Methods for computing the stability and chaotic regions of non-linear differential equations with periodic coefficients

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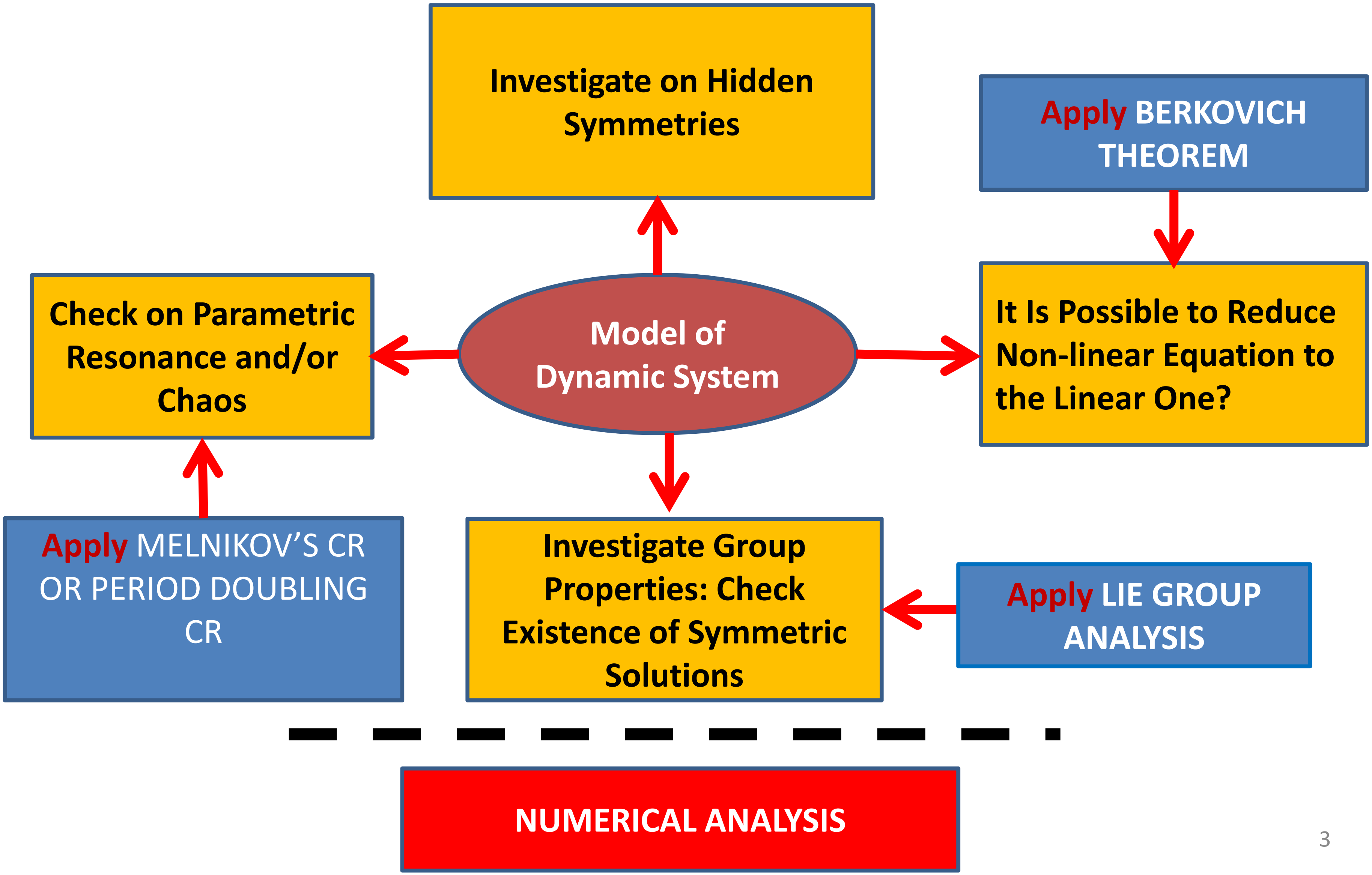
Outline

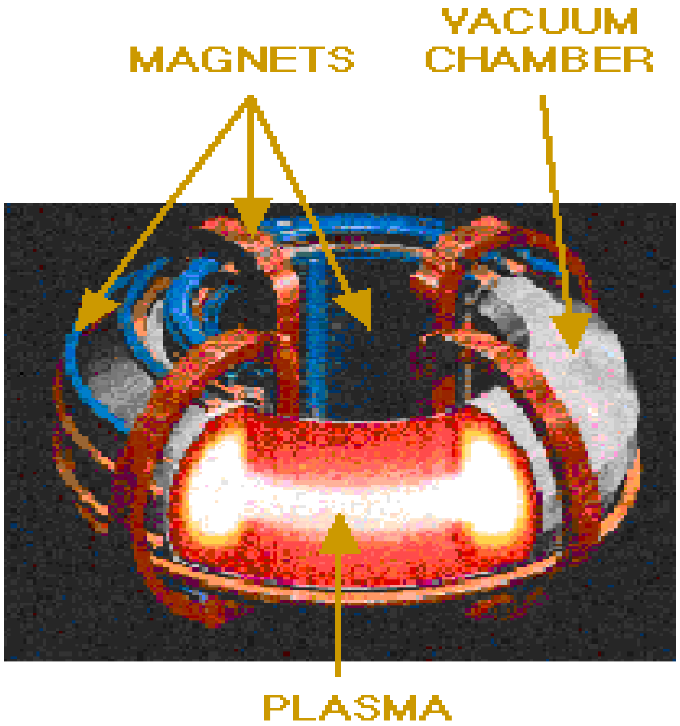
- **Motivation, Objective and Approach**
- **Field Equations**
- **Basic Assumptions**
- **Nonlinear systems in general**
- **Special cases**
- **String in magnetic field**

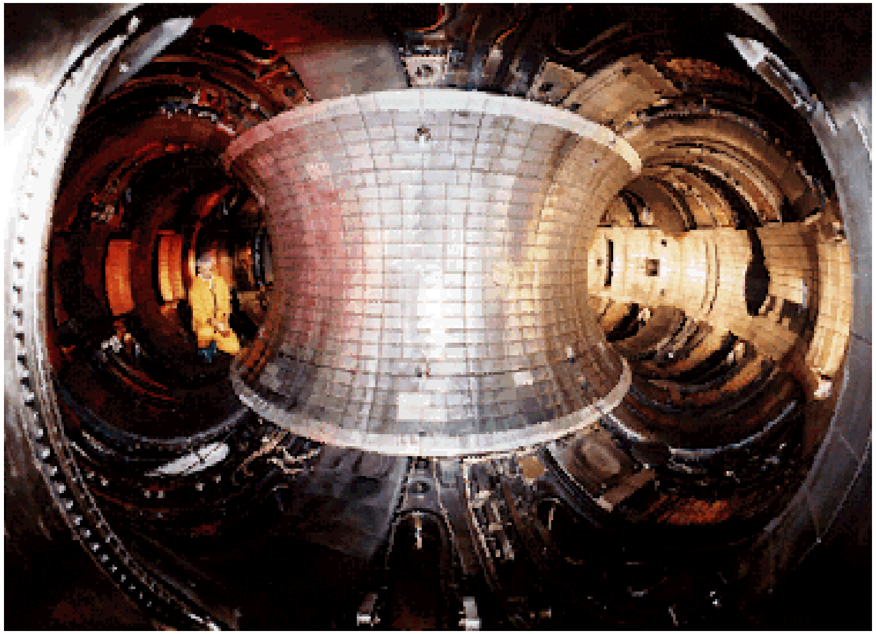
- Stability condition in static and dynamic cases
- Non-linear vibrations of a parametrically exited string in resonance cases
- Numerical examples
- Invariant-Group analysis of differential equations
- Conclusion

Chart for Dynamic Systems



Magnetic Fusion and MHD Magnets (TOKAMAK)





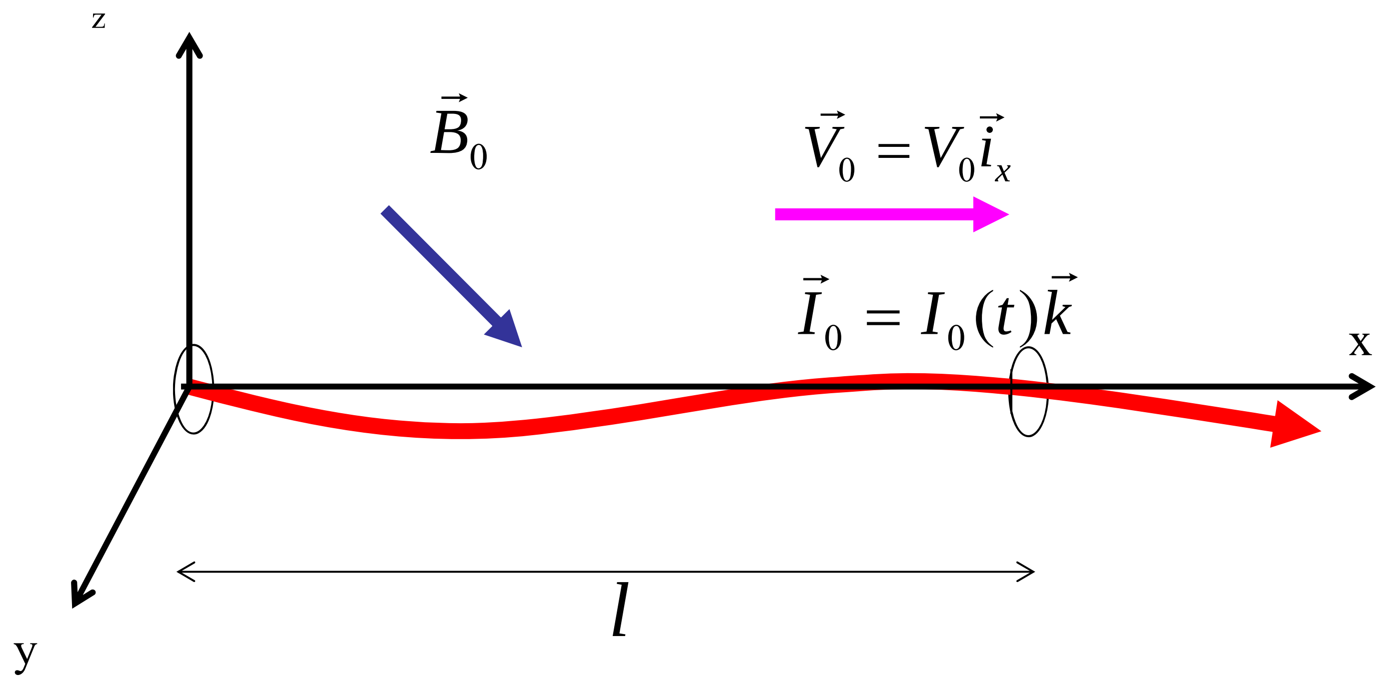
Inside the TOKAMAK Fusion Test Reactor

TOKOMAK Structure

TOKOMAK Fusion Reactor

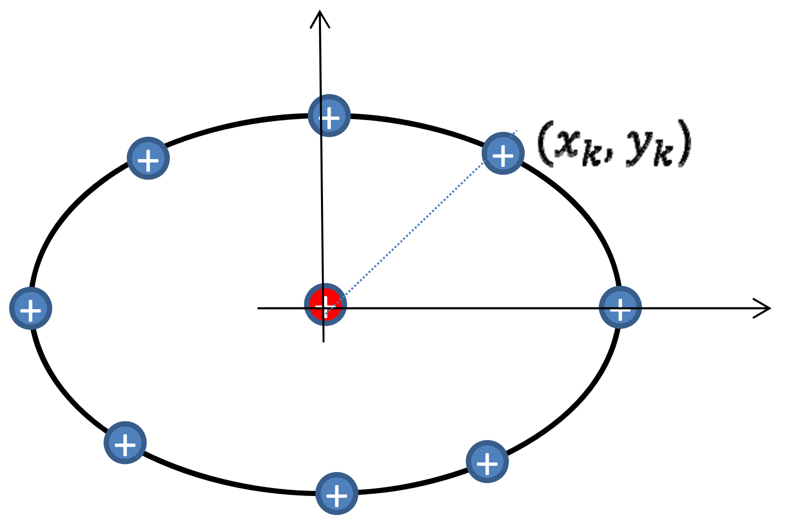
Shell Configurations

Modelling of moving current carrying string in a magnetic field $\vec{B}_0$



Interacting Strings/Beams

Magnetic Field of Currents

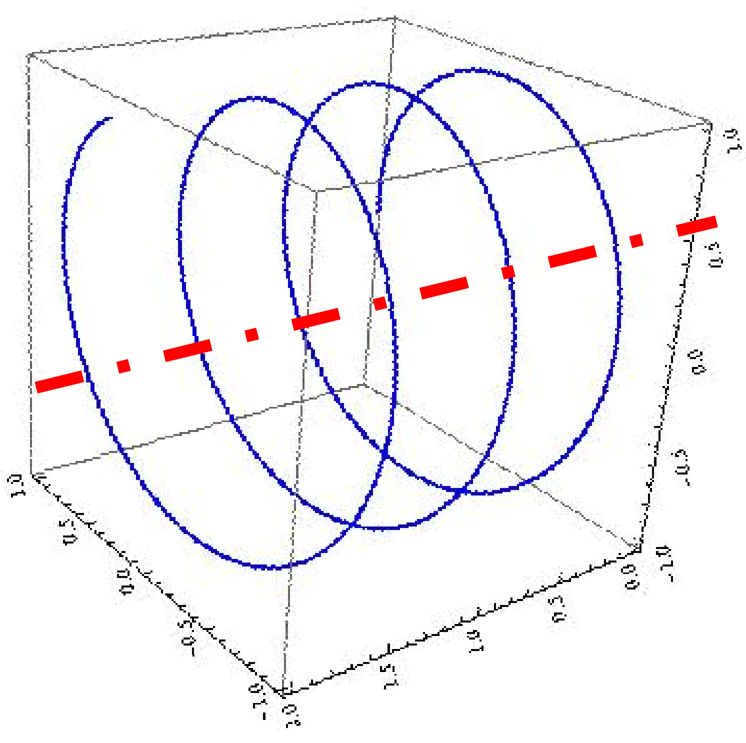


$$\vec{B}_0 = \sum_{k=1}^{N} I_k \frac{a_k(x-x_k)\vec{i} + b_k(y-y_k)\vec{j}}{\sqrt{(x-x_k)^2+(y-y_k)^2}}$$ → *N* **Distributed currents**

$$B_{\pm} = 2I_1 \frac{-y\vec{i} + (x \mp a)\vec{j}}{(x \mp a)^2 + y^2}$$ → *Two* **Distributed currents**

$$\vec{F} = I_0(\vec{k} \times \vec{B}_0)$$ → **Force on central String/Beam**

Two Interacting Strings/Beams



$$\vec{F} = I_0(\vec{k} \times \vec{B}_0)$$ ⟹ **Force on central String/Beam**

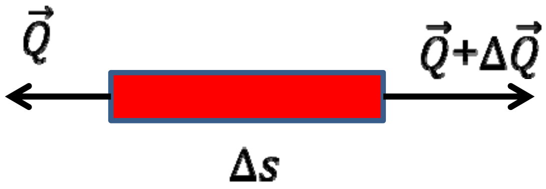
Y. Derbenev, R. Jhonson case

Definitions

$$\vec{B}_0 = B_0 \vec{i}_x$$

Ampere force acting on the string $$\vec{F} = I_0 B_0 (\vec{k} \times \vec{i}_x)$$

Vector of internal stresses $\vec{Q}$



Inertial forces $\vec{R}$

Position vector $$\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$$

Unit vector along the tangent $$\vec{k} = \frac{\partial x}{\partial s}\vec{i}_x + \frac{\partial y}{\partial s}\vec{i}_y + \frac{\partial z}{\partial s}\vec{i}_z = \frac{\partial \vec{r}}{\partial s}$$

The equation of motion of a moving current currying conductive string.

$$\frac{\partial \vec{Q}}{\partial s} + \vec{F} = \rho \vec{R} ,$$

$$\vec{Q} \times \vec{k} = 0 ,$$

$$\left(\frac{\partial \vec{r}}{\partial s} \right)^2 = 1$$

where

$$R_x = 0, \qquad R_y = \frac{\partial^2 y}{\partial t^2} + 2V_0 \frac{\partial^2 y}{\partial s \partial t} + V_0^2 \frac{\partial^2 y}{\partial s^2} + \eta \frac{\partial y}{\partial t}$$

$$R_z = \frac{\partial^2 z}{\partial t^2} + 2V_0 \frac{\partial^2 z}{\partial s \partial t} + V_0^2 \frac{\partial^2 z}{\partial s^2} + \eta \frac{\partial z}{\partial t}$$

The equation for the lateral vibrations of the string

$$\frac{\partial}{\partial s}\left[\frac{P_0 \frac{\partial u}{\partial s}}{\sqrt{1-\left|\frac{\partial u}{\partial s}\right|^2}}\right] - iI_0 B_0 \frac{\partial u}{\partial s} = \rho\left[\frac{\partial^2 u}{\partial t^2} + \eta\frac{\partial u}{\partial t} + 2V_0\frac{\partial u}{\partial s \partial t} + V_0^2\frac{\partial^2 u}{\partial s^2}\right]$$

where $$u(s,t) = y(s,t) + iz(s,t)$$

Boundary conditions: $$u(0,t) = u(l,t) = 0$$

The three wire/beam problem

$$\frac{\hat{N}(v)}{v}\mathbf{r}'' - \lambda\left[\frac{z'(x+a)\mathbf{i} + yz'\mathbf{j} + (-ax' - xx' - yy')\mathbf{k}}{(x+a)^2 + y^2}\right]$$

$$-\lambda\left[\frac{z'(x+a)\mathbf{i} + yz'\mathbf{j} + (ax' - xx' - yy')\mathbf{k}}{(x-a)^2 + y^2}\right] = \mathbf{0}.$$

$$u = \lambda L u + H(\lambda, u), \qquad \lambda = 2I_0 I_1$$

DIFFERENTIAL EQUATIONS GENERAL FORM

$$u_{tt} + a_0 u_t + a_1 u_{ts} + a_2 u_{ss} - [G(u_s^n) u_s^m]_s = K_0(t,s,u)\, u_s^q + K_1(t,s) u^p$$

$$u(s,t) = y(s,t) + iz(s,t)$$ or $$u(s,t) = (y(s,t), z(s,t))^T$$

Particular cases: -->$\{\frac{\partial}{\partial t} = 0, G = 0, K_0 = 0, K_1(t,s) = K(s)\}$**;**

Courant, Snyder;

Derbenev, Shiltsev;

Danilov, Nagaitsev;

Ermakov

$$\frac{\partial}{\partial t} \longrightarrow \beta$$

Stability of a string

1) Static instability

Solution

$$u(s) = A + B \cdot \exp[i\gamma s]$$

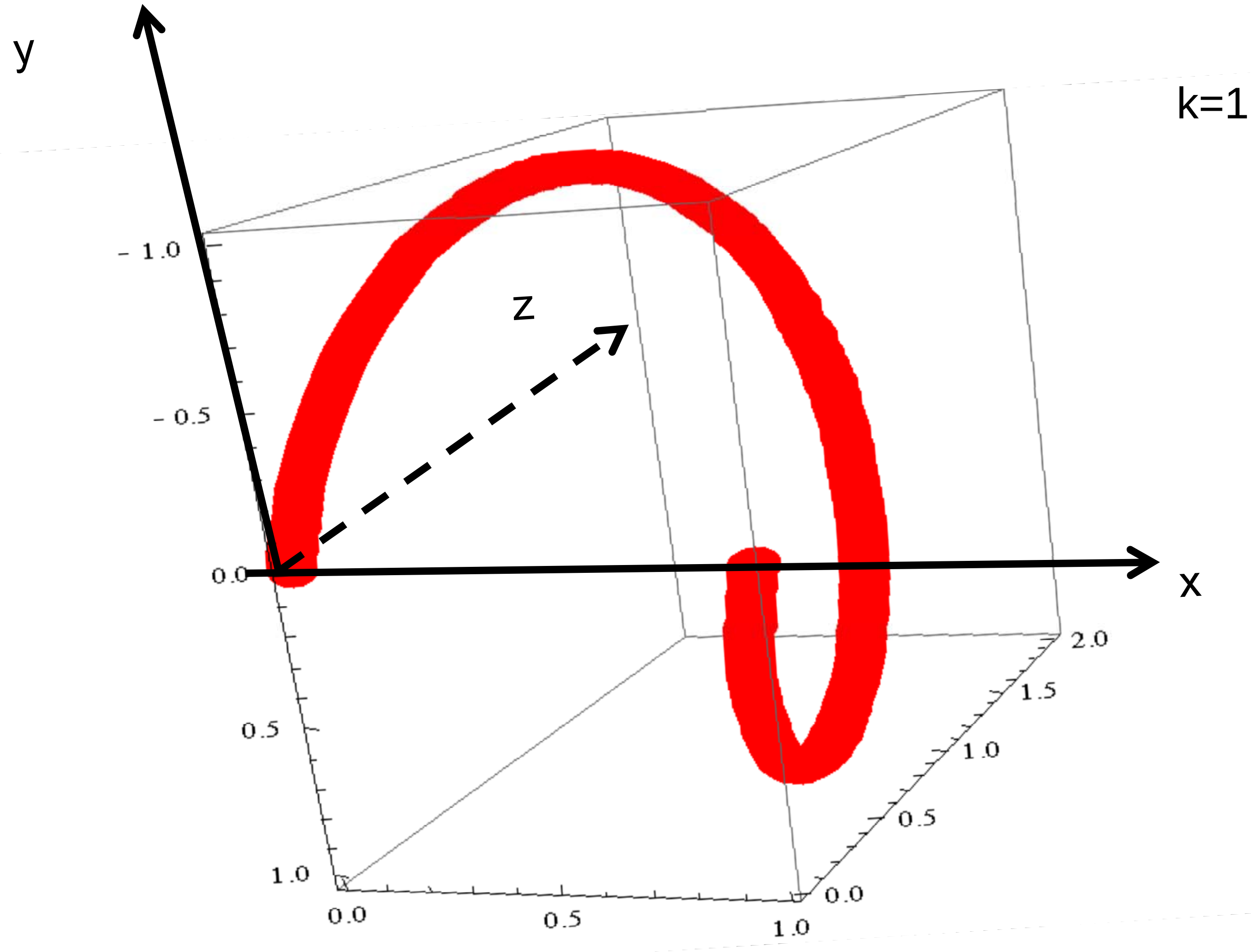
Stability condition in general

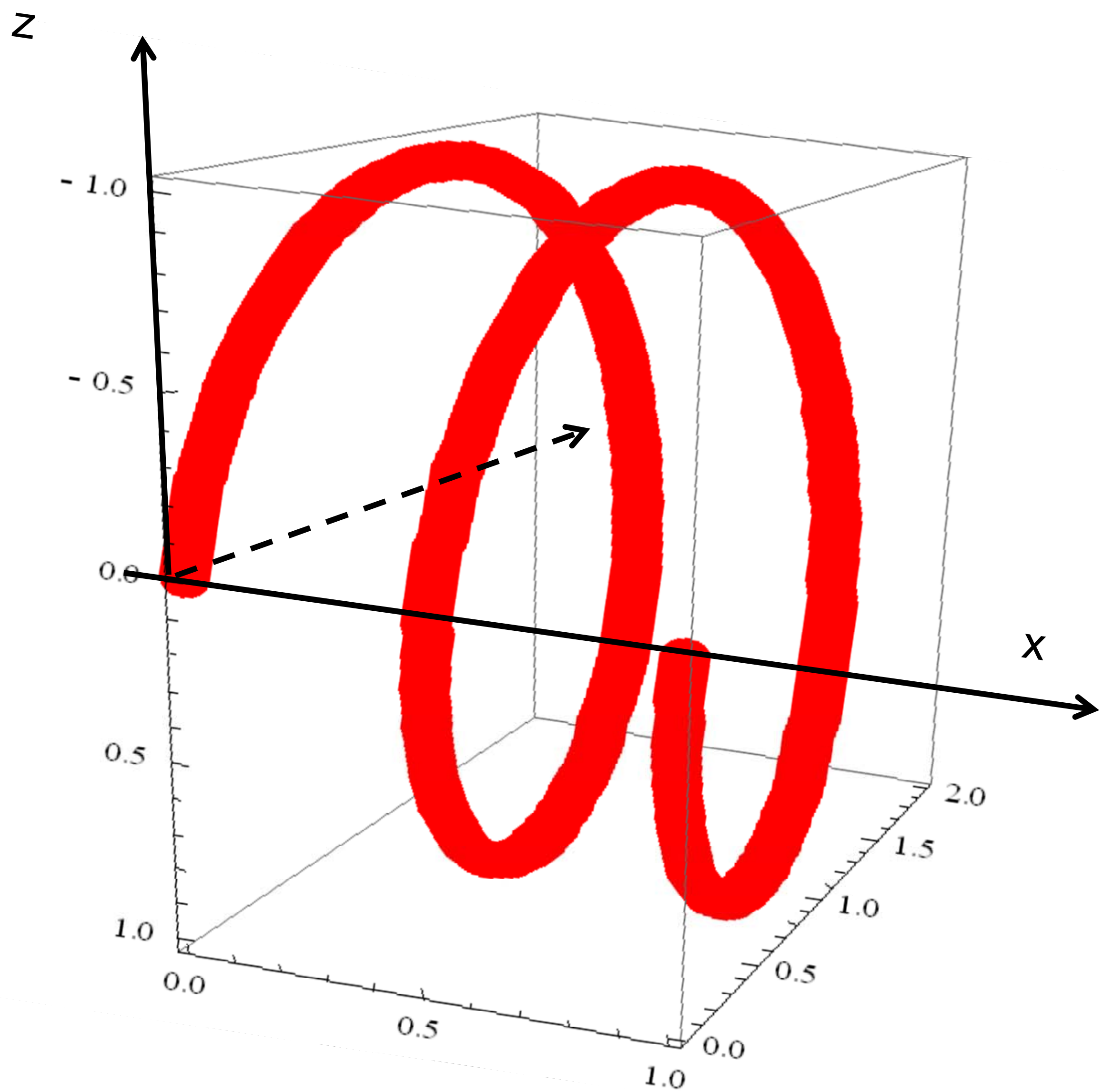
$$\frac{P_0 \gamma}{\sqrt{1 - \gamma^2 |A|^2}} = \rho V_0^2 \gamma + I_0 B_0 \qquad \text{where} \qquad \gamma = \lambda_k = \frac{2\pi k}{l},$$

Stability condition for linearized problem

$$\frac{\rho V_0^2}{P_0} + \frac{I_0 B_0}{P_0 \lambda_k} < 1$$

Form of instability of a string





k=2

2) Dynamic instability: linear problem

$$s \cong x \qquad\qquad 1 - \left|\frac{\partial u}{\partial s}\right|^2 \cong 1$$

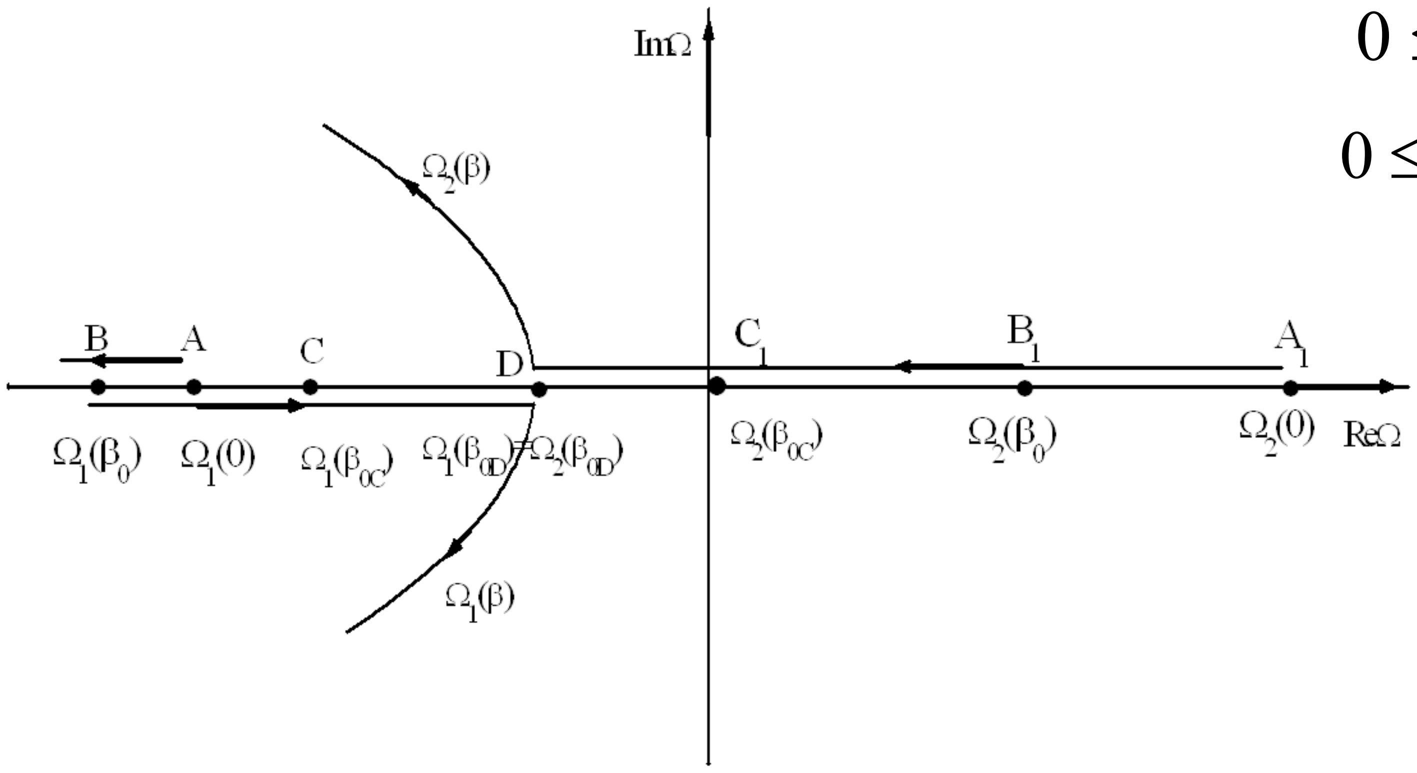
solution $$u(s,t) = A \cdot \exp[\gamma_1 s + i\omega t] + B \cdot \exp[\gamma_2 s + i\omega t]$$

where $$\gamma_{1,2} = [i(\lambda + b\omega) \pm \sqrt{-(\lambda + b\omega)^2 - 4ac\omega^2}\,]/2c,$$

Dynamic stability condition → $$\left(\frac{I_0 B_0}{\lambda_k P_0}\right)^2 + \frac{\rho V_0^{\,2}}{P_0} < 1$$

Static stability condition → $$\frac{\rho V_0^{\,2}}{P_0} + \frac{I_0 B_0}{P_0 \lambda_k} < 1$$

Trajectory of the roots of characteristic equation



$$0 \le \beta < \infty$$

$$0 \le \alpha \le 0.25$$

Notations

$$\Omega = \omega / \Omega_0 ,\ \Omega_0 = \lambda_k \sqrt{1/a} ,\ \beta = 2\lambda / \lambda_k ,\ \lambda_k = 2\pi k / l ,\ \alpha = aV_0^2$$

$$a = \rho / P_0 ,\ b = 2aV_0 ,\quad c = 1 - aV_0^{\,2} ,\quad \lambda = I_0 B_0 / 2P_0 .$$

Investigation of the equilibrium state for a moving current carrying string

$$u(s,t)=\sum_{n=0}^{\infty}A_n(t)\,\Phi_n(s)\ ,\quad \Phi_n(s)=\exp[i\lambda_n s]\ ,\ \lambda_n=2\pi n/l\quad (n=0,1\ldots)$$

$$\frac{d^2A_n}{d\tau^2}+\left(\frac{\eta}{\Omega_0}+\frac{2iV_0\lambda_n}{\Omega_0}\right)\frac{dA_n}{d\tau}-\left(\frac{V_0^2\rho}{P_0}+\frac{I_0B_0}{\lambda_nP_0}\right)A_n+\frac{A_n}{\sqrt{1-\left|A_n\right|^2\lambda_n^2}}=0.$$

$$\ddot{A}_*(\tau)+(\mu_0+i\nu_0)\dot{A}_*(\tau)-\Omega_1^2A_*(\tau)+\frac{A_*(\tau)}{\sqrt{1-\left|A_*(\tau)\right|^2}}=0,$$

$$\lambda_nA_n(\tau)=A_*(\tau),\ \ \nu_0=\frac{2V_0\lambda_n}{\Omega_0},\quad \Omega_1^2=\frac{B_0I_0}{\lambda_nP_0}+\frac{V_0^2\rho}{P_0},$$

$$\mu_0=\frac{\eta}{\Omega_0},\qquad \Omega_0^2=\frac{P_0\lambda_n^2}{\rho}.$$

$$\left.\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\mu_0 x_3 + \nu_0 x_4 + \Omega_1^2 x_1 - \frac{x_1}{\sqrt{1-\left(x_1^2 + x_2^2\right)}} \\
\dot{x}_4 &= -\mu_0 x_4 + \nu_0 x_3 + \Omega_1^2 x_2 - \frac{x_2}{\sqrt{1-\left(x_1^2 + x_2^2\right)}}
\end{aligned}\right\}$$

$x_1(\tau) = \operatorname{Re} A_*(\tau)$, $x_2 = \operatorname{Im} A_*(\tau)$, $x_3 = \operatorname{Re} \dot{A}_*(\tau)$, $x_4 = \operatorname{Im} \dot{A}_*(\tau)$

Equilibrium Points

The equilibrium points are as follows:

$\Omega_1^2 - 1 < 0,$ $\qquad M_0(x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0)$

$\Omega_1^2 - 1 > 0,$ $\qquad M_1(x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0)$

$$M_*^{\pm}(x_1 = 0, x_2 = \pm x_0, x_3 = 0, x_4 = 0)$$

$$M_{**}^{\pm}(x_1 = \pm x_0, x_2 = 0, x_3 = 0, x_4 = 0)$$

$$x_0 = \sqrt{\frac{\Omega_1^4 - 1}{\Omega_1^4}}$$

Non-linear vibrations of a parametrically excited string in resonance cases.

$$I_0 = I_{01} + I_{02}\cos(\Omega\tau)$$

$$\ddot{X}(\tau) + \omega_0^2 X(\tau) = -\varepsilon\left[2(\mu + i\nu)\dot{X}(\tau) + g\cos(\Omega\tau)X(\tau) + \alpha X(\tau)\left|X(\tau)\right|^2\right]$$

$$\mu = \frac{\mu_0}{2\varepsilon}, \qquad \nu = \frac{\nu_0}{\varepsilon}, \qquad \omega_0^2 = 1 - \frac{I_{01}B_0}{\lambda_n P_0} - \frac{V_0^2\rho}{P_0},$$

$$g = \frac{B_0 I_{02}}{\varepsilon P_0 \lambda_m}, \alpha = \frac{\lambda_m^2 \varepsilon}{2}, \ \varepsilon << 1$$

Resonance vibrations of a string near the basic state $\left(\Omega \cong \omega_0\right)$

$$\Omega^2 = \omega_0^2 + \varepsilon\sigma_1$$

$$\ddot{X} + \Omega^2 X = \varepsilon\left\{\sigma_1 X - 2(\mu + i\upsilon)\dot{X} - g\cos\left(\Omega\tau\right)X - \alpha X^3\right\}$$

$$X(\tau,\varepsilon) = X_0\left(T_0,T_1,T_2\right) + \varepsilon X_1\left(T_0,T_1,T_2\right) + \ldots \qquad T_n = \varepsilon^n\tau\,(n = 0,1,..)$$

A second order approximation to the solution

$$X(\tau,\varepsilon) \equiv X_{\omega_0}(\tau,\varepsilon) = Ae^{i\Omega\tau} + Be^{-i\Omega\tau} +$$

$$+\varepsilon\left\{-\frac{g\left(A+B\right)}{2\Omega^2} + \frac{\alpha A^2\overline{B}}{8\Omega^2}e^{3i\Omega\tau} + \frac{\alpha B^2\overline{A}}{8\Omega^2}e^{-3i\Omega\tau} + \frac{gA}{6\Omega^2}e^{2i\Omega\tau} + \frac{gB}{6\Omega^2}e^{-2i\Omega\tau}\right\} + 0\left(\varepsilon^2\right)$$

Resonance vibrations of a string near the main state $\left(\Omega \cong 2\omega_0\right)$

$$\Omega^2 = 4\left(\omega_0^2 + \varepsilon\sigma_2\right)$$

$$X(\tau,\varepsilon) \equiv X_{2\omega_0}(\tau,\varepsilon) = Ae^{\frac{i\Omega\tau}{2}} + Be^{-\frac{i\Omega\tau}{2}} + \varepsilon\left\{\left(\frac{\alpha A^2\overline{B}}{2\Omega^2} + \frac{gA}{4\Omega^2}\right)e^{\frac{3i\Omega\tau}{2}} + \left(\frac{\alpha B^2\overline{A}}{2\Omega^2} + \frac{gB}{4\Omega^2}\right)e^{-\frac{3i\Omega\tau}{2}}\right\} + 0\left(\varepsilon^2\right)$$

Stability of the solutions near $\Omega \cong \omega_0$ and $\Omega \cong 2\omega_0$

The stability of the trivial solution

$$-2i\Omega\dot{A} + \varepsilon\left[\sigma_1 - 2i\Omega(\mu + i\nu)\right]A +$$

$$\varepsilon^2\left[(\mu + i\nu)^2 + \frac{\sigma_1^2}{4\Omega^2} + \frac{g^2}{4\Omega^2}\right]A + \frac{\varepsilon^2 g^2 B}{4\Omega^2} = 0$$

$$2i\Omega\dot{B} + \varepsilon\left[\sigma_1 + 2i\Omega(\mu + i\nu)\right]B +$$

$$\varepsilon^2\left[(\mu + i\nu)^2 + \frac{\sigma_1^2}{4\Omega^2} + \frac{g^2}{4\Omega^2}\right]B + \frac{\varepsilon^2 g^2 A}{4\Omega^2} = 0$$

The trivial solution is stable when $\operatorname{Re}\lambda < 0$

Instability starts when $\operatorname{Re}\lambda > 0$

The non-trivial curve when $\nu = 0$

$$(A_r, A_i, B_r, B_i) = (a_r, a_i, b_r, b_i)e^{\varepsilon\lambda t}$$

The equation for the definition of a non-trivial curve

$$l_{2\omega_0} = \left[\Omega^2\mu^2 + \delta_2^2\right]^2 - \left(\frac{g}{2}\right)^2 \left\{2\left(\Omega^2\mu^2 + \delta_2^2\right) - \left(\frac{g}{2}\right)^2\right\} = 0$$

$$\delta_1 = \sigma_1 + \varepsilon\left(\mu^2 + \frac{\sigma_1^2}{4\Omega^2} + \frac{g^2}{6\Omega^2}\right), \quad \delta_2 = \sigma_2 + \varepsilon\left(\mu^2 + \frac{\sigma_2^2}{4\Omega^2} - \frac{g^2}{6\Omega^2}\right)$$

Non-linear Vibrations: Resonance Cases

$$I_0 = I_{01} + I_{02}\cos(\Omega\tau)$$

$$\omega_0^2 = 1 - \frac{I_{01}B_0}{\lambda_n P_0} - \frac{V_0^{\,2}\rho}{P_0}$$

1) Resonance vibrations of a string near the basic state $(\Omega \cong \omega_0)$

The stability of the solution

$$\Delta_{\omega_0}(\lambda, g, \Omega, \mu, \gamma) = \det \overset{\frown}{Q} = 0$$

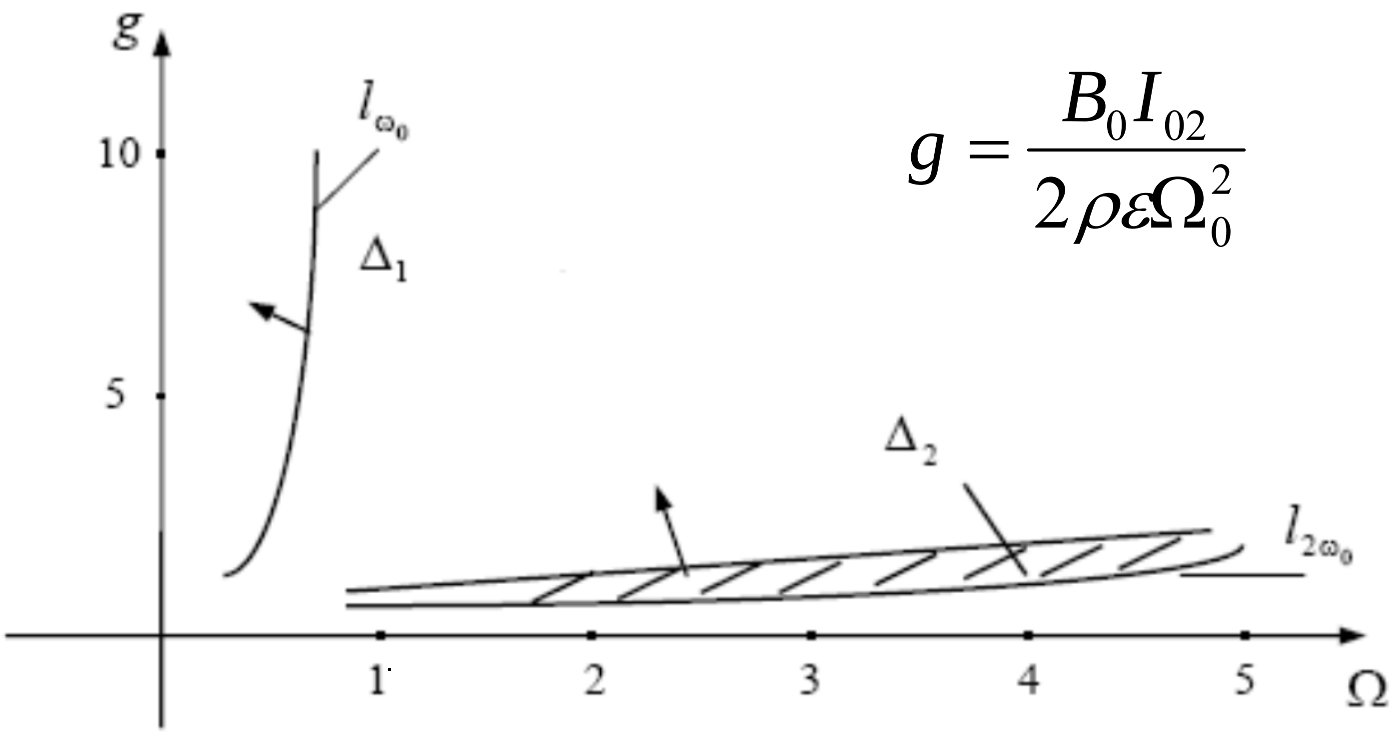
2) Resonance near the main state $(\Omega \cong 2\omega_0)$

The stability of the solution

$$\Delta_{2\omega_0}(\lambda, g, \Omega, \mu, \gamma) = \det \overset{\frown}{R} = 0$$

Pre-chaotic state

$$g = \frac{B_0 I_{02}}{2\rho\varepsilon\Omega_0^2}$$



Δ_1 **is unstable domain near the frequency** $\Omega = \omega_0$

Δ_2 **is unstable domain near the frequency** $\Omega = 2\omega_0$

is the region for parameters when the system falls into pre-chaotic state

Melnikov’s Method for Observing Chaotic motion

These method can be applied to problems where dissipation is small and equations for the manifolds of the zero dissipation problem are known

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} + \varepsilon g_1(\mathrm{p,q,t}) \\ \dot{p} = -\dfrac{\partial H}{\partial q} + \varepsilon g_2(\mathrm{p,q,t}) \end{cases}$$

$$g_k(\mathrm{p,q,t+T}) = g_k(\mathrm{p,q,t})$$

$H = H(p, q)$, is a Hamiltonian for undamped, unforced problem

Melnikov’s Function

$M(\tau)=\int_{-\infty}^{\infty} \vec{g}_0 \cdot \nabla H(p_0, q_0)dt$ where

$\vec{g}_0=\vec{g}(p_0, q_0, t+\tau)$; p_0 *and* q_0 *are the solutions for the unperturbed homoclinic orbit originaiong at* the saddle point of the Hamiltonian problem.

Examples

$$\ddot{\theta} + \gamma\dot{\theta} + Sin\theta = f_1 \dot{C}os\theta Cos\omega t + f_0, \qquad q = \theta, p = \dot{\theta}$$

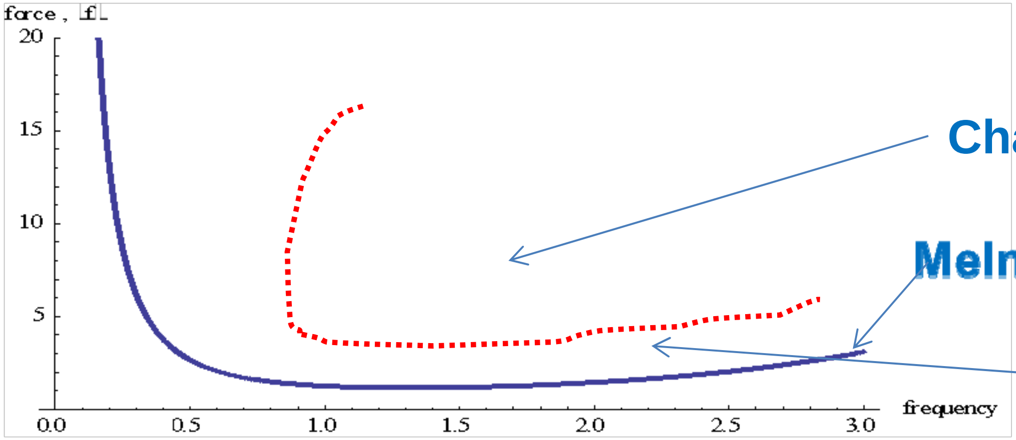
$$H = \frac{1}{2}p^2 + (1 - Cos\theta) \rightarrow$$ undamped, unforced problem

Saddle point: $\theta = 0, \dot{\theta} = 0$

Unperturbed homoclinic orbit: $q_0 = 2Tan^{-1}(Sinht);\ p_0 = 2Secht$

Melnikov function: $M(\tau) = -8\gamma + 2\pi f_0 + 2\pi f_1 \omega^2 Sech(\frac{\pi\omega}{2})\ Cos\omega\tau$

Condition for chaotic motion: $f_1 > \left|\frac{4\gamma}{\pi} - f_0\right| \frac{Cosh(\pi\omega/2)}{\omega^2}$



Chaotic region: experiment

Melnikov's line: γ=0.5

Periodic region

Chaotic Motion for a String in a Magnetic Field

Particular Case: $\nu_0 = \frac{2V_0\lambda_n}{\Omega_0} = 0$

$$I_0 = I_{01} + I_{02}\cos(\Omega\tau)$$

$$B_0 = B_{01} + B_{02}\cos(\Omega t)$$

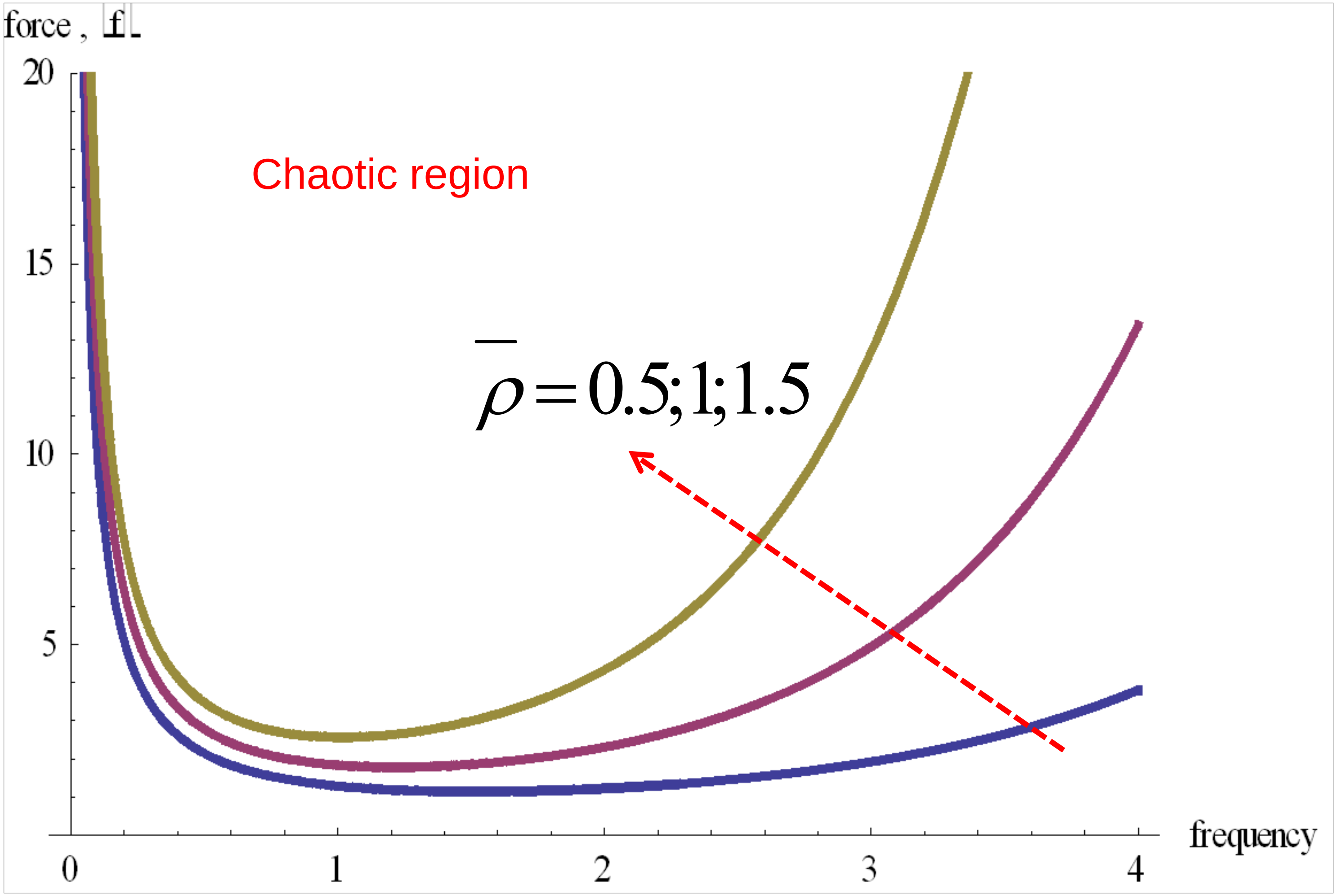
Melnicov Function: $M(\tau) = -\frac{\pi\Omega_1^2\beta_1}{sh(\pi\Omega_1/2)}\sin(\Omega_1\tau) + \alpha$

Homoclinical structures exist in the phase space if the Melnicov function has a simple roots, i.e.- *Strange Attractor*

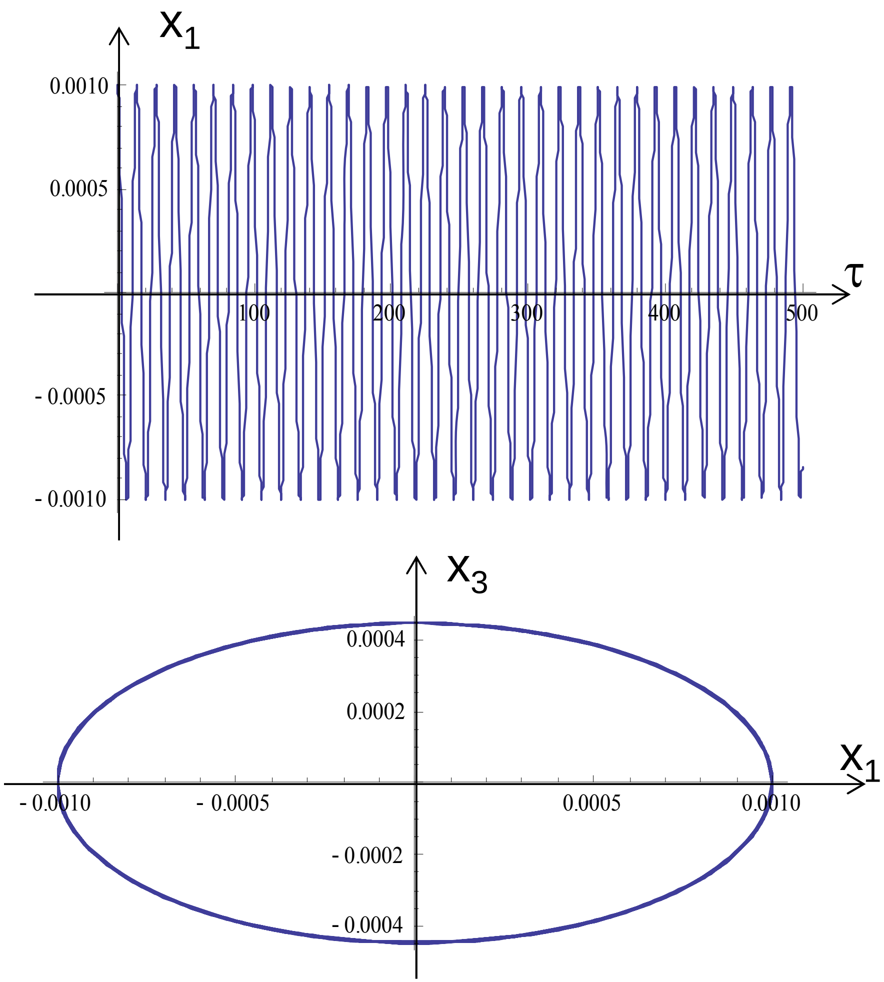
Conditions for chaotic motion

$$\frac{I_{02}B_0}{l} > \frac{4\eta J_{01}\sqrt{J_{01}}}{3\pi\Omega^2\sqrt{2\rho}}\, sh\left(\frac{\pi}{2}\sqrt{\frac{2\rho}{J_{01}}}\Omega\right), \qquad J_{01} = \frac{2\pi}{l}\left(I_{01}B_0 - \frac{2\pi P_0}{l}\right)$$

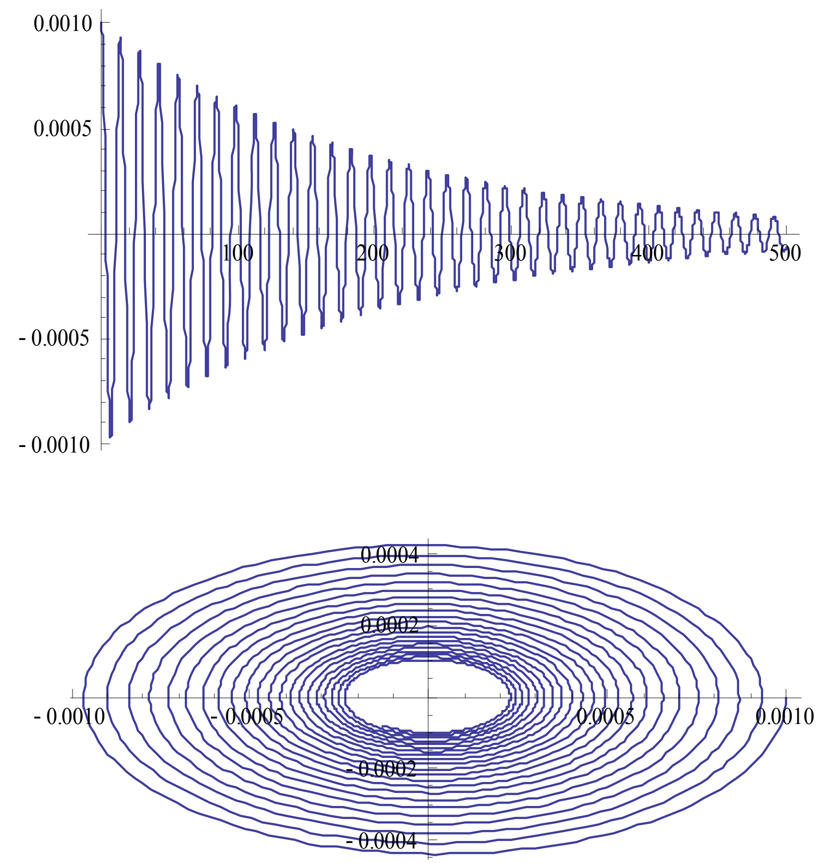
Melnikov’s lines for different nondimensional densities



Numerical Calculations: Case $\nu_0 = \dfrac{2V_0\lambda_n}{\Omega_0} = 0$



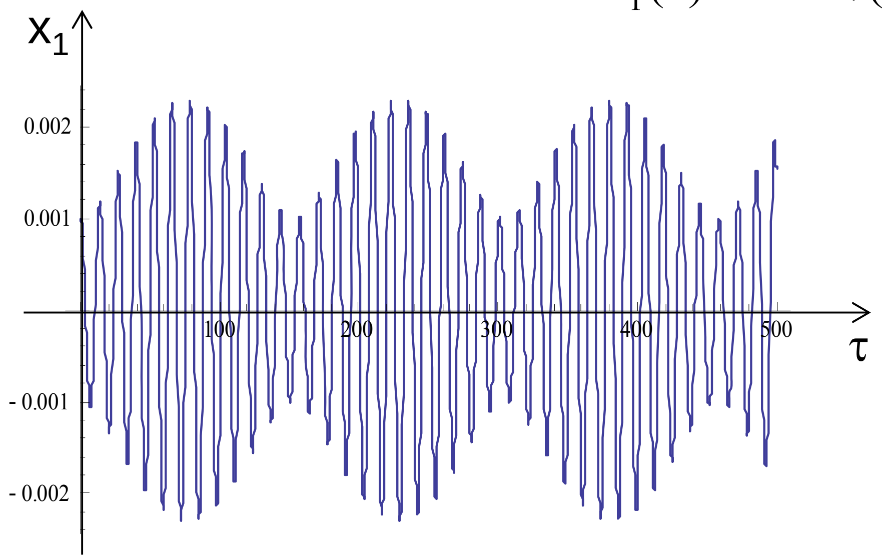
No damping

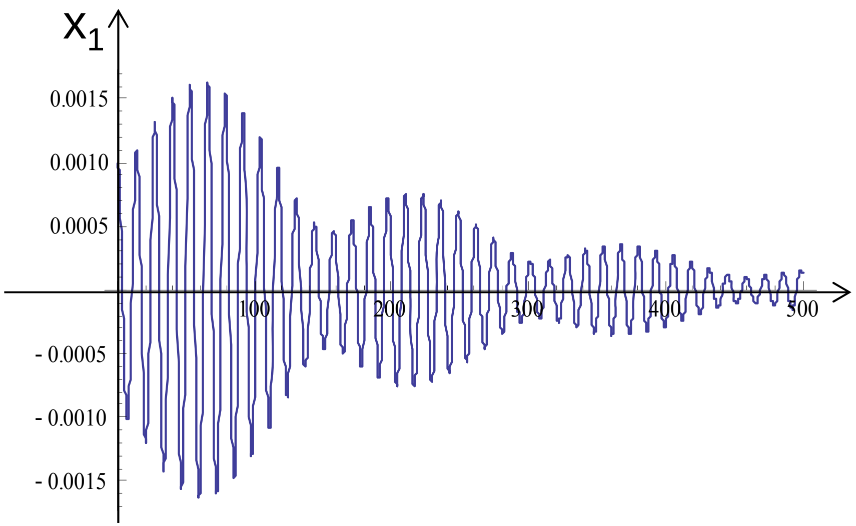


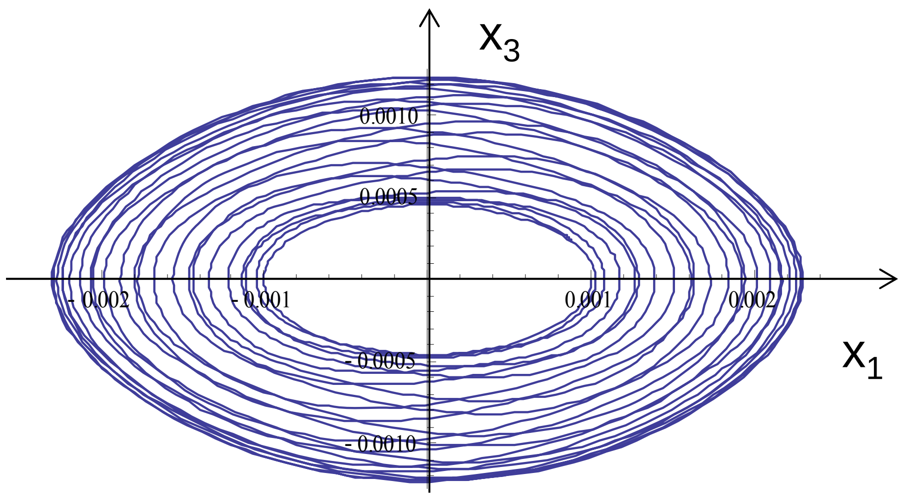
damping

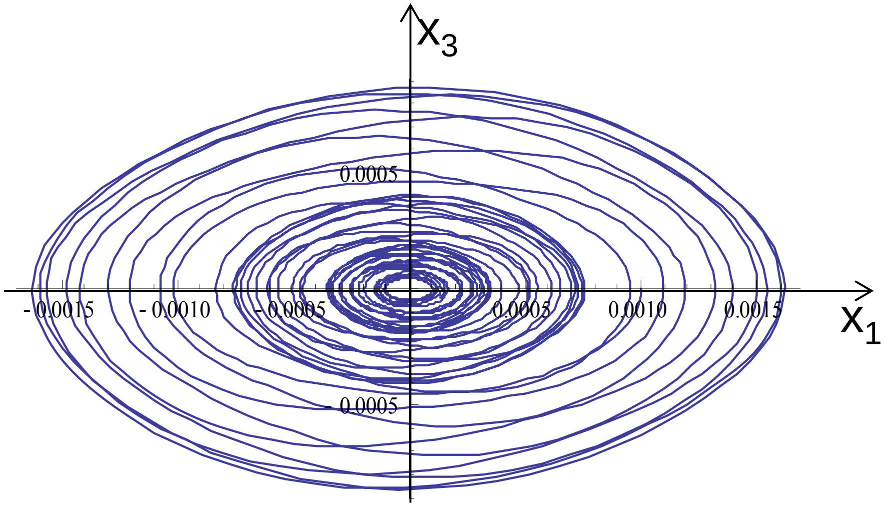
Numerical Calculations: Case $\nu_0=0.2$

$$x_1(\tau) = \mathrm{Re}\, A_*(\tau) \qquad x_3 = \mathrm{Re}\, \dot{A}_*(\tau)$$





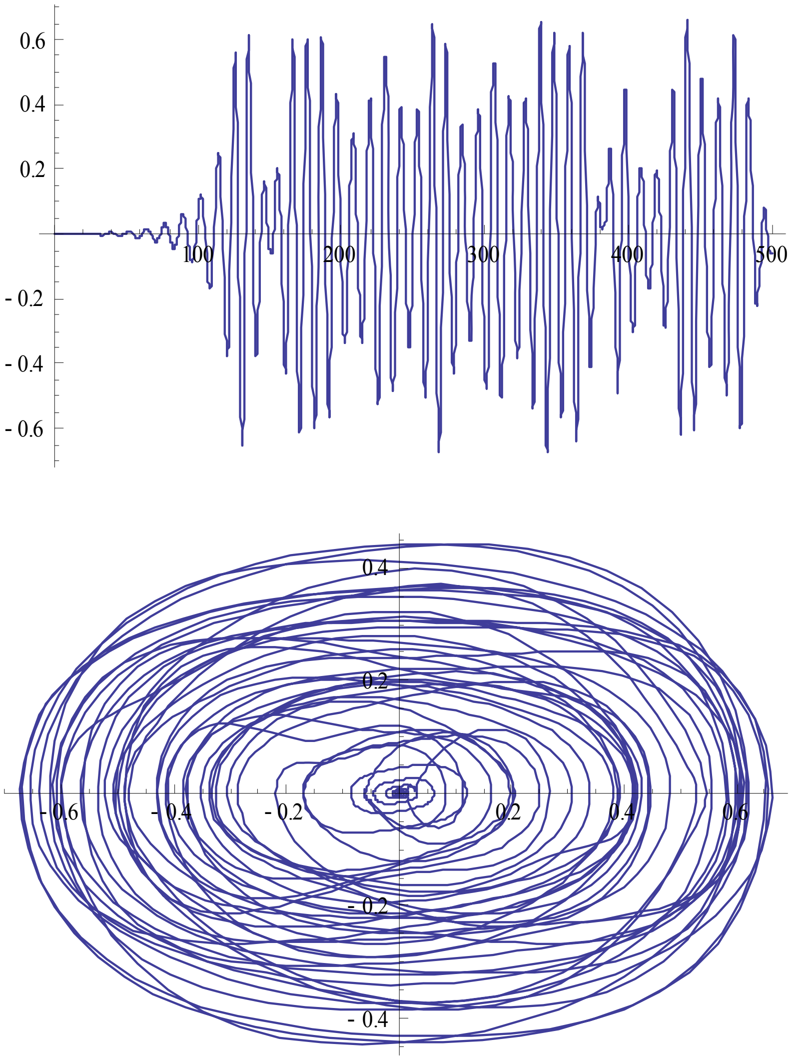




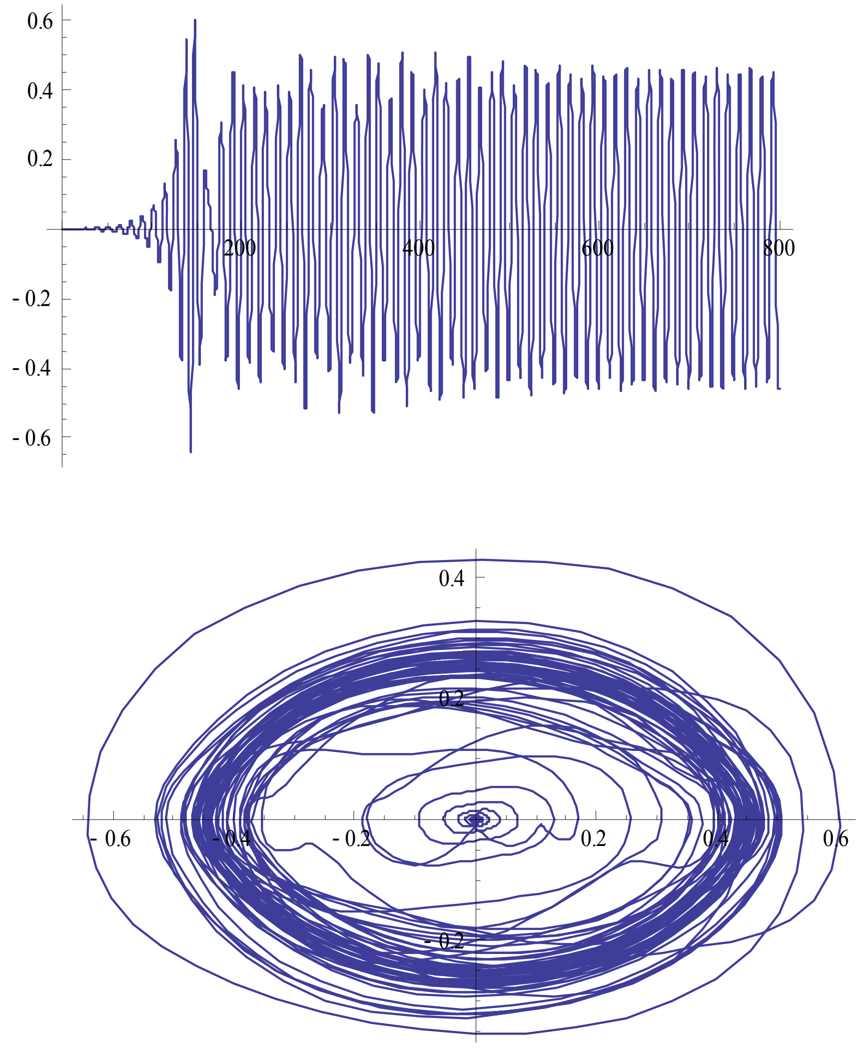
No damping

damping

Numerical Calculations: Case $\nu_0 = \frac{2V_0\lambda_n}{\Omega_0} = 0.3$



No damping



Damping

Computing the stability regions of Hill's equation

A linear second-order ordinary differential equation with a periodic coefficient of the form

$$y'' = \omega\phi(t, \beta_1, \beta_2, \ldots, \beta_{n-1})y, \quad y \in C^1(0, 2T),$$
$$\phi(t+T, \beta_1, \beta_2, \ldots, \beta_{n-1}) = \phi(t, \beta_1, \beta_2, \ldots, \beta_{n-1}),$$

$$y(0) = y(2T),$$
$$y'(0) = y'(2T).$$

Example

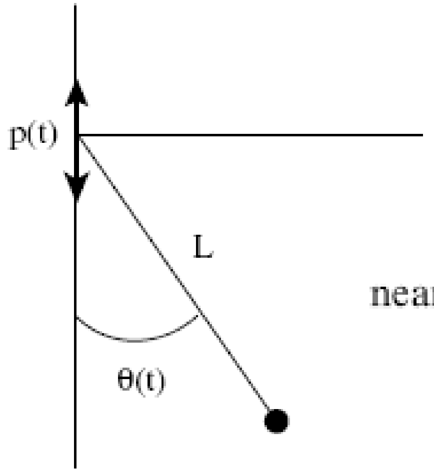


Fig. 1. An inverted pendulum with an oscillating pivot point.

$$\theta'' + \left[\frac{g}{L} + \frac{1}{L}r(t)\right]\sin\theta = 0.$$

near $(0,0)$ and $(\pi,0)$ leads to the linearized equations (Hill's equations)

$$\theta'' \pm \left[\frac{g}{L} + \frac{1}{L}r(t)\right]\theta = 0,$$

With the new function, our equation becomes

$$\frac{1}{\omega}z(t) = f(t)\left[\int_0^t z(s)\,\mathrm{d}s + y(0)\right] - \int_0^t f(s)z(s)\,\mathrm{d}s + \frac{1}{\omega}y'(0).$$

where

$$f(t) = \int_0^t \phi(s)\,\mathrm{d}s.$$

$$y'(t) = z(t) \Rightarrow y(t) = \int_0^t z(s)\,\mathrm{d}s + y(0).$$

$$\mathscr{A}z := f(t)\int_0^t z(s)\,\mathrm{d}s - \int_0^t f(s)z(s)\,\mathrm{d}s + C_1 + Cf(t),$$

$$H = \left\{x \mid x \in L_2(0,2T),\ \int_0^{2T} x\,\mathrm{d}t = 0\right\},$$

$$E = \left\{x \mid x \in C(0,2T),\ \int_0^{2T} x\,\mathrm{d}t = 0\right\}.$$

$$f(t) = \int_0^t \phi(s)\,\mathrm{d}s, \quad f(2T) \neq 0, \quad \phi \in L_2(0,2T).$$

Property 2. *The operator $\mathscr{A} : H \to E$ is compact, i.e., maps any bounded set from H to a compact set $D \subset E$.*

Property 3. *$\mathscr{A}$ is self-adjoint on the set H.*

Property 4. *The spectral problem*

$$\mathscr{A}z = \lambda z, \quad z \in E$$

is equivalent to (18) *and* (19) *if we denote $\lambda = 1/\omega$ and introduce notation*

$$y(t) = \int_0^t z(s)\,\mathrm{d}s + C,$$

Examples: Instability regions of $y''(t) + (\delta + \varepsilon r(t))y(t) = 0,$

(a) $r(t) = \sin t - \dfrac{2}{\pi}, \qquad 0 \leqslant t < \pi,$

(b) $r(t) = t - \dfrac{\pi}{2}, \qquad 0 \leqslant t < \pi,$

(c) $r(t) = \begin{cases} t - \pi/4, & 0 \leqslant t < \pi/2, \\ \frac{3}{4}\pi - t, & \pi/2 \leqslant t < \pi. \end{cases}$

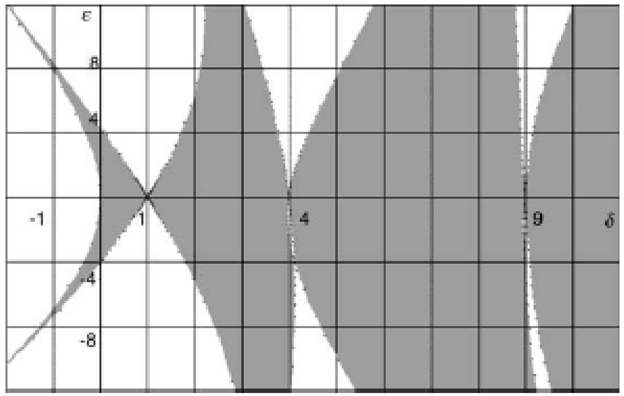


Fig. 5. Stability diagram of Hill's equation with periodic coefficient given by (47)(a).

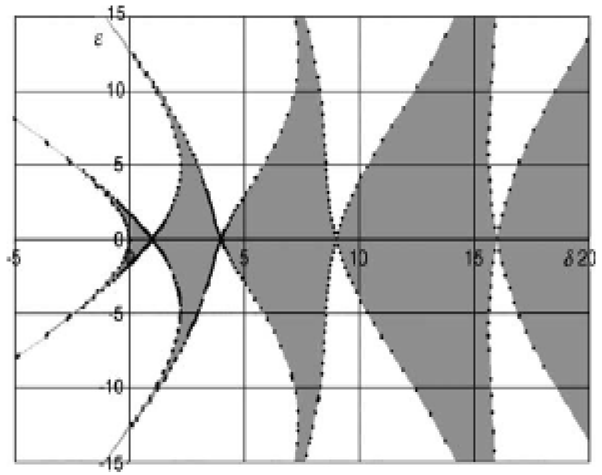


Fig. 6. Stability diagram of Hill's equation with periodic coefficient given by (47)

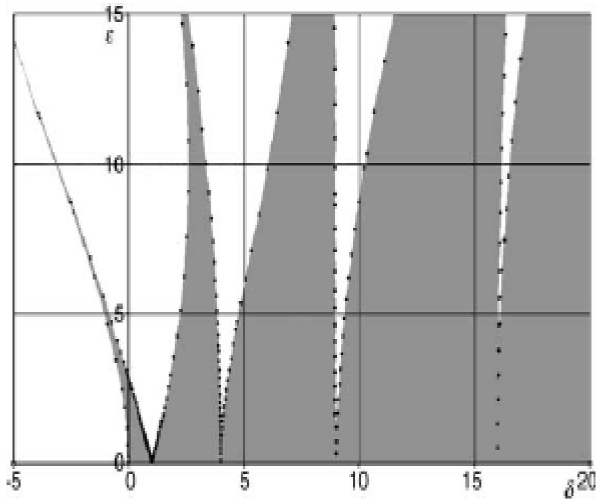


Fig. 7. Stability diagram of Hill's equation with periodic coefficient given by (47)(c).

Group Properties of Some Differential and Integro-Differential Equations

Discussion of some equations. Examples:

1) $$F(x, y, y', ..., y^{(n)}) = 0$$

2) $$u_{tt} - [G(u)u_t^m u_s^n]_s = F(u)u_t^p u_s^q,$$

3) **Vlasov-Maxwell Equation**

EXAMPLE: Reduction of arbitrary non-linear equation to the linear one: Berkovich Thorem

Theorem *In order for the equation $y'' = f(x, y, y')$ reduce to the linear form (4.2) by transformation (1.2), it is necessary that it admits the commutative factorization:*

$$\left[\frac{1}{u_1 + u_2 y'} D - \frac{v_x + v_y y'}{v(u_1 + u_2 y')} - r_2\right] \left[\frac{1}{u_1 + u_2 y'} D - \frac{v_x + v_y y'}{v(u_1 + u_2 y')} - r_1\right] y + cv = 0,$$

or noncommutative factorization:

$$\left[D - \frac{D(u_1 + u_2 y')}{u_1 + u_2 y'} - \frac{v_x + v_y y'}{v} - r_2(u_1 + u_2 y')\right] \left[D - \frac{v_x + v_y y'}{v} - r_1(u_1 + u_2 y')\right] y +$$

$$c(u_1 + u_2 y')^2 v = 0, \quad D = \frac{d}{dx},$$

where r_k, $k = 1, 2$, satisfy the characteristic equation

$$r^2 + b_1 r + b_0 = 0.$$

$$\frac{d^2 Y}{dX^2} + b_1 \frac{dY}{dX} + b_0 Y + c = 0, \quad b_1, b_0, c = \text{const}, \tag{4.2}$$

$$y = v(x, y) z, \quad dt = u_1(x, y) dx + u_2(x, y) dy, \tag{1.2}$$

EXAMPLE

In order for the equation

$$y'' + f(y)y'^2 + b_1\varphi(y)y' + \psi(y) = 0, \quad y' = \frac{dy}{dx},$$

to be linearized by $y = v(y)z, \quad dt = u(y)dx$

$$\ddot{z} + b_1\dot{z} + b_0 z + c = 0, \quad \dot{z} = \frac{dz}{dt},$$

it is necessary and sufficient that it should be presented

$$y'' + fy'^2 + b_1\varphi y' + \varphi\exp\Big(-\int f(y)dy\Big)\left[b_0\int\varphi\exp\Big(\int f(y)dy\Big)dy + \frac{c}{\beta}\right] = 0,$$

EXAMPLE

$$y'' + F(y, y') = 0$$

can be linearized by the convertible (in some domain $\Gamma(x, y)$) transformation

$$y = v(y)z, \quad \mathrm{d}t = u(y)\,\mathrm{d}x$$

Kummer-Liouville Transformation

it is necessary and sufficient that the equation is of the form

$$y'' + f(y)y'^2 + b_1\varphi(y)y' + \psi(y) = 0$$

EXAMPLE

In order that the equation

$$y'' + a_1 y' + a_0 y + f(x) y^n = 0, \quad n \neq 0; 1$$

by the KLT (2) *be reduced to the form*

$$\ddot{z} \pm b_1 \dot{z} + b_0 z + k z^n = 0$$

it is necessary and sufficient that the KLT satisfied the conditions (5), (6) *and also*

$$f(x) = k u^2 v^{1-n}.$$

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 - \frac{1}{4}\delta u^2 = A_0(x), \quad \delta = b_1^2 - 4b_0 \quad (5)$$

where $A_0(x) = a_0 - 1/4a_1^2 - 1/2a_1'$, *and*

$$v(x) = |u|^{-1/2} \exp\left(-\frac{1}{2}\int a_1 \,\mathrm{d}x \pm \frac{1}{2} b_1 \int u \,\mathrm{d}x\right). \ (6)$$

Example: ERMAKOV'S Equation

$$v'' + a_0(x)v - b_0 v^{-3} = 0.$$

$$v(x) = \sqrt{Ay_2^2 + By_2y_1 + Cy_1^2}, \quad \delta = B^2 - 4AC = -4b_0,$$

where

$$y_1,\, y_2 = y_1 \int y_1^{-2} \mathrm{d}x$$

Local one-parameter Lie group of transformations. Invariant condition

Transformation for

$y^{(n)}$=F(x,y,y', ... $y^{(n-1)}$)

$$T_\varepsilon = \begin{cases} \bar{x} = \varphi(x,y,\varepsilon) \approx x + \xi(x,y)\varepsilon, \\ \bar{y} = \psi(x,y,\varepsilon) \approx y + \eta(x,y)\varepsilon \end{cases}$$

First order linear differential operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial x}$$

If $I_0(\bar{x},\bar{y}) = I_0(x,y)$, *then* I_0 should satisfy linear partial differential equation

$$\xi(x,y)\frac{\partial I_0}{\partial x} + \eta(x,y)\frac{\partial I_0}{\partial x} = 0$$

CHECK: McMillan; Courant-Snyder; Danilov, Nagaitsev invariant

For Differential Equation $y^{(n)}$-F=0

$$X_n[y^{(n)} - F(x,y,y',\dots y^{(n-1)})]\Big|_{y^{(n)}-F} = 0$$

Example: GROUP PROPERTIES OF $u_{xx} - u_y^m u_{yy} = f(u)$

LIE GROUPS AND LIE ALGEBRA

generator Γ is taken to be

$$\Gamma = \xi(x, y, u)\frac{\partial}{\partial x} + \eta(x, y, u)\frac{\partial}{\partial y} + \sigma(x, y, u)\frac{\partial}{\partial u}.$$

The generators $\overset{1}{\Gamma}, \overset{2}{\Gamma}$ of the once and twice extended groups are

$$\overset{1}{\Gamma} = \Gamma + \sigma_{[x]}\frac{\partial}{\partial u_x} + \sigma_{[y]}\frac{\partial}{\partial u_y},$$

$$\overset{2}{\Gamma} = \overset{1}{\Gamma} + \sigma_{[xx]}\frac{\partial}{\partial u_{xx}} + \sigma_{[xy]}\frac{\partial}{\partial u_{xy}} + \sigma_{[yy]}\frac{\partial}{\partial u_{yy}}.$$

invariant condition $\overset{2}{\Gamma} F \equiv 0$

Table 1. Infinitesimal Lie groups for choices of $f(u)$

$f(u)$	m	Infinitesimal groups
$\lambda^2 u$	$\neq -4$	$\xi = c_2, \quad \eta = c_3 y + c_4, \quad \sigma = \frac{m+2}{m} c_3 u + c_5 e^{\lambda x} + c_6 e^{-\lambda x}$
	$= -4$	$\xi = \frac{A}{\lambda} e^{2\lambda x} - \frac{B}{\lambda} e^{-2\lambda x} + c_2, \quad \eta = c_3 y + c_4$
		$\sigma = (Ae^{2\lambda x} + Be^{-2\lambda x} + c_3/2)u + Ce^{\lambda x} + De^{-\lambda x}$
$-\lambda^2 u$	$\neq -4$	$\xi = c_2, \quad \eta = c_3 y + c_3,$
		$\sigma = \frac{m+2}{m} c_3 u + c_5 \sin(\lambda x) + c_6 \cos(\lambda x)$
	$= -4$	$\xi = \frac{A}{\lambda} \sin(2\lambda x) - \frac{B}{\lambda} \cos(2\lambda x) + c_2, \quad \eta = c_3 y + c_4$
ku^p	$\neq -4, \quad p = p^*$	$\xi = c_1 x + c_2, \quad \eta = c_3 y + c_4, \quad \sigma = \left(\frac{m+2}{m} c_3 - \frac{2}{m} c_1\right) u$
	$\neq -4, \quad$ for all p	$\xi = c_2, \quad \eta = c_4, \quad \sigma = 0$
	$= -4, \quad p = -3$	$\xi = c_0 x^2 + 2c_1 x + c_2, \quad \eta = c_4, \quad \sigma = (c_0 x + c_1)u$
	$= -4, \quad p \neq -3$	$\xi = -\frac{p-1}{p+3} c_3 x + c_2, \quad \eta = c_3 y + c_4, \quad \sigma = \frac{2c_3 u}{p+3}$

Reduction to the Ordinary Differential Equation

Case B: f(u) *arbitrary*

m $\neq -2$. We obtain the similar solution $u = F(\omega)$; where $\omega = x + ay$; $a = -\dfrac{c_2}{c_4}$ and the ODE reduction in this case becomes

$$F'' - a^{m+2}F'^{m}F'' = f(F).$$

Case C: $f(u) = ku^p$; $k, p \in R$, $k, p \neq 0$

m $\neq -4$. For the case when $(m+2)c_3 \neq 2c_1$, by setting $c_2 = c_4$ we obtain the similar solution form

$$u = x^{-\left[\frac{(m+2)a+2}{m}\right]}F(\omega),$$

where $\omega = yx^a$; $a = -\dfrac{c_3}{c_1}$ and $F(\omega)$ satisfying

$$\left[\frac{(m+2)a+2}{m}\right]\left[\frac{(m+2)a+m+2}{m}\right]F - a\left[\frac{(m+4)(a+1)}{m}\right]\omega F'$$

$$+ a^2\omega^2F'' - F'^{m}F'' = kF^p.$$

Example: LIE GROUP ANALYSIS

Symmetries of radial multi-component plasma

purely radial motion the Vlasov–Maxwell system of equations for collision-less, multi-component, plasmas without magnetic field has the following form

$$\partial_t f_\alpha + u\partial_r f_\alpha + \frac{q_\alpha}{m_\alpha} E\partial_u f_\alpha = 0,$$

$$\partial_r E + \frac{2}{r} E - \sum_\alpha \frac{q_\alpha}{\epsilon_0} \int_0^\infty du\, u^2 f_\alpha = 0,$$

$$\partial_t E + \sum_\alpha \frac{q_\alpha}{\epsilon_0} \int_0^\infty du\, u^3 f_\alpha = 0,$$

where $E = E(t,r)$ is the radial component of electric field vector,

$f_\alpha = f_\alpha(t,r,u)$ is the radial distribution function of α-plasma component

q_α, m_α are charge and mass

u is the radial component of vector velocity

Generator of Group

$$G = \tau\partial_t + \xi\partial_r + \rho\partial_u + \sum_\alpha \eta_\alpha \partial_{f_\alpha} + \zeta\partial_E.$$

infinitesimal criterion of invariance

$$G^{(m)}F = 0$$

where $G^{(m)}$ is the extended to m-th order generator of the point transformation

Solutions of the determining equations lead to the following three generators

$$G_1 = \partial_t, \qquad G_2 = -t\partial_t - 2r\partial_r - u\partial_u + 5\sum_\alpha f_\alpha \partial_{f_\alpha},$$

$$G_3 = -3t\partial_t - r\partial_r + 2u\partial_u + 5E\partial_E,$$

Groups and Invariant Solutions

classification of essentially independent invariant solutions

No	Subgroup	Form of the solution
1	G_1	$f_\alpha(r,u)\,,\ E(r)$
2	G_2	$t^{-5}f_\alpha\big(t^2r^{-1},r^{-1}u^2\big),\quad E\big(t^2r^{-1}\big)$
3	G_3	$f_\alpha\big(tr^{-3},r^{-1}u^2\big),\quad t^{-1}r^{-2}E\big(tr^{-3}\big)$
4	$\pm G_1-3G_2+G_3$	$e^{\mp 15t}f_\alpha\big(re^{\mp 5t},ue^{\mp 5t}\big),\quad e^{\pm 5t}E\big(re^{\mp 5t}\big)$
5	$a_2G_2+a_3G_3$	$r^{-2}uf_\alpha\big(t^{(2a_2+a_3)}r^{-(a_2+3a_3)},tr^{-1}u\big),$ $t^{-2}rE\big(t^{(2a_2+a_3)}r^{-(a_2+3a_3)}\big)$

CONCLUSIONS

- Derived and modeled an equation of motion for conductive string in a magnetic field;
- Investigated bifurcation and parametric resonance;
- Implemented new numerical method for Hill's type equation for finding regions of parametric resonance;
- Applied Melnikov's method to determine regions of parameters for chaotic motion;
- For general nonlinear equations, applied Lie group method to find invariant solutions;
- Implemented examples of nonlinear dynamics;
- Discussed future work related to accelerators and beam dynamics.

Thank you for your attention!