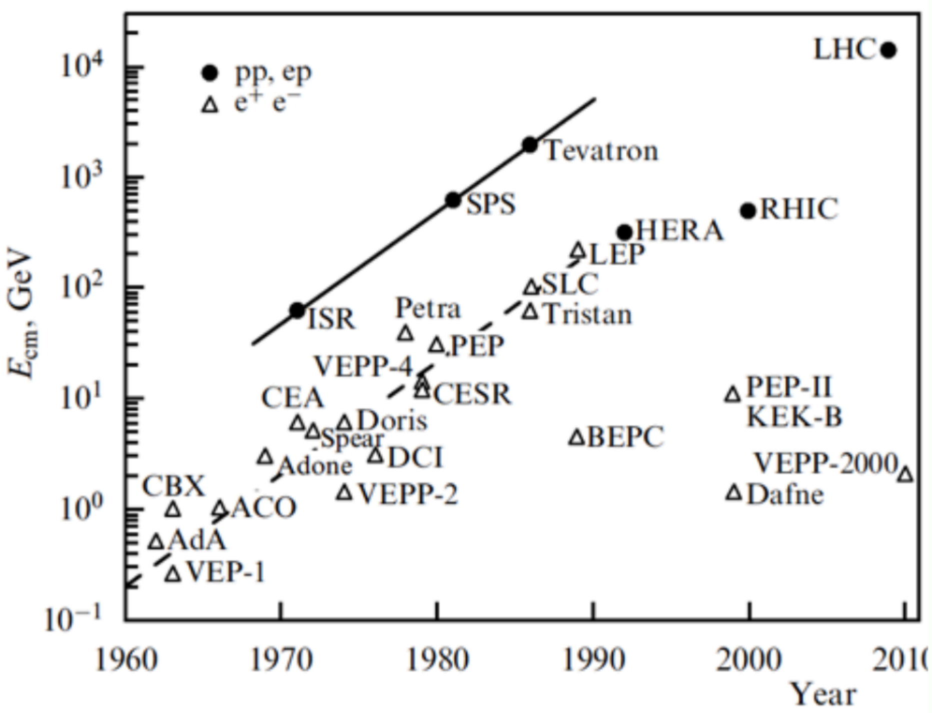
For Gaussian beams, if $\sigma_{1x} = \sigma_{2x} = \sqrt{\varepsilon_x \beta_x^*}$, $\sigma_{1y} = \sigma_{2y}\sqrt{\varepsilon_y \beta_y^*}$,

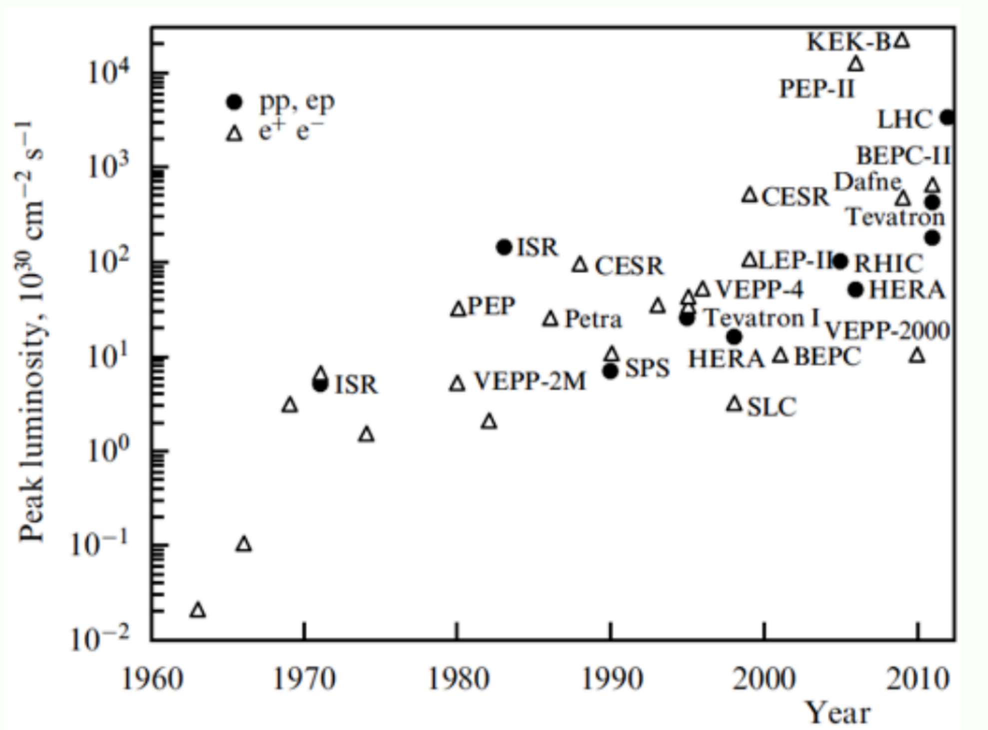
$$L = \frac{f_{rep} N_1 N_2}{4\pi\sigma_x\sigma_y} = \frac{f_{rep} N_1 N_2}{4\pi\sqrt{\varepsilon_x \beta_x^* \varepsilon_y \beta_y^*}}$$

The goal of pushing for high luminosity is to design optics with small beta*, and deliver high charge beam with small emittance at interaction point.

Path Toward Higher Luminosity

V D Shiltsev, "High energy particle colliders: past 20 years, next 20 years and beyond", Physics-Uspekhi 55 (10) 965-976 (2012)

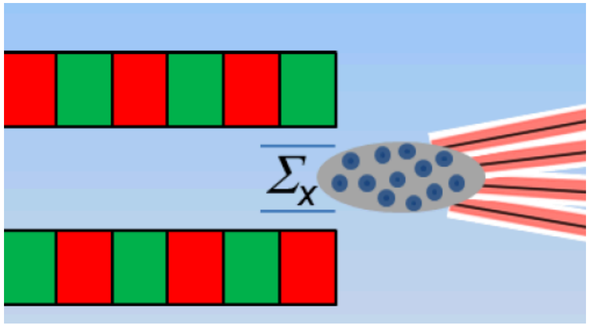




- With the increased understanding of mechanism of emittance growth and mitigation method, and advances in technology and simulations, the colliders continue to achieve higher luminosity performances.

Role of Emittance in Undulator Radiation

- Brightness of a photon beam



Photon pulse generated by single electron

$$B_0(\vec{x},\vec{x}')=\frac{F_1(\omega)}{(\lambda/2)^2}e^{-\vec{x}^2/2\sigma_r^2-\vec{x}'^2/2\sigma_r'^2}, \quad \text{with} \quad \varepsilon_\gamma=\sigma_r\sigma_r'=\frac{\lambda}{4\pi}$$

Photon pulse generated by beam of electrons

$$B(\vec{x},\vec{x}')=\int B_0(\vec{x}-\vec{x}_1,\vec{x}'-\vec{x}_1',z)\,f(\vec{x}_1,\vec{x}_1')\,d\vec{x}_1 d\vec{x}_1'$$

$$B(0,0)=\frac{N_e F_1(\omega)}{(2\pi)^2\Sigma_x\Sigma_{x'}\Sigma_y\Sigma_{y'}}$$

$$\text{for} \quad \Sigma_{x,y}=\sqrt{\sigma_{x,y}^2+\sigma_r^2}, \quad \Sigma_{x',y'}=\sqrt{\sigma_{x',y'}^2+\sigma_r'^2}$$

Radiation dominated regime (diffraction-limited regime)

$$B\sim N_e B_0 \quad \text{when} \quad \varepsilon_{x,y}\le\varepsilon_\gamma$$

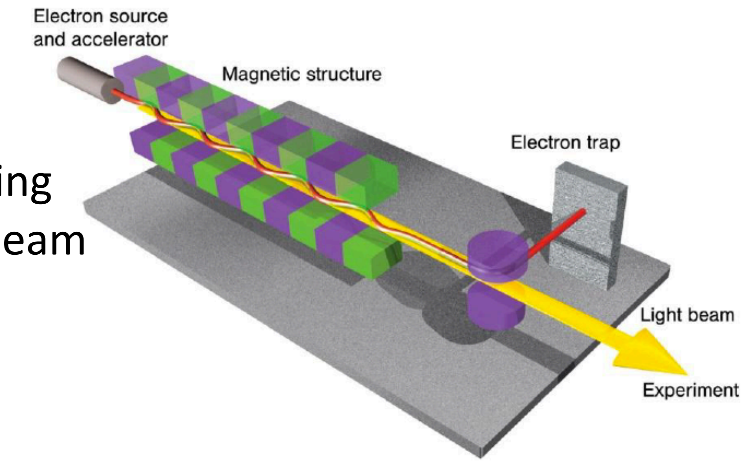
Requirement on emittance

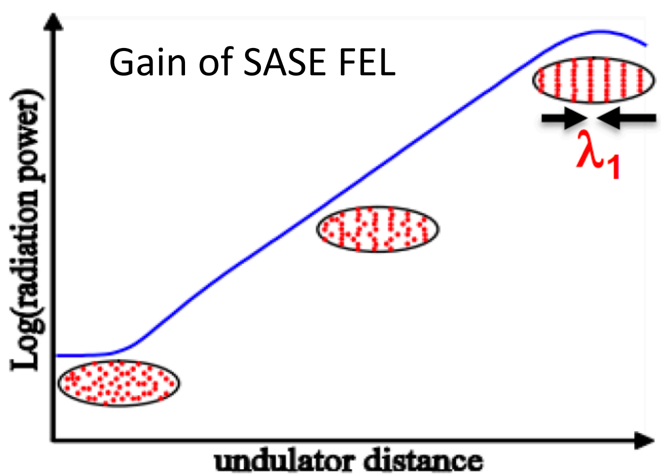
Role of Emittance in FEL Performance

- Gain performance for SASE X-ray FEL
  - Transverse divergence will change the average longitudinal velocity of the electron bunch, causing change of synchronism of e-beam with photon beam
  - FEL gain performance requires

$$\frac{\varepsilon_n}{\gamma} \leq \frac{\lambda_r}{4\pi} \quad \text{for} \quad \lambda_r = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$

Larger emittance means longer gain length and the need for longer undulator.

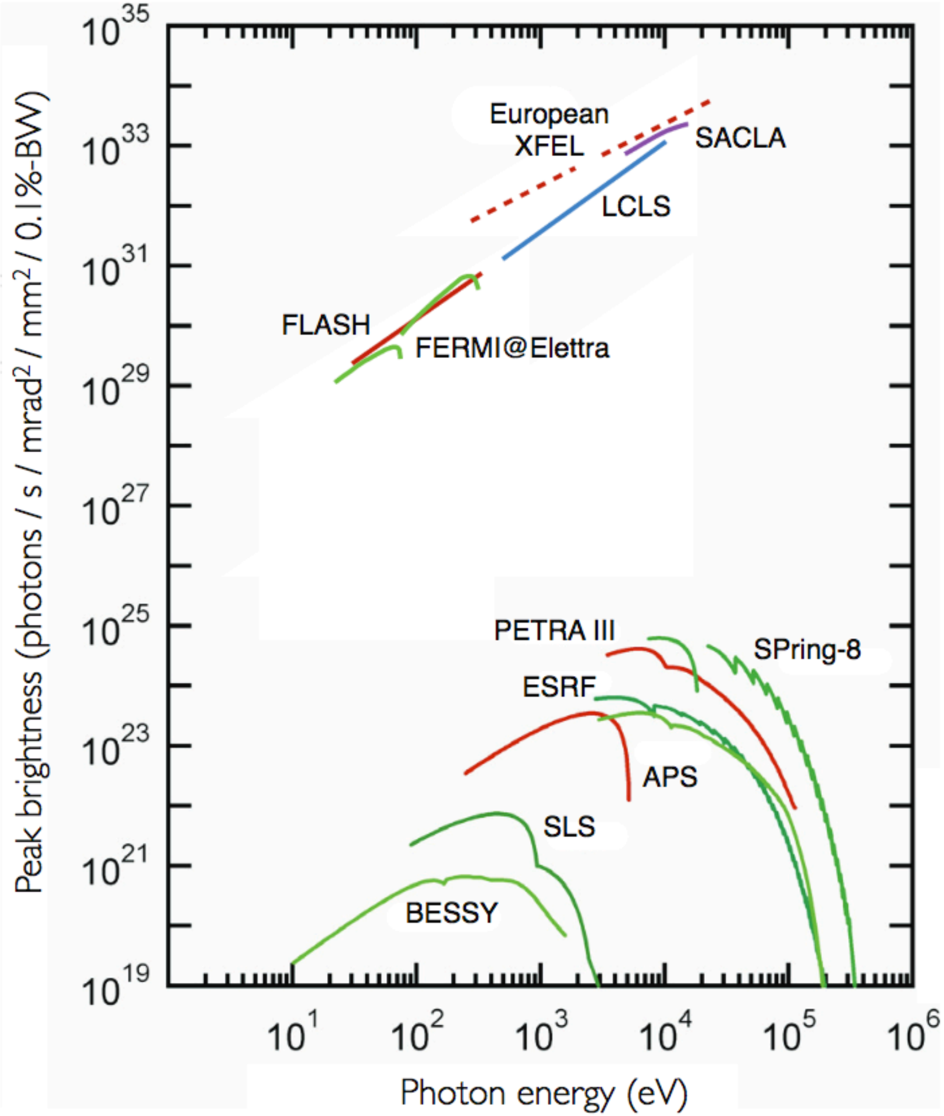




- Example

LCLS: $\varepsilon_{nx} = \gamma_0 \varepsilon_x \sim 1 \text{ mm-mrad}$

Path toward Higher Brightness



III. Emittance Dilution and Mitigation Methods

- Sources of emittance dilution
- Mechanisms for emittance dilution
  - Mismatch
  - Filamentation from nonlinear optics
  - Collective effects
  - Halo formation
  - Scattering
- Examples of Mitigation Method
  - Space charge compensation
  - Electron cooling to suppress IBS effects

Sources of Emittance Dilution

Hamiltonian

- Accelerator system
  - Beam mismatch
  - Nonlinear optics
  - Errors, misalignments
- Collective effects
  - Space charge
  - Coherent synchrotron radiation (CSR)
  - Wakefield (impedance)
- Two-beam effects
  - Beam-beam
  - Electron cloud for positively charged beam
  - Ion effects for electron beams

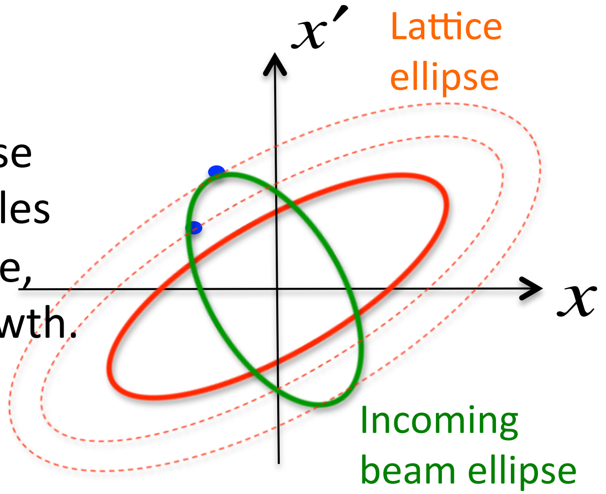
Non-Hamiltonian

- Synchrotron radiation
- Scattering
  - Residual gas scattering
  - Intrabeam scattering
  - Touschek scattering

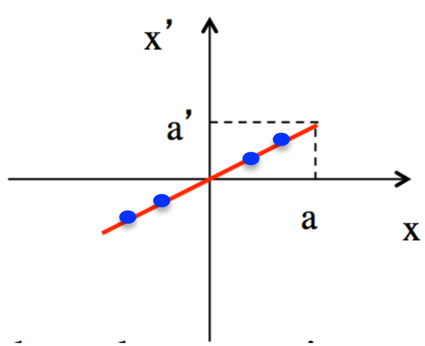
Example Mechanisms of Emittance Dilution

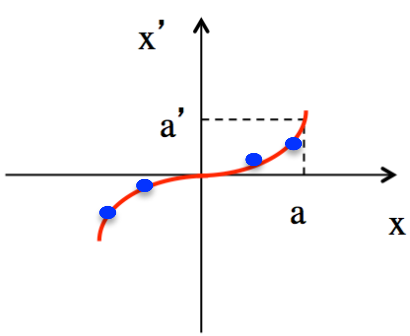
- Mismatch of beam and lattice ellipse

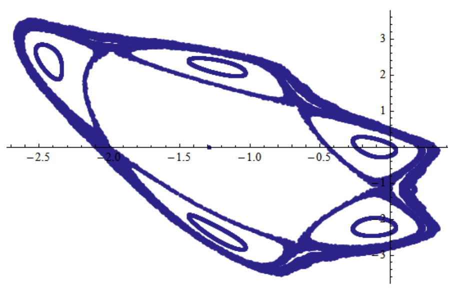
For a periodic lattice, when the incoming beam ellipse differs from machine lattice ellipse, the beam particles phase space trajectory will follow the machine ellipse, causing enlarged phase space area or emittance growth.



- Nonlinear optics
  - Nonlinear optics maps a straight line in phase space to a curved line. It features tune dependence on the betatron amplitude.
  - Phase space may have resonance islands at large amplitudes that trap particles, and separatrix may lead particle to chaotic motion



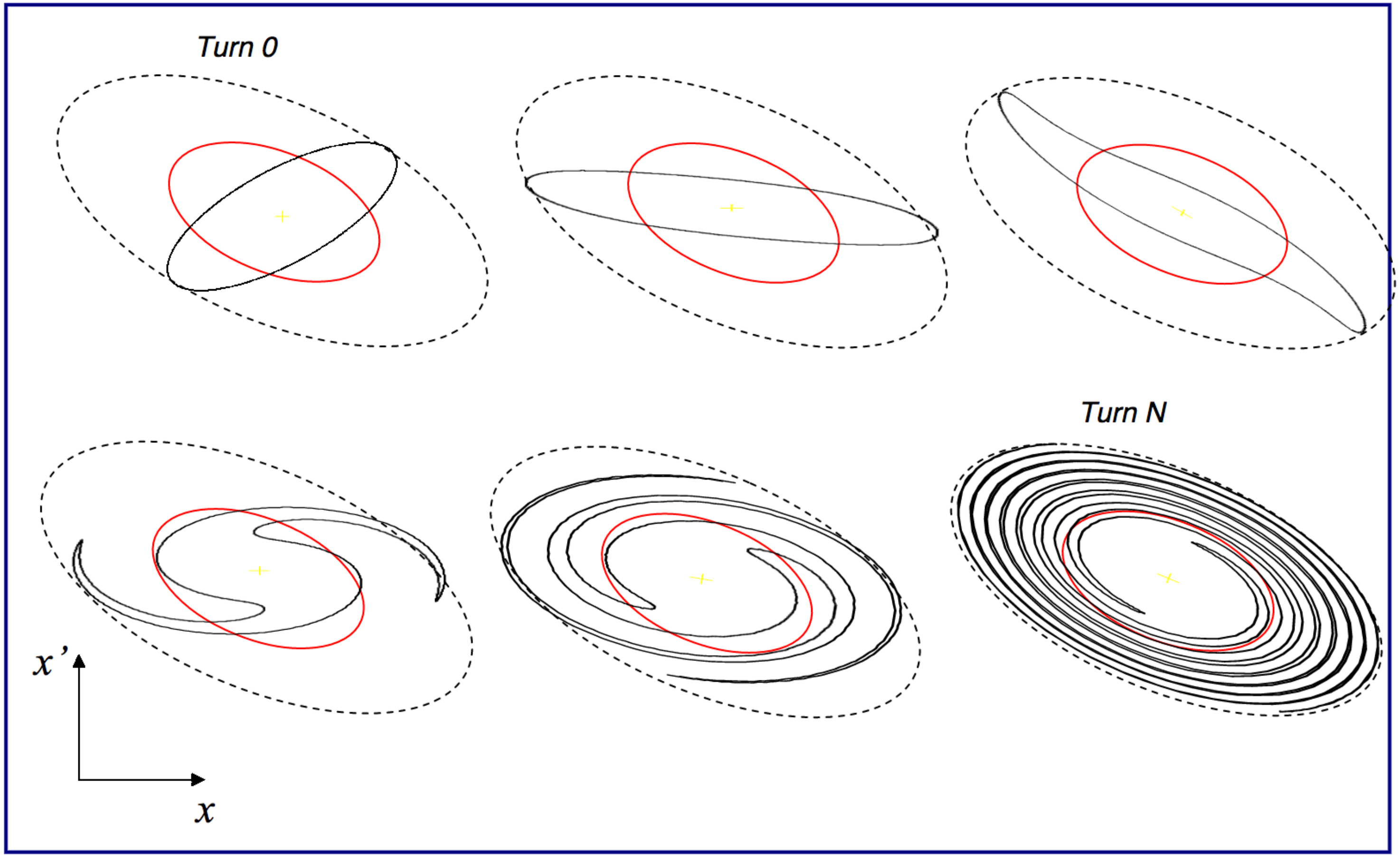




Example Mechanisms of Emittance Dilution

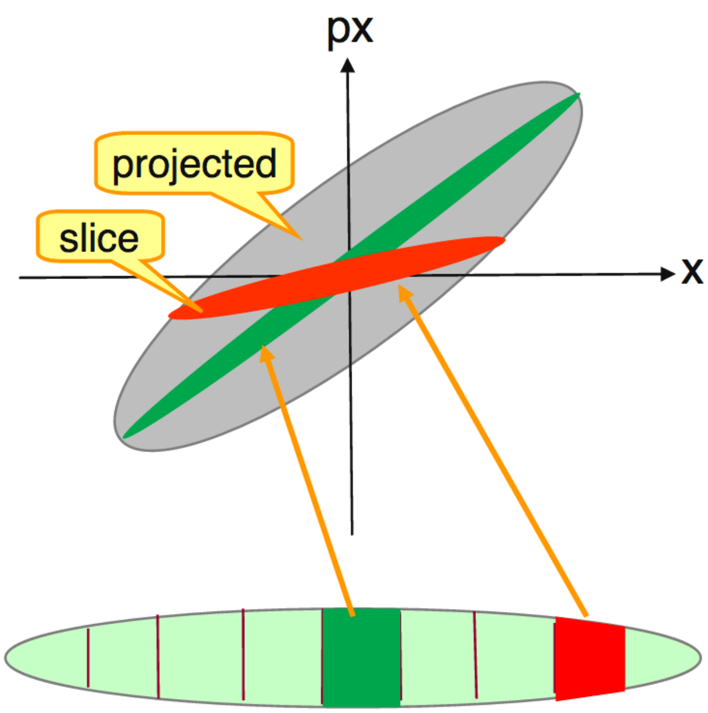
- Optical mismatch at injection

Filamentation fills larger ellipse with same shape as matched ellipse



Collective Effect on Emittance Dilution

- Example: Space charge effects at injector



- Linear transverse space charge force
- Different slices experience different transverse phase space transport
- The slice emittance is the same, but the projected emittance can be much larger.

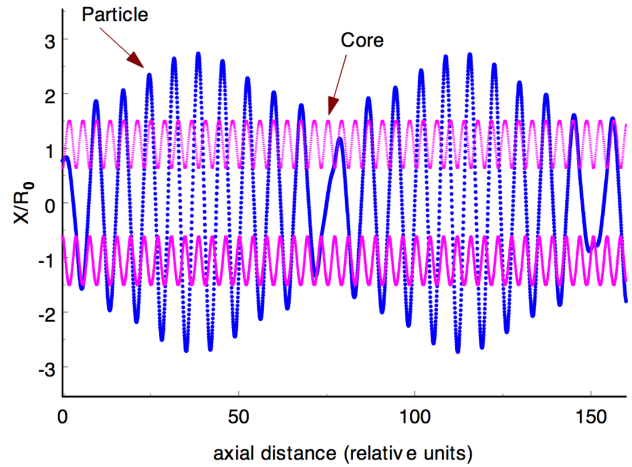
Halo Formation

– Mismatched beam cause breathing mode for the beam core due to space charge interaction

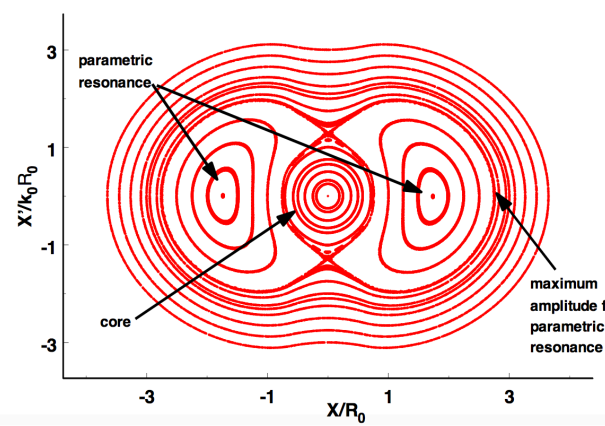
– Breathing of beam core drives particles to parametric resonance, forming halo particles

– Halo causes beam loss at aperture, that can cause radioactivation at the pipe wall

Example: Displacement versus distance for oscillating uniform-density core and a resonant particle.
(uniform core, mismatch parameter μ=1.5, tune depression=0.5)

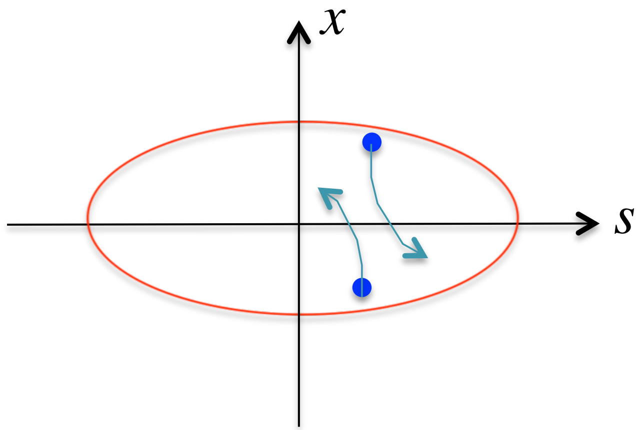


Stroboscopic phase-space plot
Particle-Core Model - breathing mode excitation of uniform core.
(mismatch parameter μ = 1.5, tune depression=0.5)



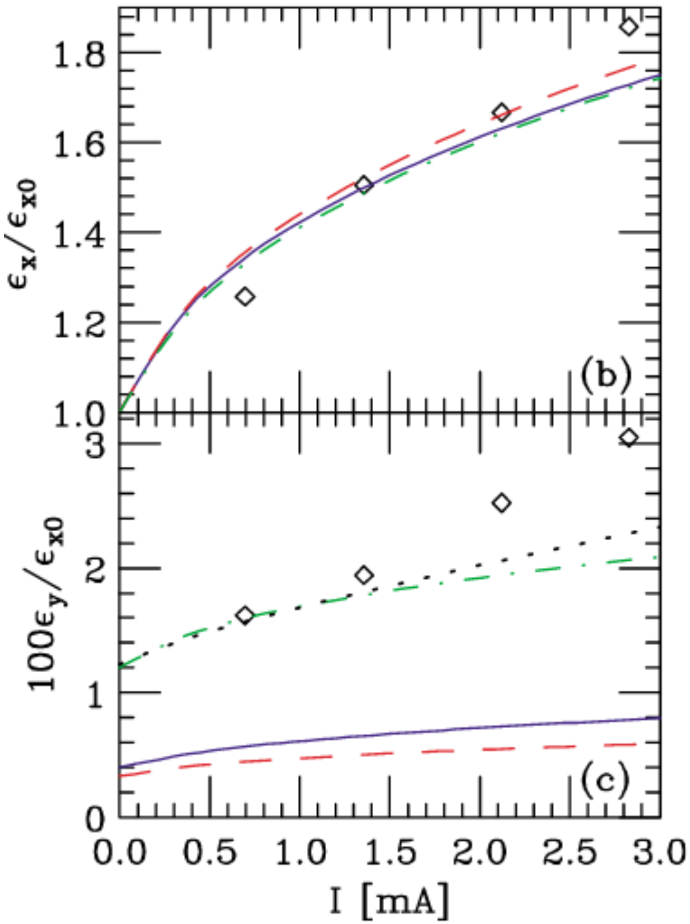
Scattering Effects on Emittance Dilution

- Intrabeam Scattering (IBS)
  - Particles within the beam can have Coulomb collision with small angle that could transfer the transverse momentum to longitudinal ones.
  - With dispersion (x-dE/E) and x-y coupling in the lattice, this could lead to 6D emittance growth.

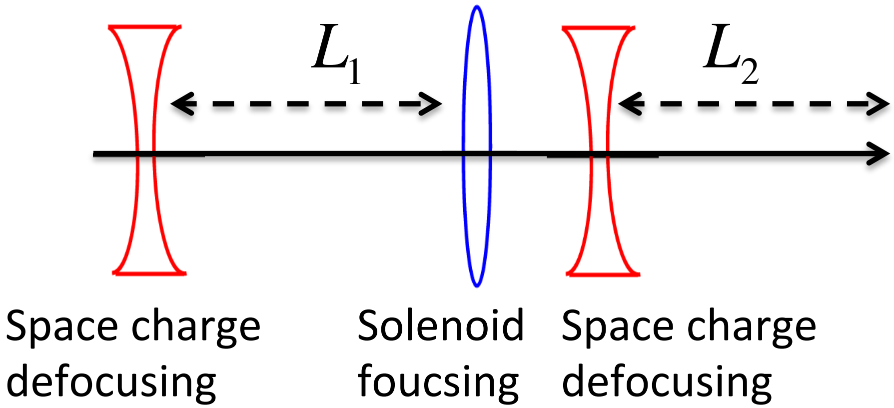


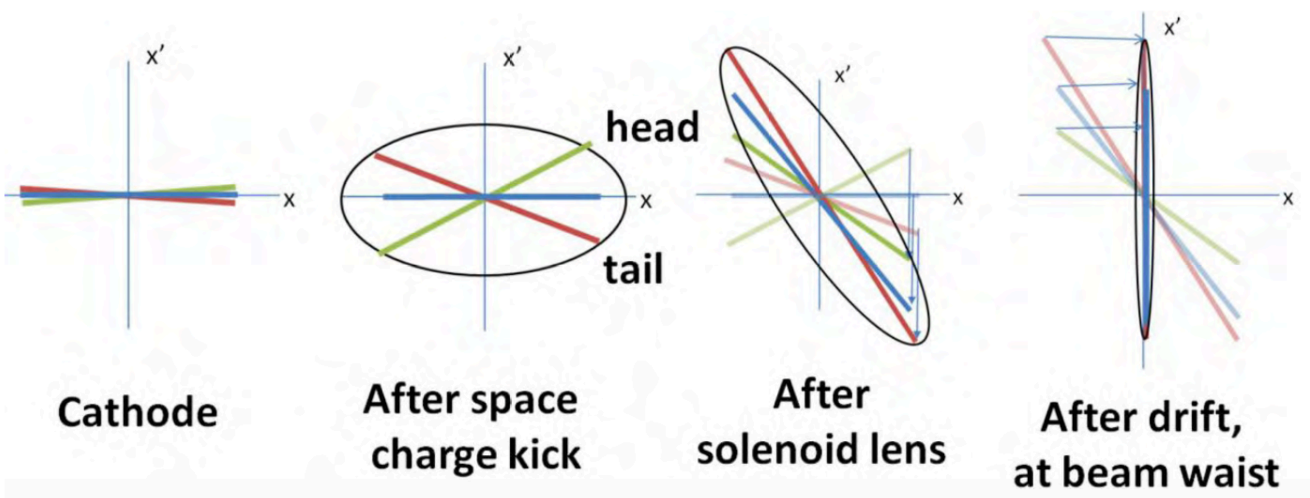
Particle scattering in the beam comoving frame

Measured and simulated emittance growth due to IBS at KEK-ATF



Emittance Compensation Method

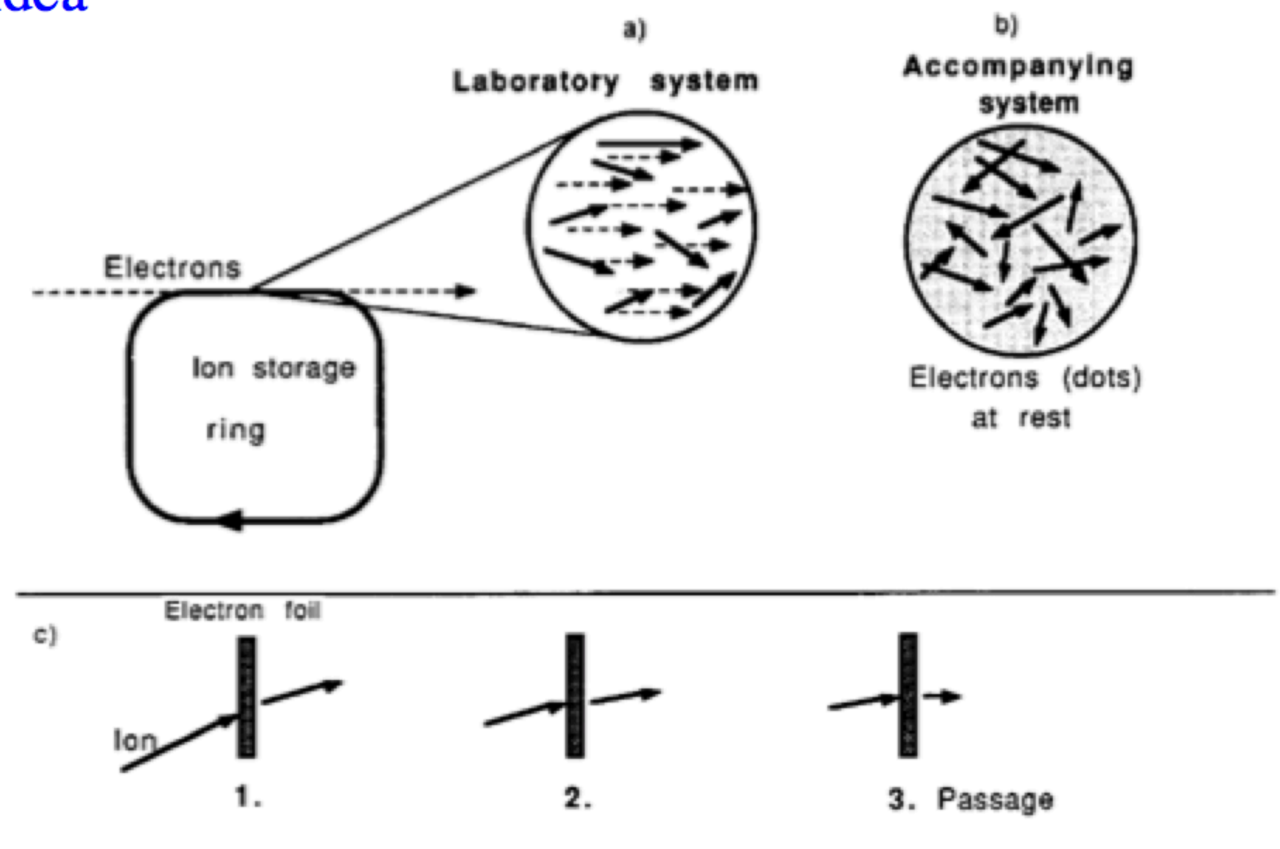




Electron Cooling to Suppress IBS Effects

H. Poth, Electron cooling: theory, experiment, application

Basic Idea

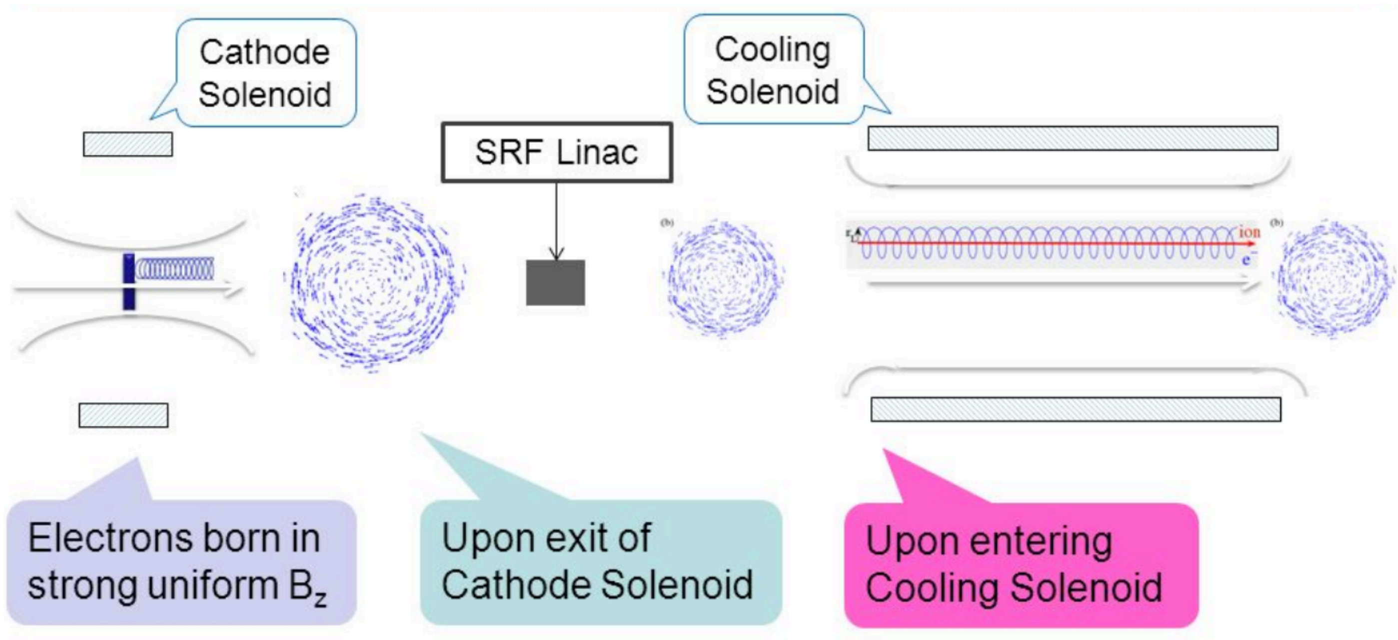


IV. Electron Cooling and Magnetized Beam

- Benefit of using magnetized beam
- Generation of magnetized beam
- Characterization of magnetized beam
- Transport of Magnetized
- Simulation results for JLEIC

Magnetized Beam for Electron Cooling

- Illustration of high-energy cooling using magnetized beam



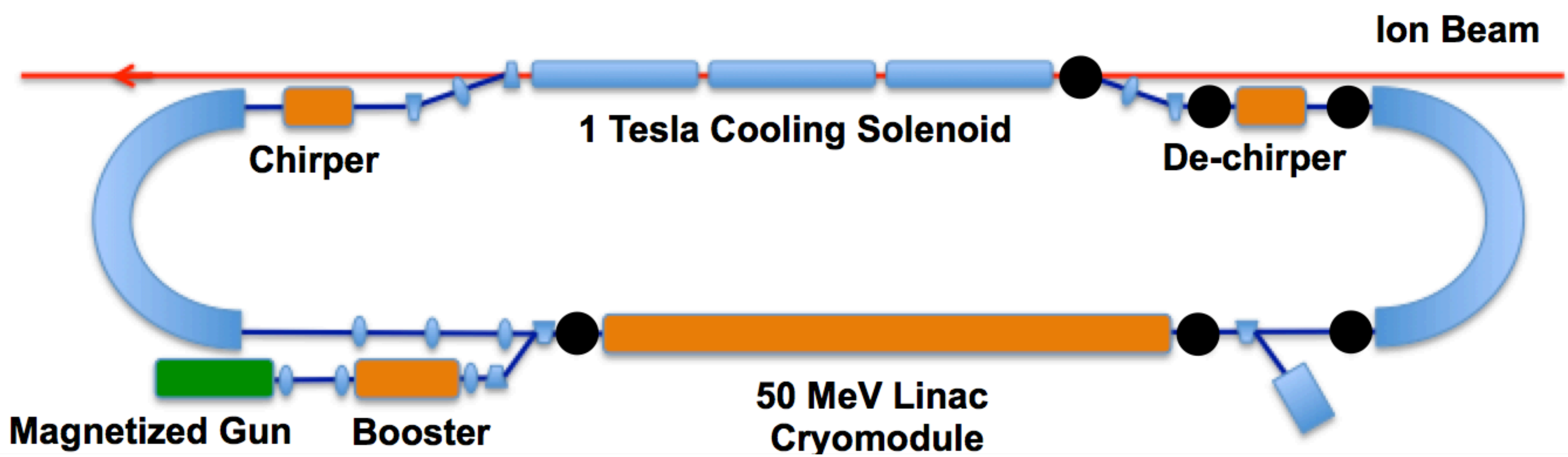
Features:

- Electron cathode immersed in solenoid
- Optical matching between two solenoids through all the beam lines
- Electron beam is in calm state (almost-parallel beam with large transverse size) during cooling process $\left(\sigma_{\perp} \gg \rho_L = \frac{\gamma m_e v_{\perp} [c]}{e B_s} \right)$

Benefits and Challenges of Using Magnetized Beam

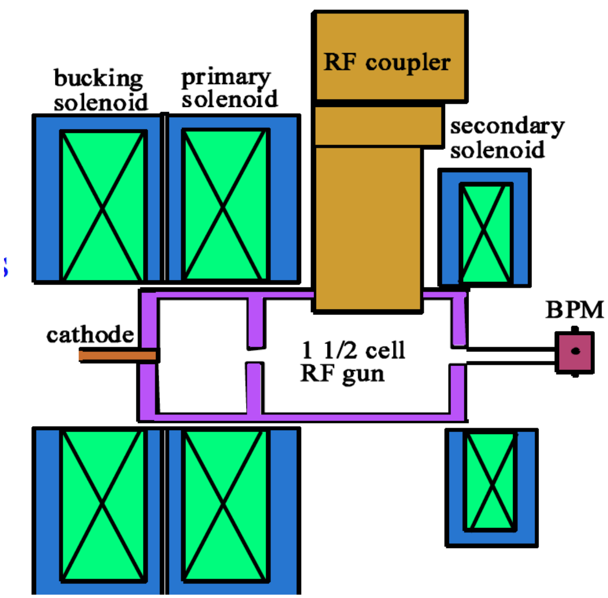
- Benefit
  - Strong reduction of e-beam divergence
  - Due to the freezing of the electron transverse motion, cooling efficiency is significantly improved since it is now only limited by the longitudinal temperature $\left(\text{usually } T_{\parallel} \ll T_{\perp}\right)$
  - Strong reduction of misalignment impact on cooling rate
  - Cyclontron motion of electron in solenoid also suppresses recombination during cooling process

Electron Cooler Design for JLEIC



- Challenges
  - Requires generation and transport of magnetized beam
  - Need optical solution of beamline design that is different from those for usual uncoupled beams
  - Rotating beam transported through lattice with axial-symmetric focusing

Generation of Magnetized Beam



FNPL 1.625-cell RF gun, 1.3 GHz

Canonical angular momentum (CAM) is conserved:

$$L = \vec{r} \times (\gamma m \vec{v} + e\vec{A}/c) = \gamma m r^2 \dot{\theta} + \frac{eB_z}{2[c]} r^2$$

Inside solenoid

$$A_\theta = \frac{1}{2} B_z r^2$$

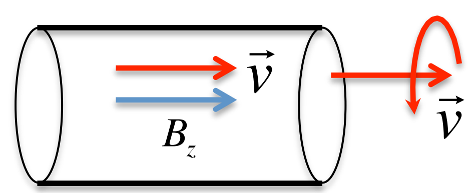
At cathode

$$\dot{\theta} = 0$$

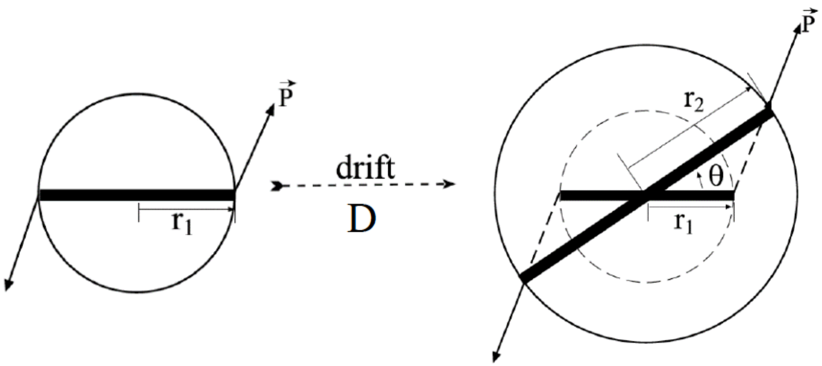
CAM vs. rms size at cathode:

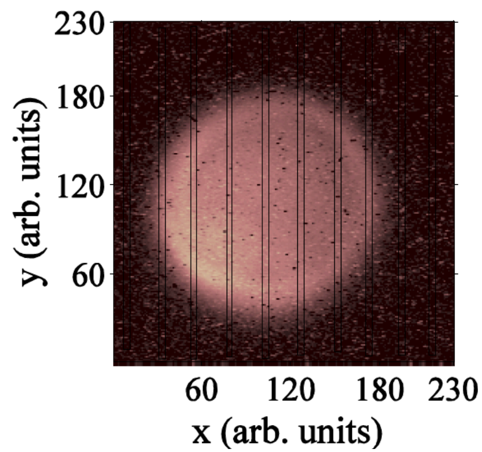
$$\langle L \rangle = \frac{eB_z}{[c]} \sigma_c^2$$

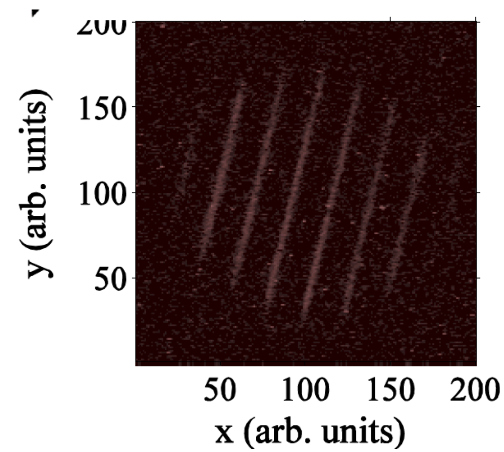
Kinetic angular momentum acquired at the Exit fringe of the solenoid:



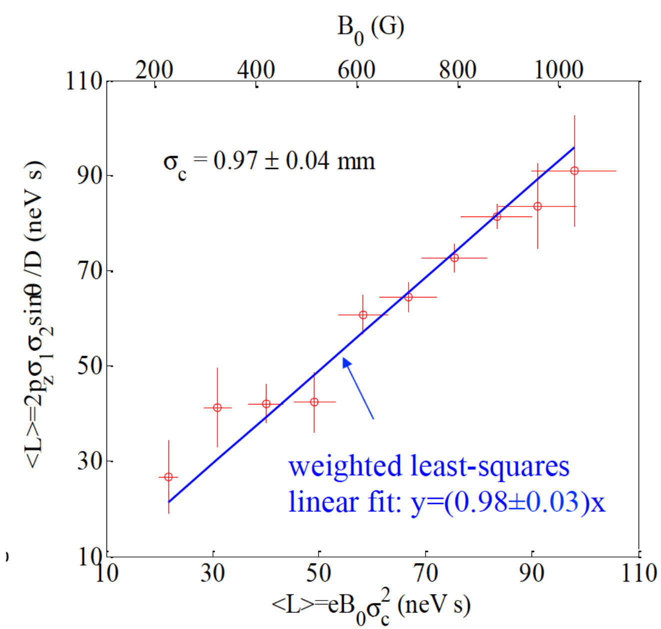
Measurement of Angular Momentum

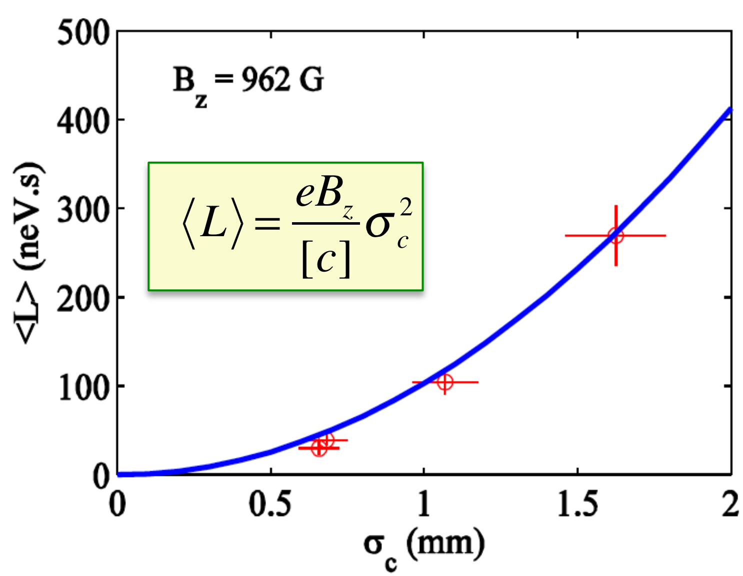






$$\langle L \rangle = 2p_z \frac{\sigma_1 \sigma_2 \sin\theta}{D}$$



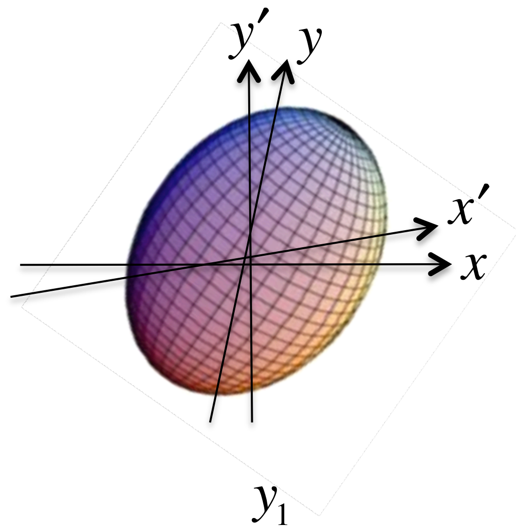


Characterization of Magnetized Beam

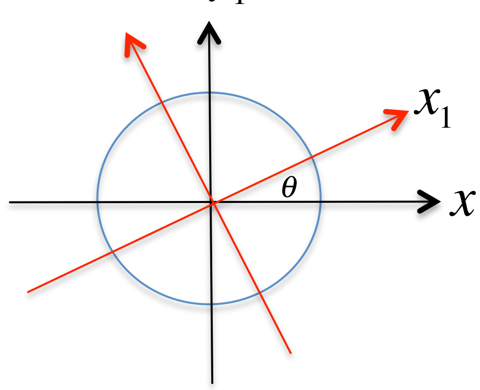
- 4D ellipsoid in (x,x',y,y') phase space

For projected 2D phase space vectors $X=\begin{pmatrix} x \\ x' \end{pmatrix}$, $Y=\begin{pmatrix} y \\ y' \end{pmatrix}$,

$$\Sigma=\left\langle\begin{pmatrix} X \\ Y \end{pmatrix}\begin{pmatrix} \tilde{X} & \tilde{Y} \end{pmatrix}\right\rangle=\begin{pmatrix} \langle X\tilde{X}\rangle & \langle X\tilde{Y}\rangle \\ \langle \tilde{X}Y\rangle & \langle Y\tilde{Y}\rangle \end{pmatrix}=\left(\begin{array}{cc|cc} \langle x^2\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle x'x\rangle & \langle x'^2\rangle & \langle x'y\rangle & \langle x'y'\rangle \\ \hline \langle yx\rangle & \langle yx'\rangle & \langle y^2\rangle & \langle yy'\rangle \\ \langle y'x\rangle & \langle y'x'\rangle & \langle y'y\rangle & \langle y'^2\rangle \end{array}\right)$$



- Rotational symmetry



Rotation in x,y: $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}=R\begin{pmatrix} X \\ Y \end{pmatrix}$ for $R=\begin{pmatrix} I\cos\theta & I\sin\theta \\ -I\sin\theta & I\cos\theta \end{pmatrix}$

symmetry: $$\Sigma_1=\left\langle\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}\begin{pmatrix} \tilde{X}_1 & \tilde{Y}_1 \end{pmatrix}\right\rangle=R\Sigma\tilde{R}=\Sigma$$

Characterization:

$$\Sigma=\begin{pmatrix} \varepsilon_{\text{eff}}T & \mathrm{L}J \\ -\mathrm{L}J & \varepsilon_{\text{eff}}T \end{pmatrix} \quad \text{with} \quad J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \det T=1, \quad \mathrm{L}=L/2$$

volume of 4D phase space

$$\varepsilon_{4D}=\sqrt{\det\Sigma}=\varepsilon_u^2$$

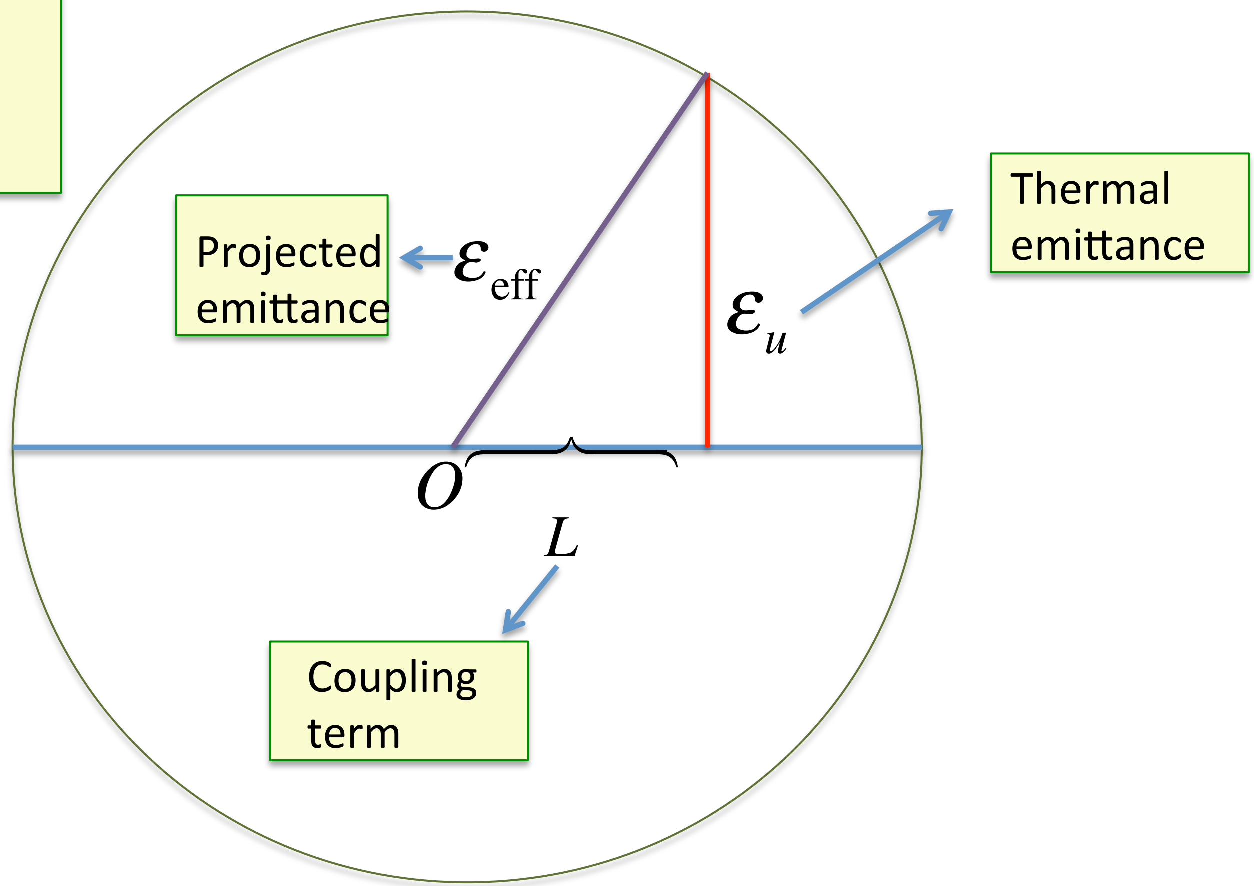
$$\varepsilon_u^2=\varepsilon_{\text{eff}}^2-\mathrm{L}^2$$

Relationship of Various Emittances

volume of 4D phase space

$$\varepsilon_{4D} = \sqrt{\det \Sigma} = \varepsilon_u^2$$

$$\varepsilon_u^2 = \varepsilon_{\text{eff}}^2 - \text{L}^2$$



Eigen-emittance and Invariance

- Eigen emittance

The symmetric matrix Σ can be diagnolized by a symplectic matrix:

$$\Sigma = M\,\Sigma_0\,\tilde{M}, \quad \text{with} \quad \Sigma_0 = \begin{pmatrix} \varepsilon_+ T_+ & 0 \\ 0 & \varepsilon_- T_- \end{pmatrix} \quad \text{for} \quad T_\pm = \begin{pmatrix} \beta_\pm & 0 \\ 0 & 1/\beta_\pm \end{pmatrix}$$

Here ε_+ and ε_- are called eigen emittance, with $\varepsilon_u^2 = \varepsilon_+ \varepsilon_-$
with $\varepsilon_+ = \varepsilon_{\text{eff}} + \mathrm{L}$ and $\varepsilon_- = \varepsilon_{\text{eff}} - \mathrm{L}$

Geometric mean

- Invariance of symplectic transformation

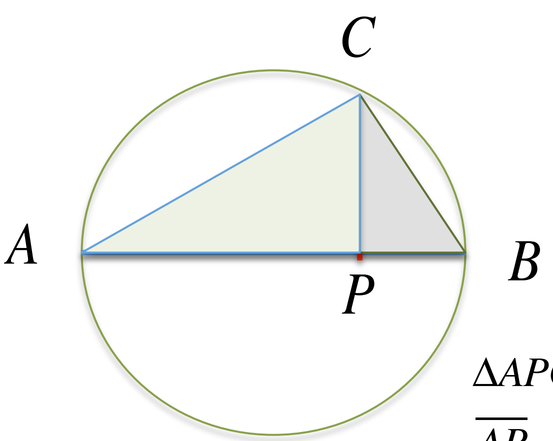
1st invariance: ε_u^2 or ε_{4D} (Liouville Theorem)

2nd invariance: $\varepsilon_+^2 + \varepsilon_-^2 = 2\left(\varepsilon_{\text{eff}}^2 + \mathrm{L}^2\right)$

Relationship of Various Emittances

Geometric mean

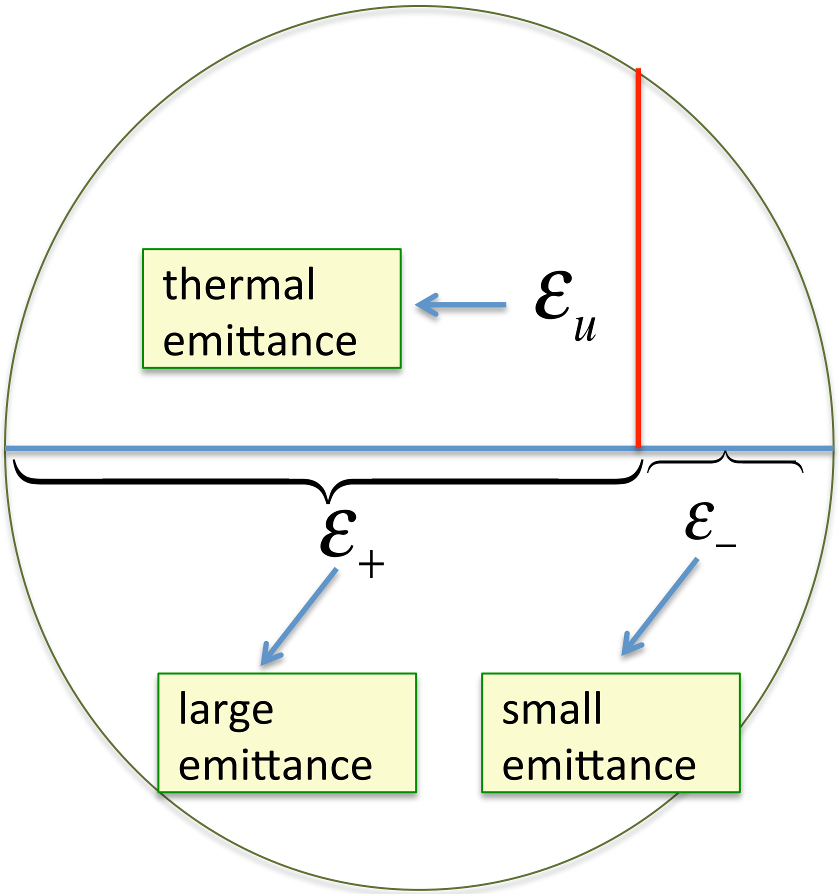
$$\varepsilon_u^2 = \varepsilon_+ \varepsilon_-$$



$$\Delta APC \sim \Delta CPB$$

$$\frac{\overline{AP}}{\overline{CP}} = \frac{\overline{CP}}{\overline{BP}}$$

$$\overline{CP} = \sqrt{\overline{AP} \cdot \overline{BP}}$$



Relationship of all the Emittances

ε_u : 4D emittance $\varepsilon_u = \sqrt{\varepsilon_{4D}}$

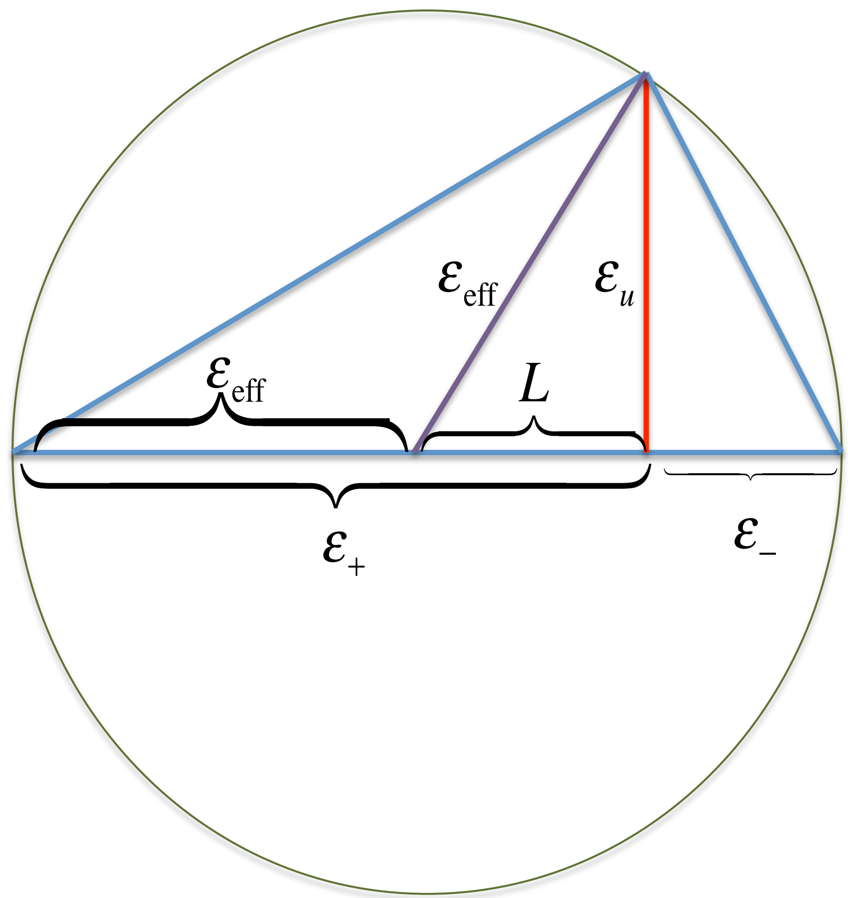
$\varepsilon_\pm$: eigen emittance

ε_{eff} : projected emittance on x-x' or y-y' plane

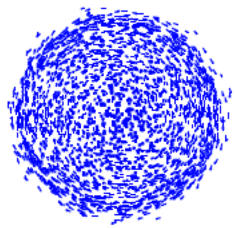
L: coupling term

$$\varepsilon_u^2 = \varepsilon_+ \varepsilon_-$$

$$\begin{cases} \varepsilon_+ = \varepsilon_{\text{eff}} + \text{L} \\ \varepsilon_- = \varepsilon_{\text{eff}} - \text{L} \end{cases}$$



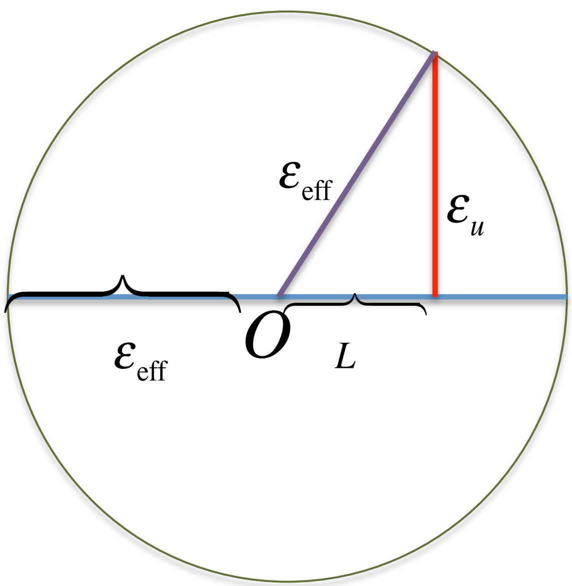
Round to Flat Transformation

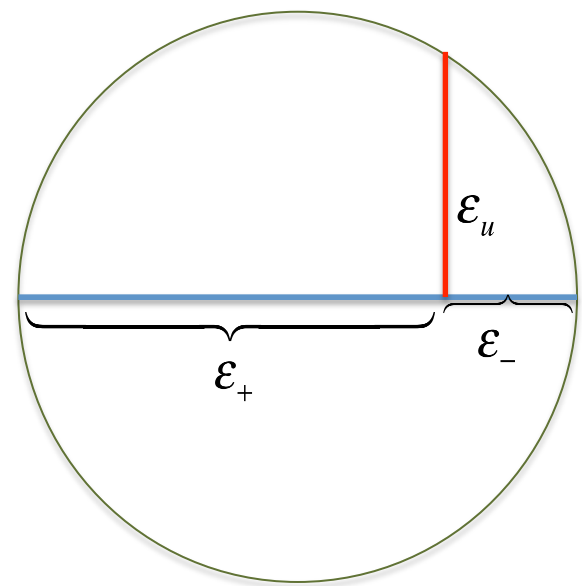


$$\Sigma = \begin{pmatrix} \varepsilon_{\text{eff}} T & \mathcal{L} J \\ -\mathcal{L} J & \varepsilon_{\text{eff}} T \end{pmatrix}$$

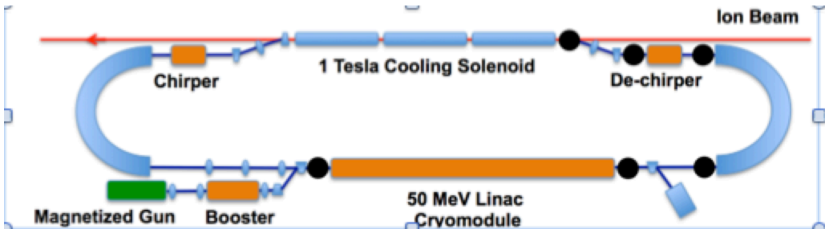
$$\Sigma = M \, \Sigma_0 \, \tilde{M}$$

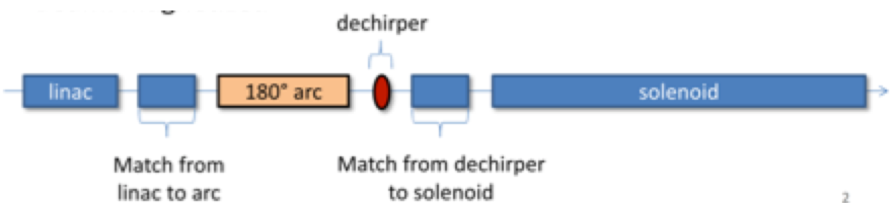
$$\Sigma_0 = \begin{pmatrix} \varepsilon_+ T_+ & 0 \\ 0 & \varepsilon_- T_- \end{pmatrix}$$

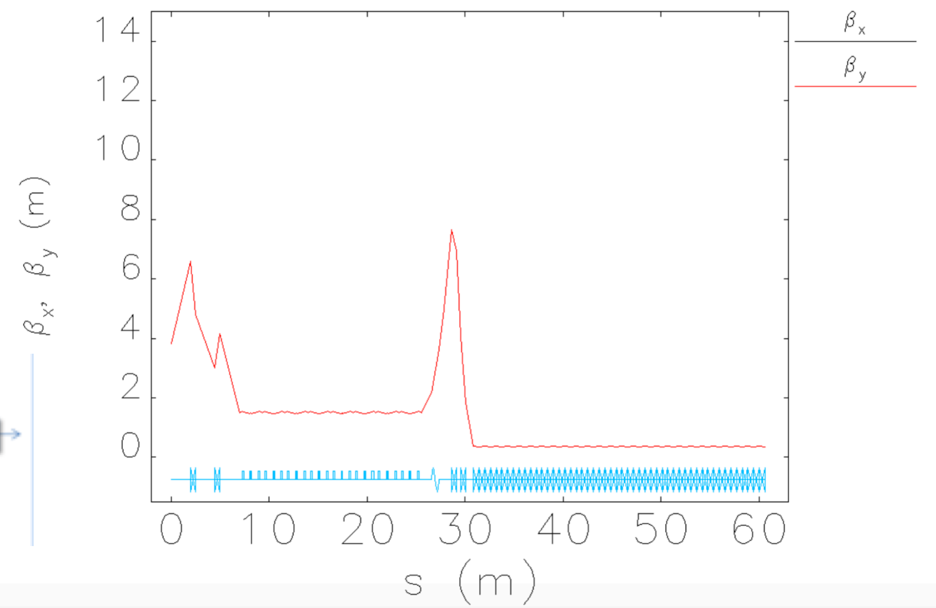




Transport of Magnetized Beam to the Cooling Channel







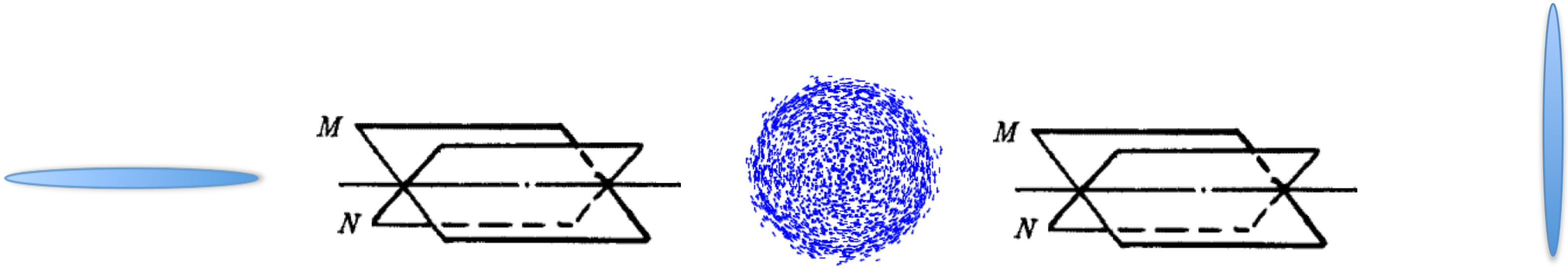
Features:

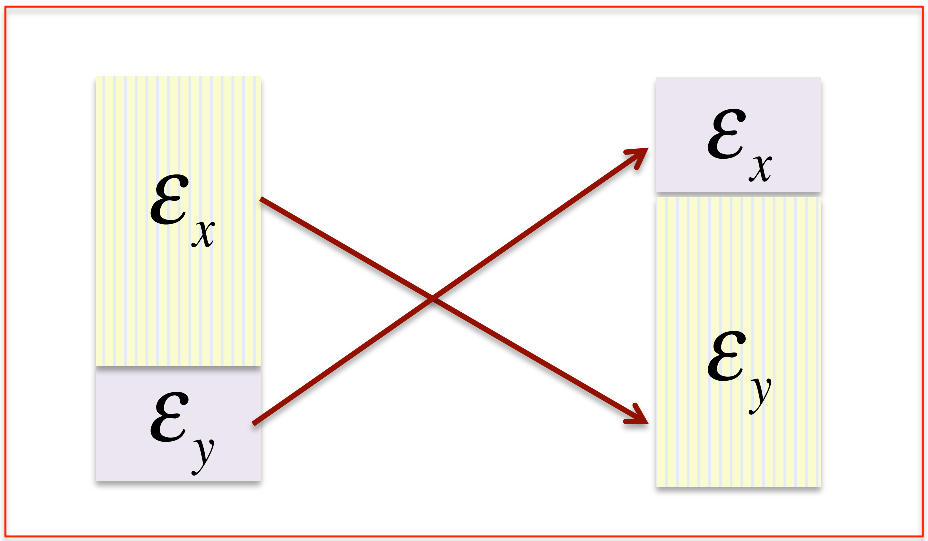
- Lattice focusing is axial symmetric, with index ½ in arc dipoles
- Matching sections are applied before the arc and before the cooling solenoid
- Beam has angular momentum during the transport

V. Phase Space Manipulation

- Horizontal to vertical emittance exchange
- Longitudinal to transverse emittance exchange
- Emittance partitioning

X-Y Emittance Exchange

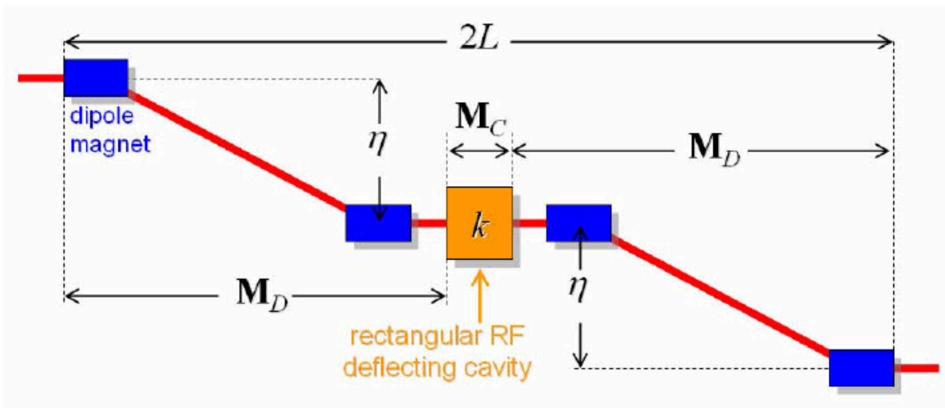




$$\varepsilon_{4D} = \varepsilon_{x1}^2 \varepsilon_{y1}^2 = \varepsilon_{x2}^2 \varepsilon_{y2}^2$$

$$I^{(2)} = \varepsilon_{x1}^2 + \varepsilon_{y1}^2 = \varepsilon_{x1}^2 + \varepsilon_{y1}^2$$

Longitudinal to Transverse EEX



$$X = \begin{pmatrix} x \\ x' \\ z \\ \delta \end{pmatrix}, \quad X \to MX$$

$$, = \begin{pmatrix} 0 & L & 0 & \eta \\ 0 & 1 & 0 & 0 \\ 0 & \eta & 1 & \zeta \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{pmatrix}$$

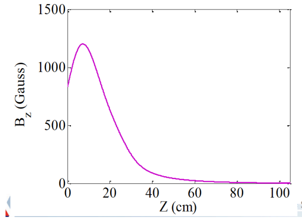
$$M_{EX} = M_D M_C M_D = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \qquad \text{when } \; 1 + k\eta = 0$$

$$\varepsilon_{4D} = \varepsilon_{x1}^2 \varepsilon_{z1}^2 = \varepsilon_{x2}^2 \varepsilon_{z2}^2$$

$$I^{(2)} = \varepsilon_{x1}^2 + \varepsilon_{z1}^2 = \varepsilon_{x1}^2 + \varepsilon_{z1}^2$$

Emittance Partitioning

Non-rotating
Round beam
Within the cathod

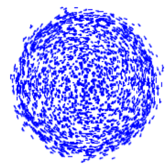


$$\Sigma = \begin{pmatrix} \sigma_c^2 & & & \\ & \sigma_{c'}^2 & & \\ & & \sigma_c^2 & \\ & & & \sigma_{c'}^2 \end{pmatrix}$$

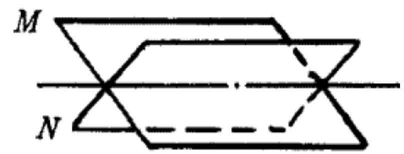
Emission
In solenoid

Non-Hamiltonian
Process to
Introduce
correlation

Rotating
Round beam



$$\Sigma = \begin{pmatrix} \varepsilon_{\text{eff}} T & \text{L}J \\ -\text{L}J & \varepsilon_{\text{eff}} T \end{pmatrix}$$

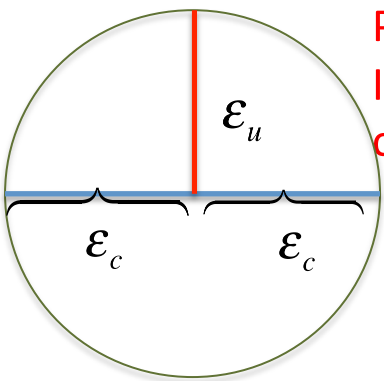


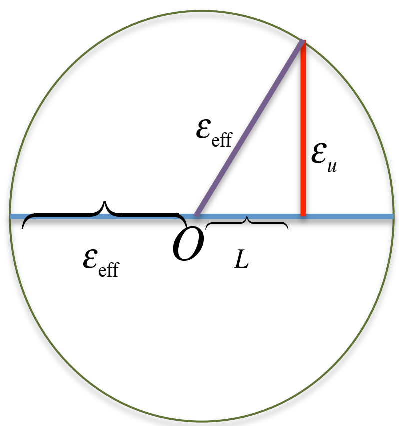
$$\Sigma = M \Sigma_0 \tilde{M}$$

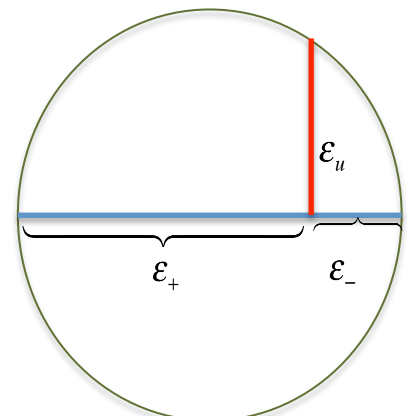
Hamiltonian
process

Flat beam

$$\Sigma_0 = \begin{pmatrix} \varepsilon_+ T_+ & 0 \\ 0 & \varepsilon_- T_- \end{pmatrix}$$





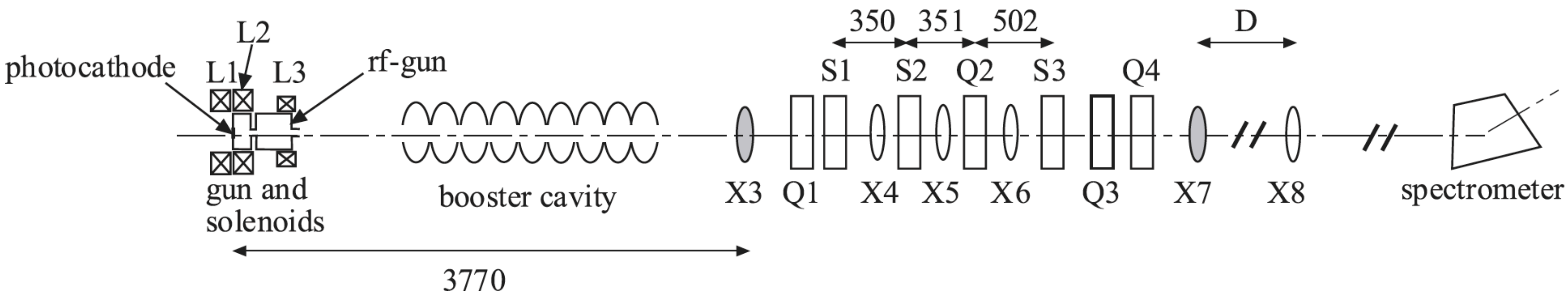


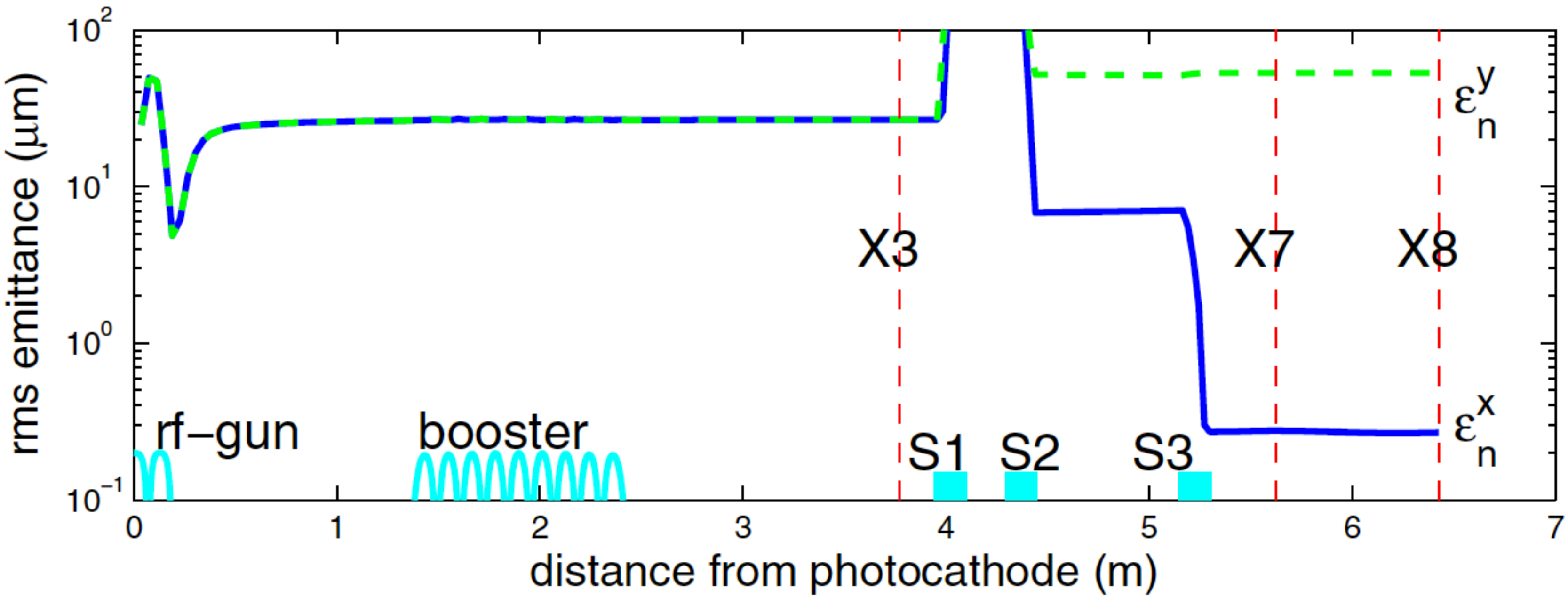
$$\varepsilon_u^2 = \varepsilon_c \cdot \varepsilon_c$$

$$\varepsilon_u^2 = \varepsilon_+ \cdot \varepsilon_-$$

Emittance Partitioning

Example of Flat Beam Generation





Summary

- We've reviewed the basic concepts of phase space flow, symplectic map and the related properties
- The Hamiltonian dynamics is fundamental for studies of single particle and bunch dynamics in accelerator physics
- Magnetized beam for electron cooling has many interesting and non-conventional features
- Recent development in phase space manipulation extends the previous concepts and opens up many new possibilities

References

- E. Courant and H. Snyder, Ann. Phys. 3,1 (1958)
- A. Dragt et al., “General moment invariants for linear Hamiltonian system”, Physical Review A, 45, 2572 (1992)
- K-J Kim, “Round-to-flat transformation of angular-momentum-dominated beams”, Physical Review Special Topics—Accelerators and Beams, 6, 104002 (2003)
- J. Buon “Beam phase space and emittance”,
- K-J Kim and A. Sessler, “Transverse-longitudinal phase-space manipulations and corrections”, AIP proceeding 821, 115 (2006)
- D. Douglas, “Life hacks for solenoids and magnetized beams”, JLEIC 2016 Spring collaboration meeting.
- Ph. Royer, Solenoidal Optics, PS/HP Note 99-12.
- M. Reiser, “Theory and design of charged particle beams”, 1994
- Y-E. Sun, “Round-to-Flat Beam Transformation and Application”,Cool15, 2015.
- P. Piot, Y-E Sun and K-J. Kim, PRSTAB 9, 031001 (2006)
- B. E. Carlsten, Nucl. Instrum. Methods Phys. Res., Sect. A 285, 313 (1989).
- B. Carlsten et al, IEEE Tran. On Nuclear Sci.,63,921 (2016)
- R. Gluckstern, Phys. Rev. Lett. 73, 1247 (1994).
- T. P. Wangler and K. R. Crandall, Phys. Rev. ST Accel. Beams 1, 084201 (1998).

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