Pre-existing betatron motion and spin flipping with RF fields in storage rings

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Physics now stands on three legs – has three branches

- Experimental physics
- Theoretical physics
- Computational physics

– FZ Juelich: publicity material about the powerful computers.

Essential, pre-requisite tools and knowledge:

- The **T-BMT equation** for spin motion:
  $d\vec{S}/ds = \vec{\Omega}(u,s) \times \vec{S}; \quad \vec{\Omega}(u,s) = \vec{\Omega}_0(s) + \vec{\omega}(u,s); \quad u = (x,\ p_x,\ y,\ p_y, z, p_z); \quad \theta = 2\pi s/C$
- The 1-turn periodic coordinate system on the CO: $\hat{l}(s), \hat{n}_0(s), \hat{m}(s)$
- The **invariant spin field** (ISF):
  $\hat{n}(u,s)$ or $\hat{n}(u,\theta)$; $\quad \hat{n}(u, s+C) = \hat{n}(u,s)$
- The **spin tune** on the closed orbit ν_0: $= a\gamma_0$ in a simple perfectly flat ring.
- The **uniform invariant frame field** (u-IFF): $\hat{n}_1(u,s)$, $\hat{n}(u,s)$, $\hat{n}_2(u,s)$
- The **equivalence class of amplitude dependent spin tunes** (ADST).
- The **naive single resonance model** (SRM) for the ISF and ADST: a single Fourier harmonic of $\vec{\omega}(u,s) \cdot (\hat{l}(s) - i\hat{m}(s))$ has dominant control of spin motion
  $\Longrightarrow$ the spin tune ν_0 on the closed orbit matches an orbital tune:
  $\nu_0 = \kappa = k_0 \pm Q_{\rm I}, \quad k_0 \pm Q_{\rm II}, \quad k_0 \pm Q_{\rm III}$ or $\nu_0 = \kappa = k_0 \pm Q_{\rm rf}$ for an integer k_0.
- The well known notation for the SRM. For example:
  $\delta = \nu_0 - \kappa$, the resonance strength ϵ_κ
- The **naive Froissart–Stora formula** for survival of vertical polarisation (1960!) when δ crosses 0:

$$\frac{S_y^{\rm final}}{|\vec{S}|} = 2\exp(-\Xi) - 1 = 2\exp\left(-\frac{\pi|\epsilon_\kappa^{\rm v}|^2}{2\alpha_{\rm FS}}\right) - 1\,, \quad \alpha_{\rm FS} = \frac{d\delta}{d\theta}$$

  Only OK if the SRM is OK!

Preamble

Spin motion in electric and magnetic fields:

The T-BMT spin precession equation:

$$\frac{d\vec{S}}{ds} = \vec{\Omega} \times \vec{S}$$

$\vec{S}$: spin expectation value

$\vec{\Omega}$: depends on $\vec{B}, \vec{E}, \vec{\beta}, \gamma$

In transverse magnetic fields:

$$\Omega \;\propto\; (a + 1/\gamma) \cdot B$$

$a = (g-2)/2$ where g is the relevant g factor.

$a = 1.793...$ for protons.

$a = -0.143$ for deuterons.

($a = 0.00115...$ for electrons.)

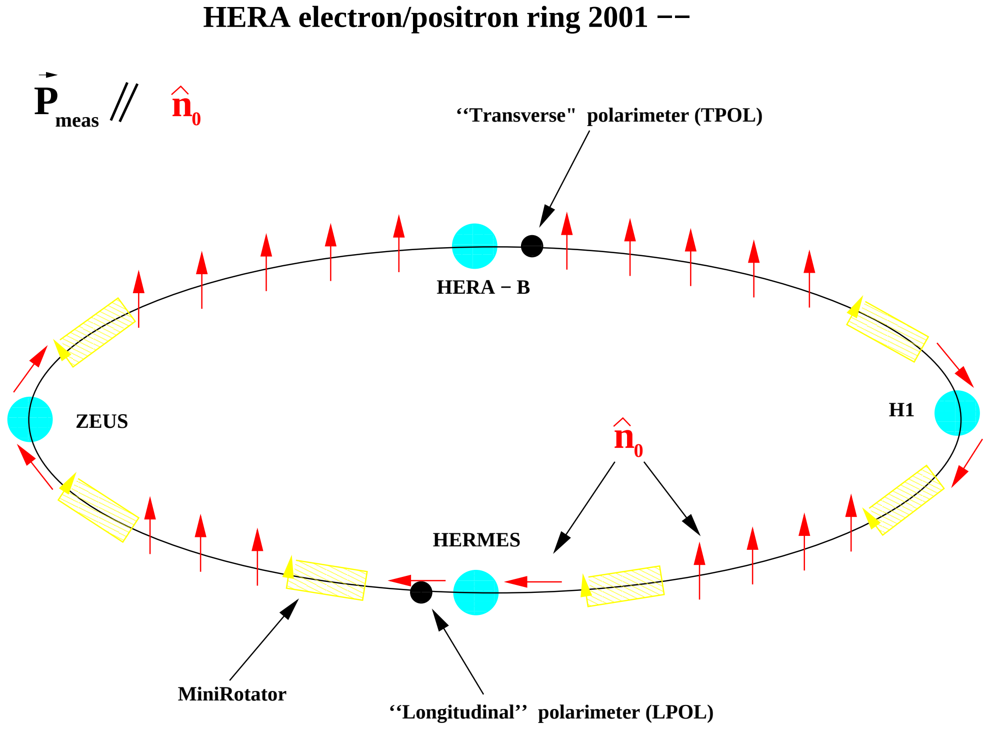
HERA

The first and only $e^{\pm}$ ring to supply longitudinal polarisation at high energy

— via the Sokolov-Ternov effect – also at 3 IP's simultaneously!

≈ 30 GeV, $\tau_{st} \approx 30$ mins. Depolarisation not too strong.

Perfectly balanced parameters



Polarisation vertical in the arcs – to drive the Sokolov-Ternov effect

Invariant fields: phase space Protons

- Canonical particle coordinates: $u \equiv (x,\ p_x,\ y,\ p_y,\ z, p_z)$ Indep. var. = azimuth, s

- For electrons at high energy: $u \equiv (x,\ p_x,\ y,\ p_y,\ z,\ p_z = \delta E/E_0)$

- Phase space density, $\rho(u; s)$: Liouville: ρ constant along paricle orbits =====>

$$\frac{\partial \rho}{\partial s} = \{H_{orb}, \rho\}$$

- Stationarity: $\rho(u; s) = \rho(u; s + C)$
  i.e. 1–turn periodicity of the (statistical) **scalar FIELD** $\rho(u; s)$

  although individual particles MOVE AROUND IN PHASE SPACE.

Invariant fields: spin

How can a proton beam be fully polarised but the polarimeter gives ZERO?

Invariant fields: spin
Protons

- Local spin polarisation $\vec{P}(u;s)$: T-BMT. =====> PARTIAL differential equation:

$$\frac{\partial \vec{P}}{\partial s} = \{H_{orb}, \vec{P}\} + \vec{\Omega}(u;s) \times \vec{P}$$

with $\vec{\Omega}(u;s) = \vec{\Omega}(u;s+C)$

- Stationarity: $\vec{P}(u;s) = \vec{P}(u;s+C)$
  i.e. 1–turn periodicity of the (statistical) **vector FIELD** $\vec{P}(u;s)$
  although individual particles MOVE AROUND IN PHASE SPACE AND THEIR SPINS MOVE TOO.

- $|\vec{P}|$ is constant along orbits: ===> $\hat{n}(u;s) = \vec{P}/|\vec{P}|$

$$\frac{\partial \hat{n}}{\partial s} = \{H_{orb}, \hat{n}\} + \vec{\Omega}(u;s) \times \hat{n}$$

- Stationarity: $\hat{n}(u;s) = \hat{n}(u;s+C)$ ===> $\hat{n}$ is called the INVARIANT SPIN FIELD.
- **Non–trivial** T–BMT solution satisfying CONSTRAINTS.
  See Barber, Ellison and Heinemann, PRST-AB, **7**, 124002 (2004).
- Solutions obeying these constraints are unstable (illdefined) at spin–orbit resonances.

And/Or

A vector field $\hat{f}$ of unit length in real 3–D space covering the 6–D phase space at each s:

.

$\hat{f}(u;s)$ with $u \equiv (x, p_x, y, p_y, z, p_z)$

$$\vec{\Omega}(u; s+C) = \vec{\Omega}(u;s)$$

$$\frac{d\hat{f}}{ds} = \frac{\partial \hat{f}}{\partial s} + \sum_{k=I,II,III} \frac{dx_k}{ds}\frac{\partial \hat{f}}{\partial x_k} + \frac{dp_k}{ds}\frac{\partial \hat{f}}{\partial p_k} = \vec{G}_{\hat{f}}(u;s) \underbrace{\Longrightarrow}_{\text{fixed length}\rightarrow\text{precession}} \vec{F}(u;s)\times\hat{f}$$

$$=====> \frac{\partial \hat{f}}{\partial s} = \{H_{orb}, \hat{f}\} + \vec{F}(u;s)\times\hat{f}$$

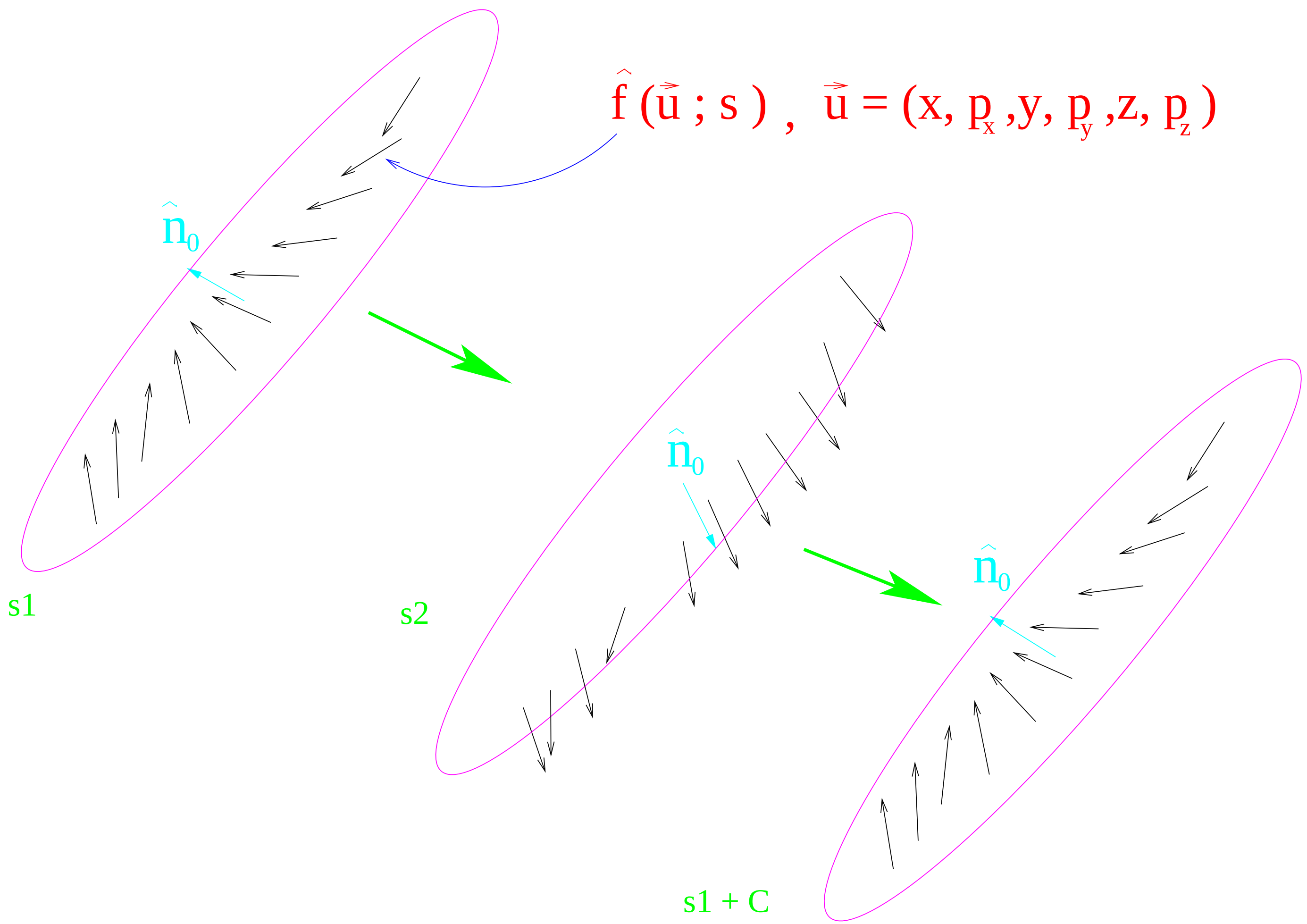
Now insist that this is the T–BMT equation! $d\hat{f}/ds = \vec{\Omega}(u;s)\times\hat{f}$

and that $\hat{f}(u; s+C) = \hat{f}(u;s)$

Rename: $\hat{f} \longrightarrow \hat{n}$

The invariant spin field (n–axis, Derbenev–Kondratenko vector)

A pre−established s−periodic unit vector field at each phase space point



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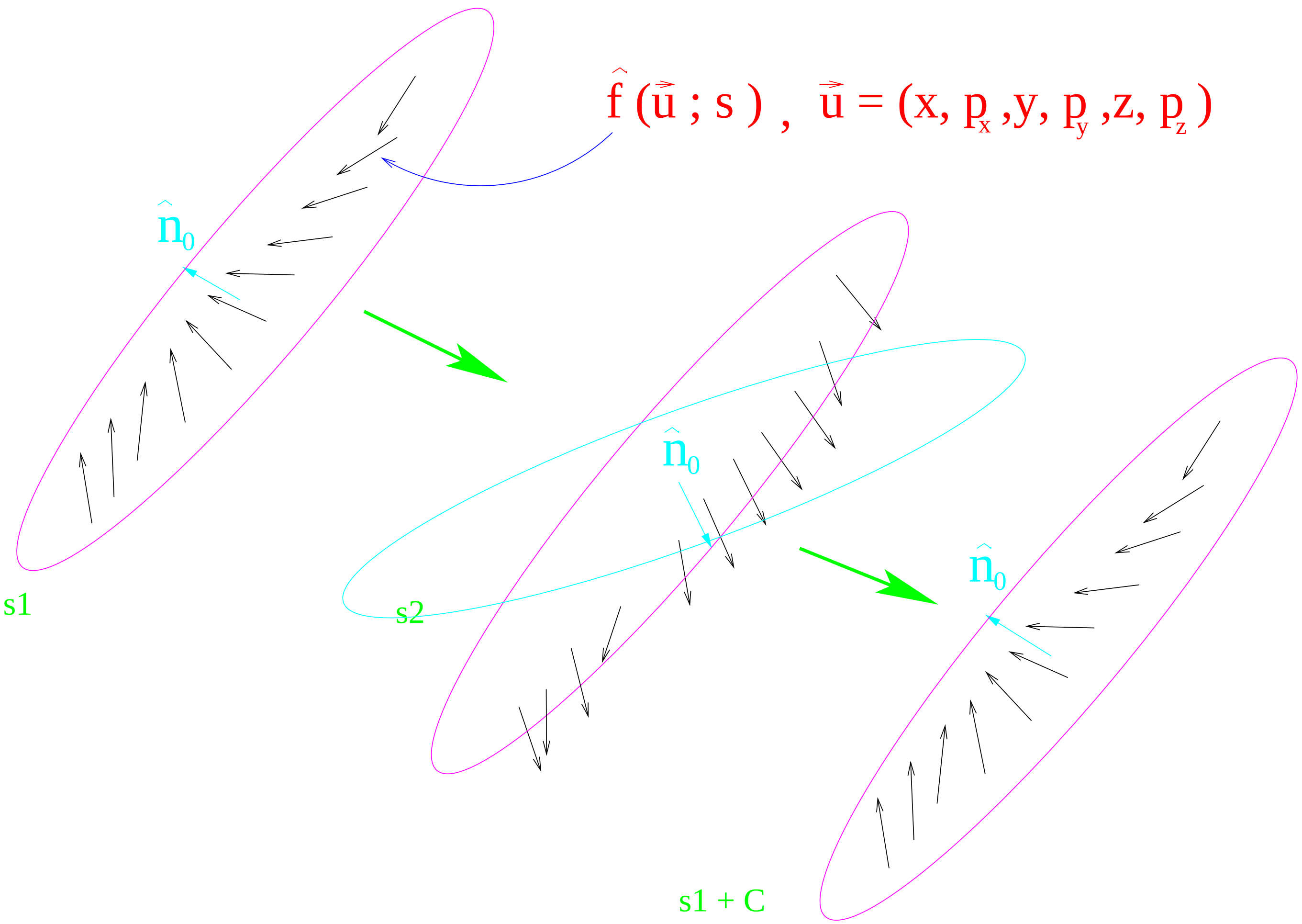

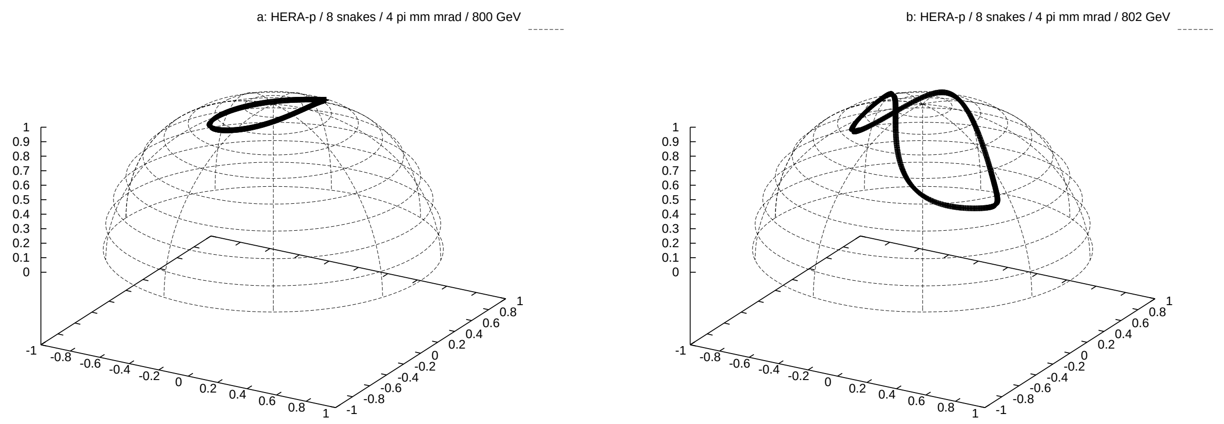

Figure 1: The $\hat{n}$–vector for the 4π mm mrad ellipse at 800 GeV (left) and 802 GeV (right).

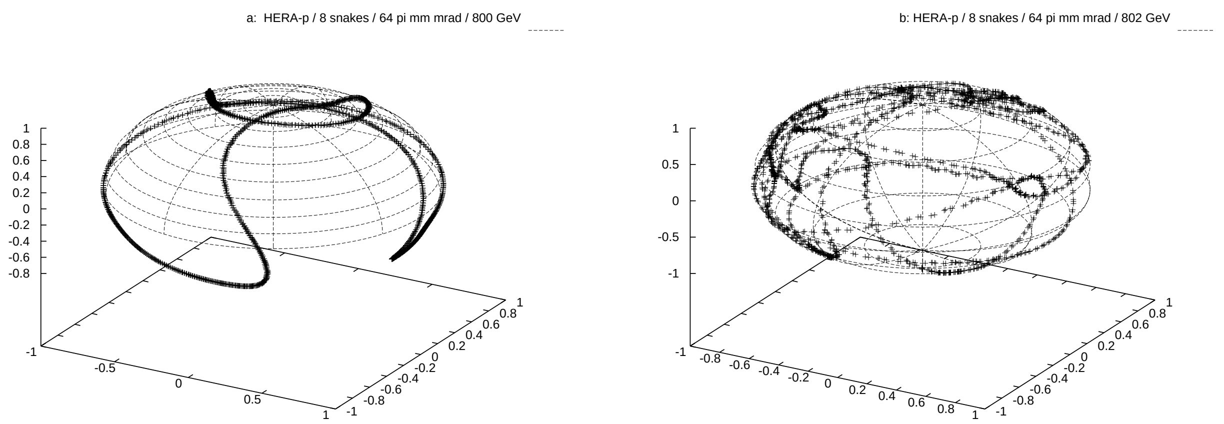

Figure 2: The $\hat{n}$–vector for the 64π mm mrad ellipse at 800 GeV (left) and 802 GeV (right).

The Invariant Spin Field, $\hat{n}$

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- $\vec{n}(M(u;s);s) = R_{3\times 3}(u;s)\vec{n}(u;s)$
  This is NOT the eigenproblem $\vec{N}(u;s) = R_{3\times 3}(u;s)\vec{N}(u;s)$
  $\hat{n}$ is NOT a "closed spin solution"!!!
  Instead, the field seen **AS A WHOLE** is invariant.

- On the closed orbit $\hat{n}(u;s) \longrightarrow \hat{n}(0;s) \equiv \hat{n}_0(s)$.
- ===> $\hat{n}$ and $\hat{n}_0(s)$ should not be confused!!!

- The invariant spin field for 1 plane of orbit motion is a smooth closed vector curve.

- For 3 planes of orbit motion $\hat{n}$ is on a smooth surface but is not closed.

The invariant spin field (ISF):

defines one axis of a local orthonormal coordinate system
at each point in phase space and azimuth for describing spin motion

— **Pre-established** at each s, u, γ_0 independently of the presence of particles or spins.

For protons: the invariant spin field defines the maximum attainable equilibrium polarisation.

$$\vec{P}_{eq}(\vec{J},\vec{\phi};s) = P(\vec{J})\ \hat{n}(\vec{J},\vec{\phi};s)$$

$$|\vec{P}_{meas}(s)| \quad = \quad |< P(\vec{J})\ \hat{n}(\vec{J},\vec{\phi};s) >_s| \quad \leq \quad |< \hat{n}(\vec{J},\vec{\phi};s) >_s|$$

Over one turn, the particles of an equilibrium phase space distribution replace each other, and spins set parallel to the local $\hat{n}$'s replace each other too.

Even if the spin field is very complicated: once in equilibrium, stay in equilibrium — but small $\vec{P}_{meas}$.

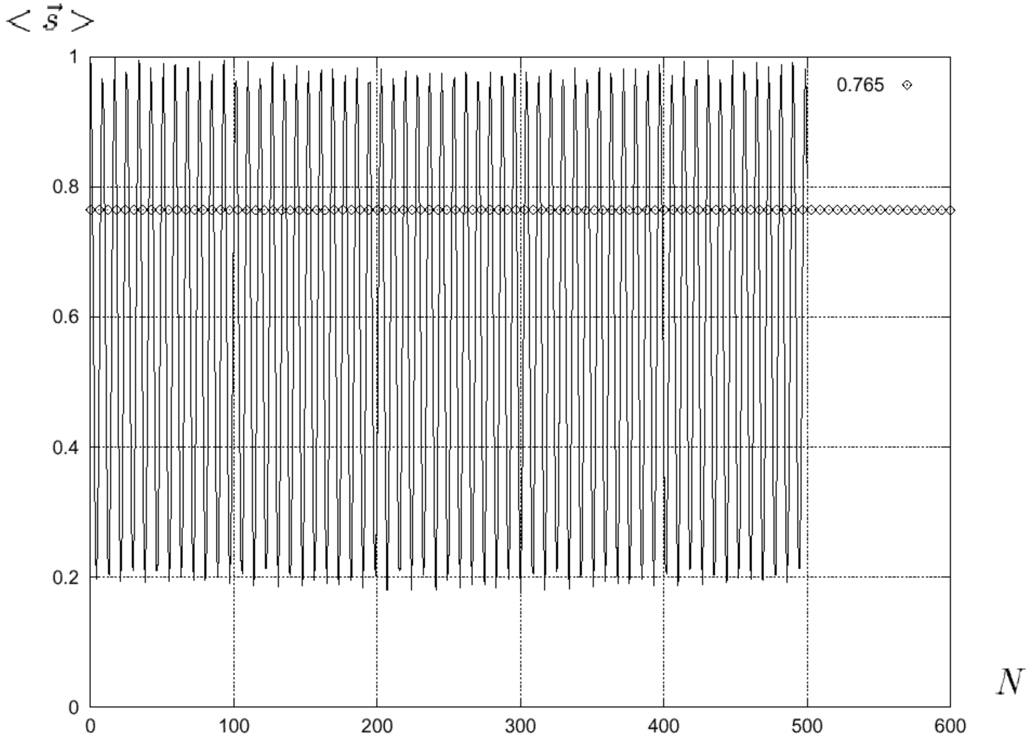


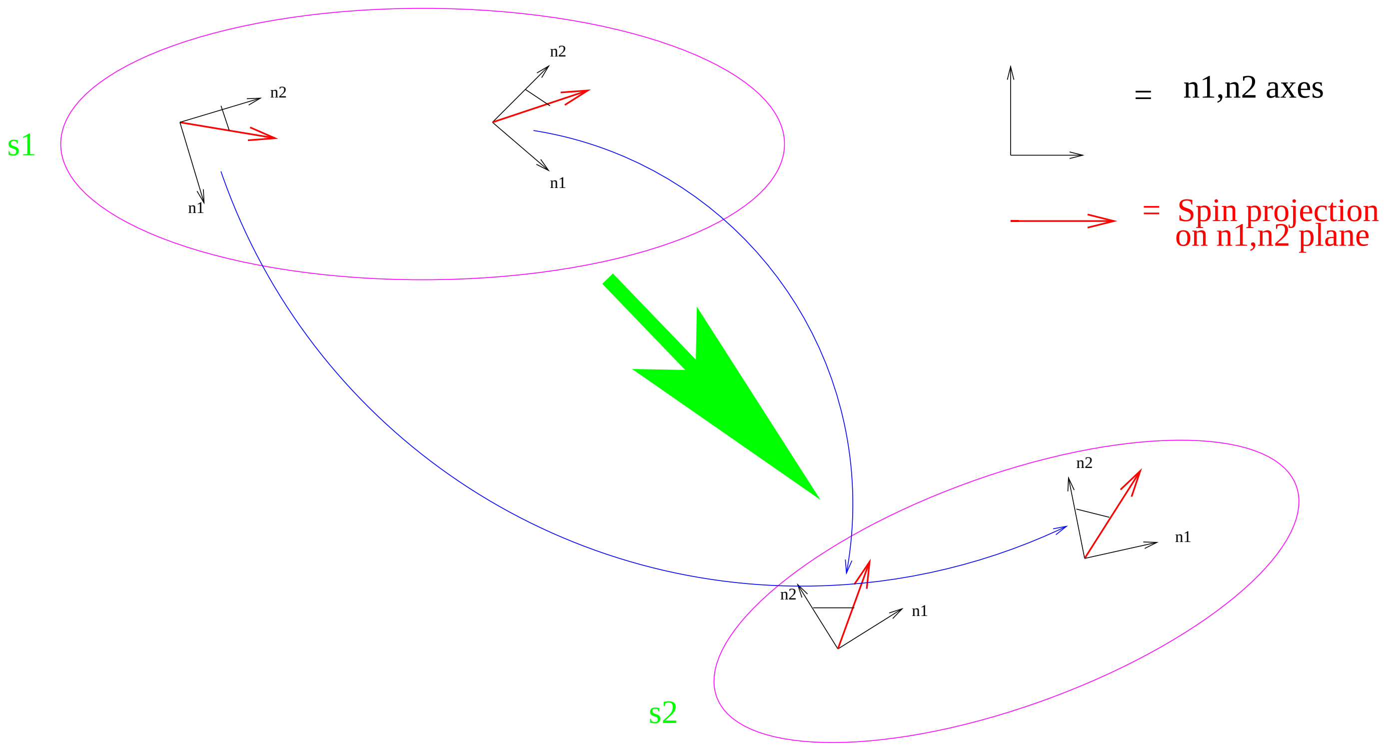
Figure 3: HERA protons at about 800 GeV: propagation of a beam that is initially completely polarised parallel to $\vec{n}_0$ leads to a fluctuating polarisation. For another beam in which the spins are initially parallel to their local $\vec{n}$ the polarisation stays constant, in this case equal to 0.765.

The stable spin direction?

- The ISF gives the stable POLARISATION directionSSSSSSSSSSSS.
- $\hat{n}_0$ gives the stable spin direction on the closed orbit.
  BUT THERE IS ONLY A TINY FRACTION OF PARTICLES ON OR NEAR THE CLOSED ORBIT!
- At very high energy
  $< \hat{n}(\vec{J}, \vec{\phi}; s) >_s$ and $< P(\vec{J})\ \hat{n}(\vec{J}, \vec{\phi}; s) >_s$ need not be parallel to $\hat{n}_0(s)$

The real spin tune: measures rate of precession around $\hat{n}$

Attaching coordinate axes to each phase space point



Spin precession rate w.r.t. n1, n2 is the same at all phase space points with same J_x , J_y , J_z .

Amplitude dependent spin tune! $\nu_{spin}(J)$

The real spin tune:

Not a single number, but an **equivalence class**

with elements related by "gauge transformations" of the **local** coordinate systems.

Without snakes, the real spin tune $\nu(\vec{J})$ does NOT oscillate with synchrotron motion: although $a\gamma$ does.

Spin–orbit resonance.

- Interleaved vertical and horizontal (quad and imperfection) fields.
- Rotations around different axes don't commute.
- If the spin and (linear) orbit motion are in resonance:

$\nu_{spin}(\vec{J}) = m + m_x \cdot Q_x + m_z \cdot Q_z + m_s \cdot Q_s$ ====> **CRAZY** spin field:

- High order resonances even for perfectly linear spin motion. (non–commutation).
- Two main groups of resonances:
  - Integer resonances due to motion along the **distorted** periodic orbit ===> strong tilt of $\hat{n}_0$ from ideal.
  - Synchro-beta ('intrinsic') resonances due to **synchro-beta oscillations** AROUND the distorted periodic orbit.
    ===> $|\hat{n}(u;s) - \hat{n}_0(s)|$ LARGE.

    ===> $|<\hat{n}(\vec{J},\vec{\phi};s)>_s|$ SMALL — geometry.

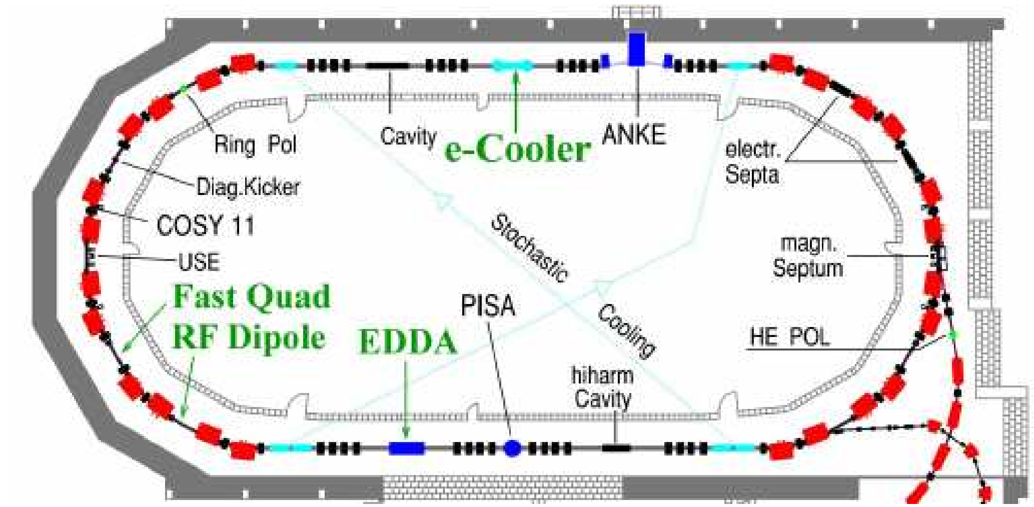
    e.g. $\approx 60^\circ$ ===> $P_{meas} \approx 0.5$!!!!

End of Preamble

COSY:

the proton/deuteron ring at Jülich

Resonance strengths – using the SLIM formalism



For students of the recent history of these matters:

Every cloud has a silver lining.

– English proverb

The SLIM/SLICK formalism

Linearised orbital and spin motion for first order analytical estimates of radiative depolarisation in electron storage rings, e.g., HERA, eRHIC, ELIC, ENC@FAIR, SuperB, LHeC.....

Attach an orthonormal 1-turn periodic coordinate system $\hat{l}(s), \hat{n}_0(s), \hat{m}(s)$ to the closed orbit.

$\hat{n}_0(s)$ obeys the BMT equation on the closed orbit – "the stable spin direction".

$$\vec{S} \approx \hat{n}_0(s) + \alpha\hat{m}(s) + \beta\hat{l}(s)$$

α, β: 2 small spin tilt angles — have subtracted out the big rotations!

$$\hat{\mathbf{M}}_{8\times 8} = \begin{pmatrix} \mathbf{M_{6\times 6}} & \mathbf{0_{6\times 2}} \\ \mathbf{G_{2\times 6}} & \mathbf{D_{2\times 2}} \end{pmatrix}$$

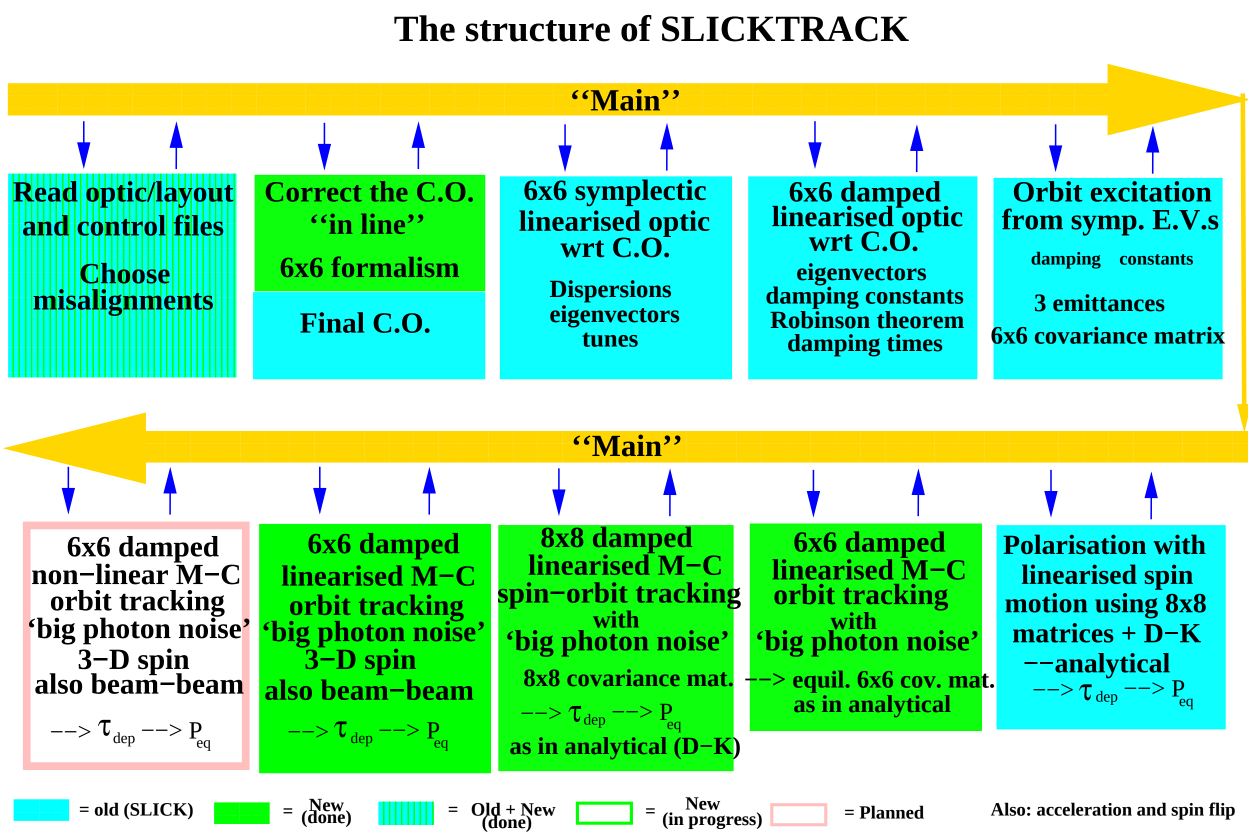
acting on $u = (x,\ p_x,\ y,\ p_y,\ z,\ p_z)$ and α, β

$\mathbf{G_{2\times 6}}$ represents the linearised solution of the T-BMT equation for α, β .

This is the **SLIM formalism**: originally A.W. Chao 1981 – working at DESY.

Theory and codes developed further by H. Mais and G. Ripken, D. Barber.

D. Barber: also a thick-lens version, SLICK and with Monte-Carlo extensions, SLICKTRACK.


The structure of SLICKTRACK
"Main"
Read optic/layout and control files
Choose misalignments
Correct the C.O. "in line"
6x6 formalism
Final C.O.
6x6 symplectic linearised optic wrt C.O.
Dispersions eigenvectors tunes
6x6 damped linearised optic wrt C.O.
eigenvectors damping constants Robinson theorem damping times
Orbit excitation from symp. E.V.s
damping constants
3 emittances
6x6 covariance matrix
"Main"
6x6 damped non−linear M−C orbit tracking 'big photon noise' 3−D spin also beam−beam
--> τ dep --> P eq
6x6 damped linearised M−C orbit tracking 'big photon noise' 3−D spin also beam−beam
--> τ dep --> P eq
8x8 damped linearised M−C spin−orbit tracking with 'big photon noise'
8x8 covariance mat.
--> τ dep --> P eq
as in analytical (D−K)
6x6 damped linearised M−C orbit tracking with 'big photon noise'
--> equil. 6x6 cov. mat. as in analytical
Polarisation with linearised spin motion using 8x8 matrices + D−K −−analytical
--> τ dep --> P eq
= old (SLICK)
= New (done)
= Old + New (done)
= New (in progress)
= Planned
Also: acceleration and spin flip

Using $G_{2\times 6}$ to get $\epsilon^{\mathrm{tot}}, \epsilon^{\mathrm{rfd}}$ etc

See for example:

- G.H. Hoffstaetter, *"High Energy Polarised Proton Beams: a Modern View"*, Springer Tract in Modern Physics, Vol 218 (2006).
- M. Vogt, PhD Thesis, University of Hamburg, Germany, DESY-THESIS-2000-054 (2000): http://www-library.desy.de/preparch/desy/thesis/desy-thesis-00-054.pdf
- D.P. Barber and G. Ripken in *"Handbook of Accelerator Physics and Engineering"*, Eds: A.W. Chao and M. Tigner, World Scientific, 3rd printing (2006).

Just need the 1-turn $G_{2\times 6}$ for free synchro-betatron motion and with the RFD, a trivial extension for the forced motion + a term for the RFD itself.

For example, for free synchro-betatron motion:

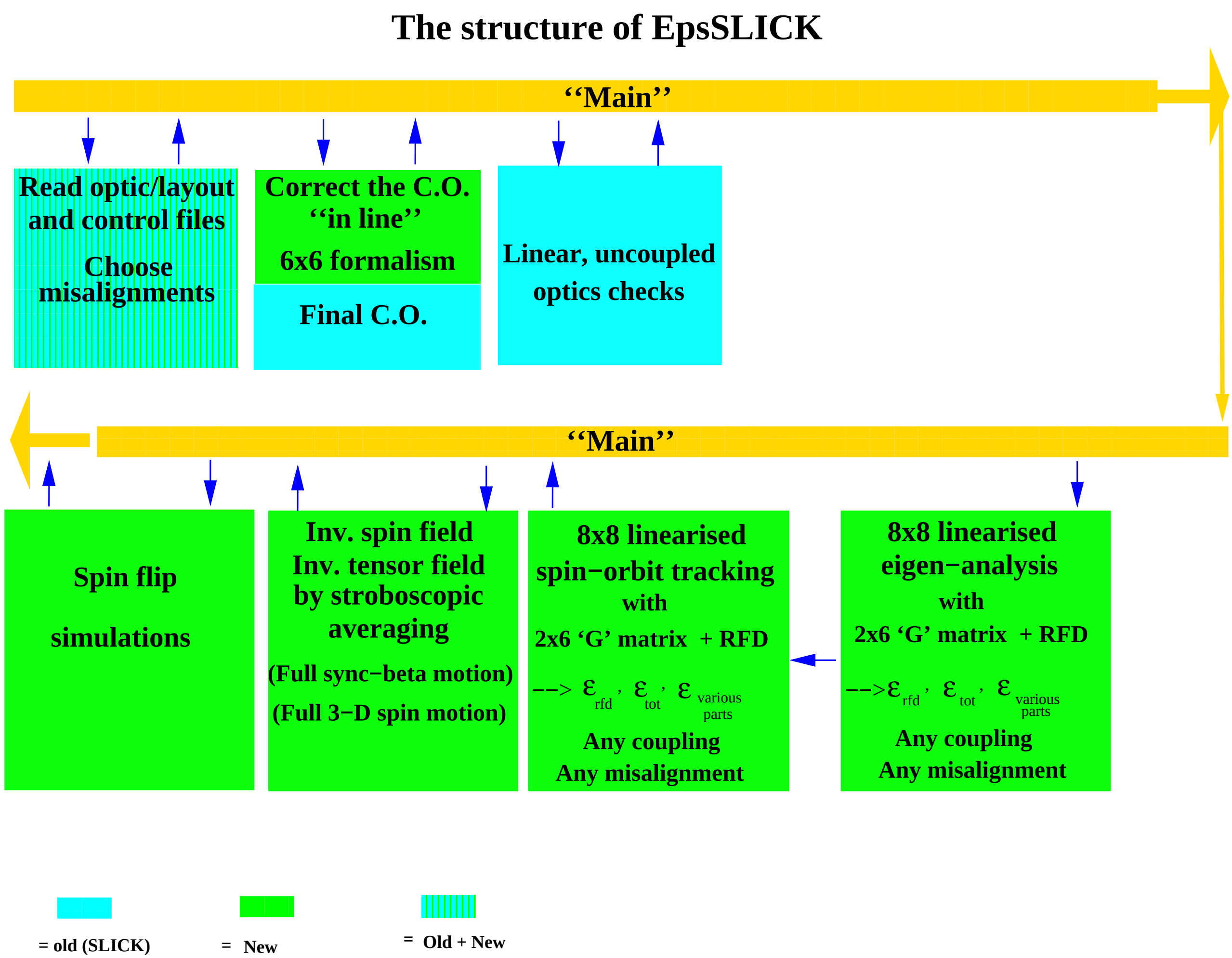
$$\epsilon_{\kappa}^{\mathrm{sb}} = \frac{1}{2\pi}(1 \quad i)\,\bar{\bar{G}}(C,0)\sqrt{2J_k}e^{-i\,\phi_k}\,\check{v}_k(0)$$

Or do long term tracking and averaging to get the Fourier integral.

Any ring geometry, any misalignment, any linear coupling.
Full 6-D orbital motion!

No need for obscene contortions!

Getting ϵ by tracking or eigenanalysis: including the (inhomogeneous) rf dipole with the matrix $G_{2\times 6}$

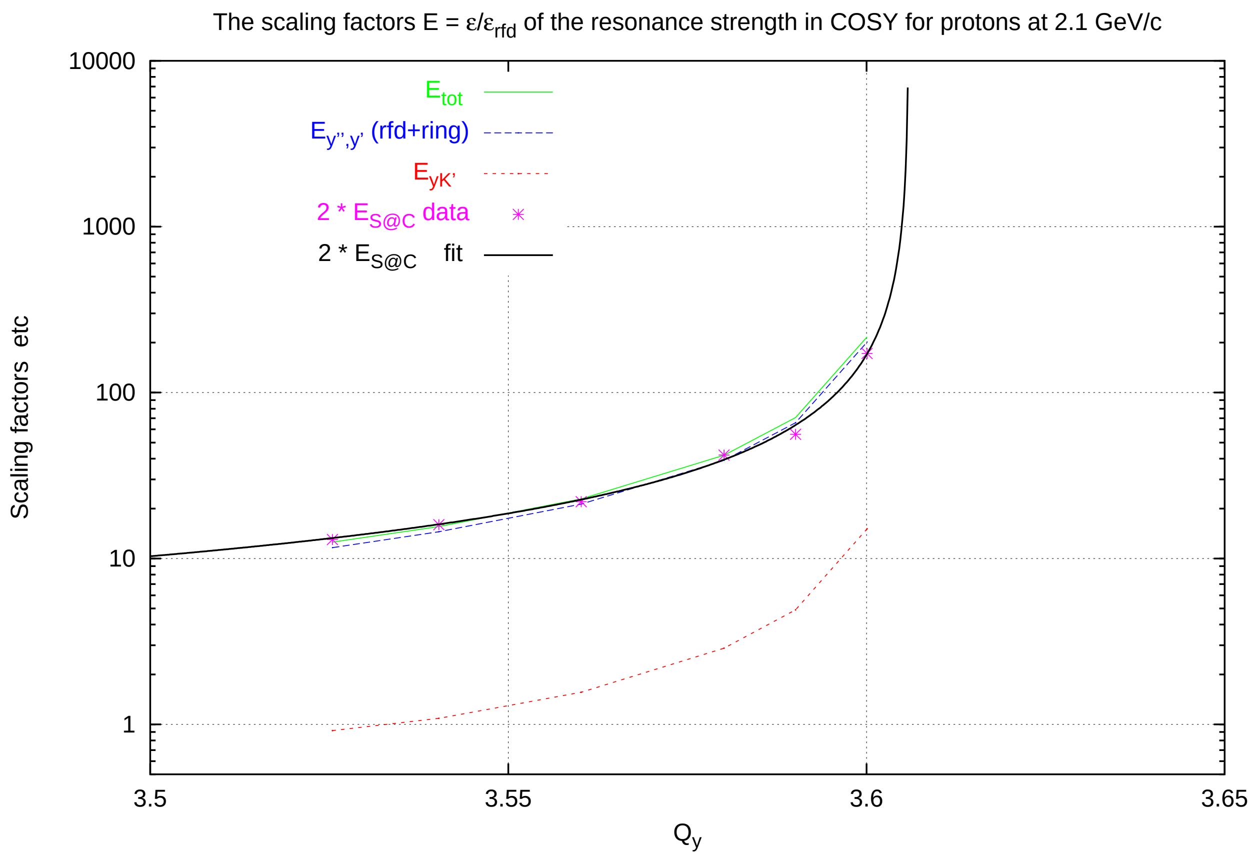


Diagnostics!

Switch spin-orbit coupling off/on to see what does what.

For example:
Check that ϵ^{rfd} comes out correctly.
Study contributions from y'', y', y.
Getting the relative signs right.

Protons



The values for $E_{y''}(\mathrm{rfd}+\mathrm{ring})$ confirm earlier preliminary results (2006!) from A. Lehrach.

The values for $E_{y'',y'}(\mathrm{rfd}+\mathrm{ring})$ are confirmed by M. Vogt with the code SPRINT.

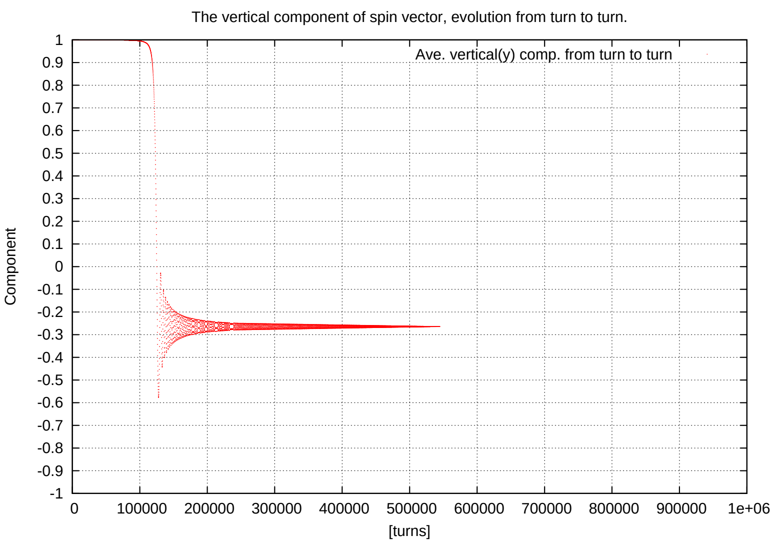
Nothing unexpected – just well known effects. Subject closed.

Spin flip simulations:

Extend the 8x8 formalism of basic SLICK to handle full 3-D spin motion. A perfect formalism for such problems.

$$\frac{S_y^{\rm final}}{|\vec{S}|} = 2\exp(-\Xi) - 1 = 2\exp\left(-\frac{\pi|\epsilon_\kappa^{\rm v}|^2}{2\alpha_{\rm FS}}\right) - 1\,, \quad \alpha_{\rm FS} = \frac{d\delta}{d\theta}$$

± 4 kHz $\equiv \pm 57|\epsilon_\kappa^{\rm rf}|$. Spins initially parallel to ISF....i.e vertical.
Run down the RF amplitude at the end. $\Xi = 1 \Longrightarrow S_{n_0}^{\rm final} = -0.264$.



Now look properly

beyond the naive SRM:

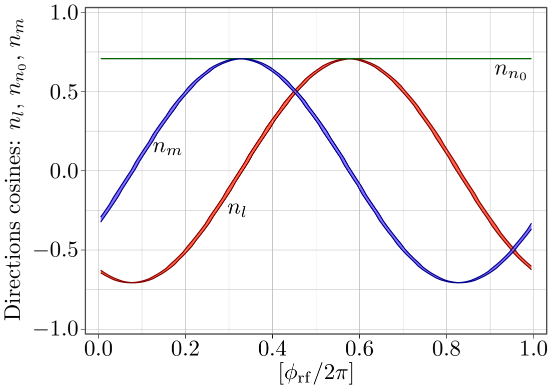
$$\hat{n}^{\rm rf}(\phi_{\rm rf}) = (\mathrm{sgn}\,\delta^{\rm rf})\frac{\left(\delta^{\rm rf}\,\hat{n}_0(s) + |\epsilon_\kappa^{\rm rf}|\,(\hat{l}(s)\cos\breve{\phi}_{\rm rf} + \hat{m}(s)\sin\breve{\phi}_{\rm rf})\right)}{\Lambda^{\rm rf}}\,,$$

where $\delta^{\rm rf} = \nu_0 - \kappa^{\rm rf}$, $\Lambda^{\rm rf} = \sqrt{(\delta^{\rm rf})^2 + |\epsilon_\kappa^{\rm rf}|^2}$ and $\breve{\phi}_{\rm rf} = \kappa^{\rm rf}\theta - \phi_\epsilon - \pi$ where ϕ_ϵ is the phase of $\epsilon_\kappa^{\rm rf}$.

$|\epsilon_\kappa^{\rm rf}| = 47.2\times 10^{-6}, \quad \kappa^{\rm rf} = 5 - Q_{\rm rf}, \quad Q_{\rm rf} \approx 0.605$

The $\hat{l}(s)$ and $\hat{m}(s)$ are chosen so that spins on the CO precess at the uniform rate ν_0 in the $(\hat{l}(s),\ \hat{n}_0(s),\ \hat{m}(s))$ frame — NOT using $(\hat{x},\ \hat{y},\ \hat{s}) \Longrightarrow$ any ring geometry.

Protons, just before the RFD. RFD + forced solution. $\delta^{\rm rf} = \pm|\epsilon_\kappa^{\rm rf}|$



– using Stroboscopic Averaging.

Include large-amplitude pre-existing vertical betatron motion.
— off spin-orbit resonance.

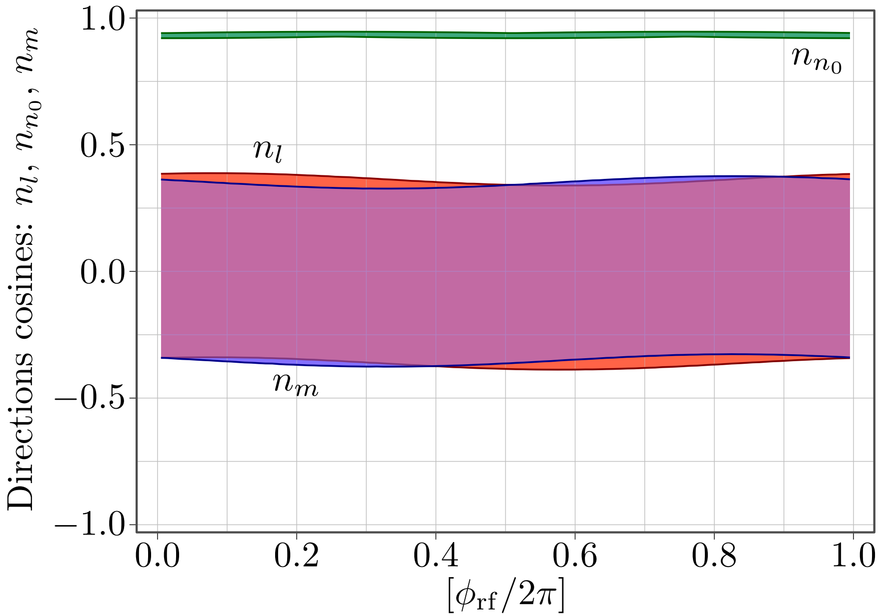
- No misalignments
- 2.1 GeV/c $\Longrightarrow \nu_0 = a\gamma_0 = 4.395$
- $Q_y = 3.580, \quad Q_x = 3.543$
- $\kappa^{\mathrm{v}} = 8 - Q_y \Longrightarrow \delta^{\mathrm{v}} = \nu_0 + Q_y - 8 = -0.02490...$
- Before cooling, the $3-\sigma$ vertical emittance for protons at 2.1 GeV/c is $2\pi - 4\pi$ mm.mrad.

  For 4π mm.mrad: $\beta_y^{\mathrm{max}} \approx 32$ metre $\Longrightarrow y_{\mathrm{max}} \approx 11$ mm.
- For this simulation, sit ON the 18π mm.mrad ellipse and imagine that the motion is still linear.

  Then $y_{\mathrm{max}} \approx 24$ mm:

  Relax! it's only a simulation!

Then:



– using Stroboscopic Averaging.

What's happening?

The "ground state" has shifted!!! and so has the spin tune – and this is not just naive interference.

The uniform Invariant Frame Field

- $\hat{n}_1(u,s), \quad \hat{n}(u,s), \quad \hat{n}_2(u,s)$ with $\hat{n}_1(u,s+C)=\hat{n}_1(u,s)$ and $\hat{n}_2(u,s+C)=\hat{n}_2(u,s)$
- – a local coordinate system attached to each (u,s) to appraise spin motion as particles flow through phase space. No History. Generalises $\hat{l}(s), \hat{n}_0(s), \hat{m}(s)$
- $\hat{n}_1(u,s)$ and $\hat{n}_2(u,s)$ are chosen so that spins precess at a uniform rate.

The proper definition of spin tune and spin-orbit resonance

See "Quasiperiodic spin–orbit motion and spin tunes in storage rings"
Barber, Ellison and Heinemann, PRST-AB, **7**, 124002 (2004).

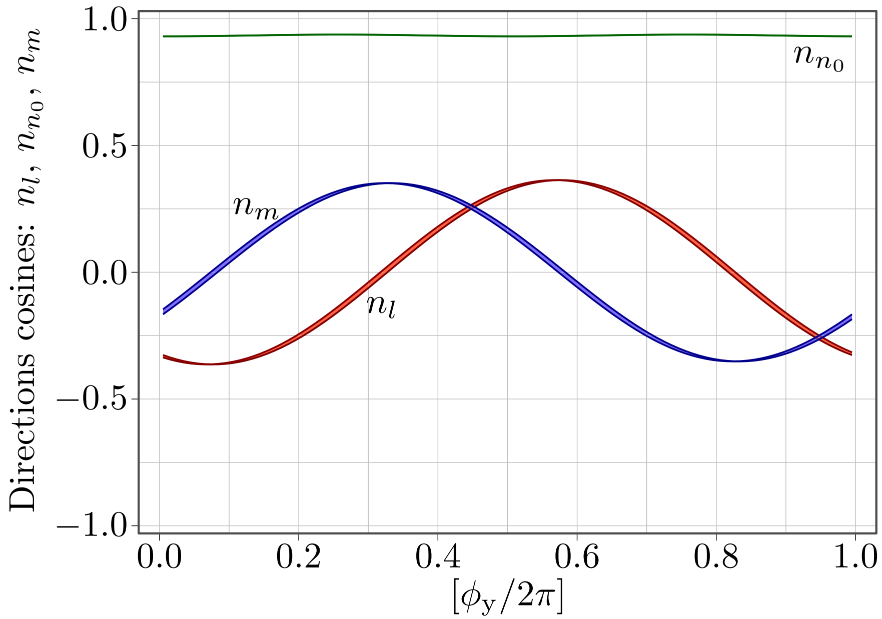
- An ADST $\nu(J)$ describes the rate of uniform spin precession around $\hat{n}$ in a u-IFF. $(\hat{n}_1(u,s),\ \hat{n}(u,s),\ \hat{n}_2(u,s))$. where J is the list of 3 amplitudes or, if an external RF field is applied, 4 amplitudes.
- The transform $\hat{n}_1 \pm i\hat{n}_2 \Longrightarrow (\hat{n}_1 \pm i\hat{n}_2)\exp\{(j_0 + j \cdot Q)\theta\}$ with $(j_0, j) \in \mathbb{Z} \times \mathbb{Z}^3$ causes the change $\nu(J) \Longrightarrow \pm\nu(J) + j_0 + j \cdot Q$
- This motivates the definition of an equivalence relation where ν_1 and ν_2 in $[0, 1)$ are said to be equivalent - and we write $\nu_1 \sim \nu_2$ - iff there exist $(\varepsilon, j_0, j) \in \{-1, 1\} \times \mathbb{Z} \times \mathbb{Z}^3$ such that $\nu_2 = \varepsilon\nu_1 + j_0 + j \cdot Q$.
- $\Longrightarrow$ The ADST is an equivalence class! – countably infinite in size.
- The system is on spin-orbit resonance if the equivalence class contains 0, i.e. if some $\nu(J) = j_0 + j \cdot Q$.
- The condition $\nu_0 = j_0 + j \cdot Q$ is NOT a spin-orbit resonance condition but the system will be near resonance if some $\nu(J) \approx \nu_0$.
- Systems tend to avoid exact resonance! – see the SRM
- The ADST does **not exist** on orbital resonance.

The preferred member of the equivalence class

- The equivalence class is countably infinite
- The member $\nu(J)$ which $\to \pm\nu_0$ as its u-IFF reduces smoothly to $(\hat{l}(s),\ \hat{n}_0(s),\ \hat{m}(s))$
  – if such a member exists.
- $\Longrightarrow$ these $(\hat{n}_1(u,s),\ \hat{n}_2(u,s))$ are the generalisation of $(\hat{l}(s),\ \hat{m}(s))$
- The new reference spin tune is the preferred member of the ADST of the **new ground state**!!

The spin motion for the pre-existing vertical betatron motion?

Just before the RFD



– using Stroboscopic Averaging.

This looks like an SRM – expected from experience.

The SRM for vertical betatron motion

$$\hat{n}^{\mathrm{v}}(s,\phi_\kappa) = (\mathrm{sgn}\,\delta^{\mathrm{v}})\frac{\Big(\delta^{\mathrm{v}}\hat{n}_0(s) + |\epsilon^{\mathrm{v}}_\kappa|\,(\hat{l}(s)\cos\breve{\phi}_\kappa + \hat{m}(s)\sin\breve{\phi}_\kappa)\Big)}{\Lambda^{\mathrm{v}}}$$

where $\Lambda^{\mathrm{v}} = \sqrt{(\delta^{\mathrm{v}})^2 + |\epsilon^{\mathrm{v}}_\kappa|^2}$ with $\delta^{\mathrm{v}} = \nu_0 - \kappa^{\mathrm{v}}$ and $\breve{\phi}_\kappa = \theta\kappa^{\mathrm{v}} - \phi_\epsilon + \pi$ where ϕ_ϵ is the phase of $\epsilon^{\mathrm{v}}_\kappa$.

- With $\delta^{\mathrm{v}} = -0.02490$ and $\hat{n}^{\mathrm{v}}_{n_0} = 0.934$, $|\epsilon^{\mathrm{v}}_\kappa|$ should be 0.009582.
- This agrees exactly with the value from the Fourier integral
- $\hat{n}^{\mathrm{v}}_{n_0} = 1/\sqrt{2}$ when $\delta^{\mathrm{v}} = \pm|\epsilon^{\mathrm{v}}_\kappa|$ etc etc.

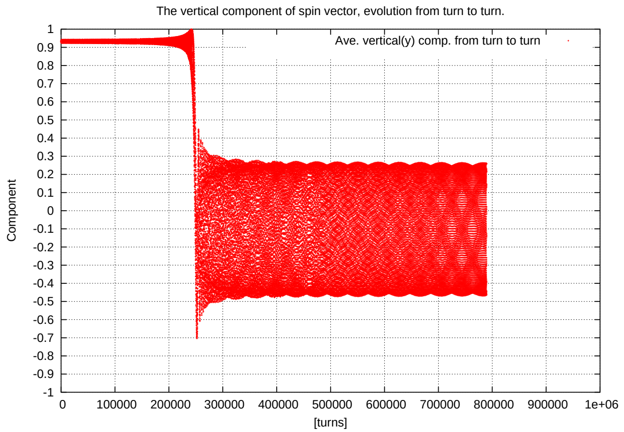
$\Longrightarrow$ spin motion for the pre-existing vertical betatron motion is well described by an SRM — expected!

The preferred member of the ADST for the SRM

The preferred member of the ADST for the SRM is

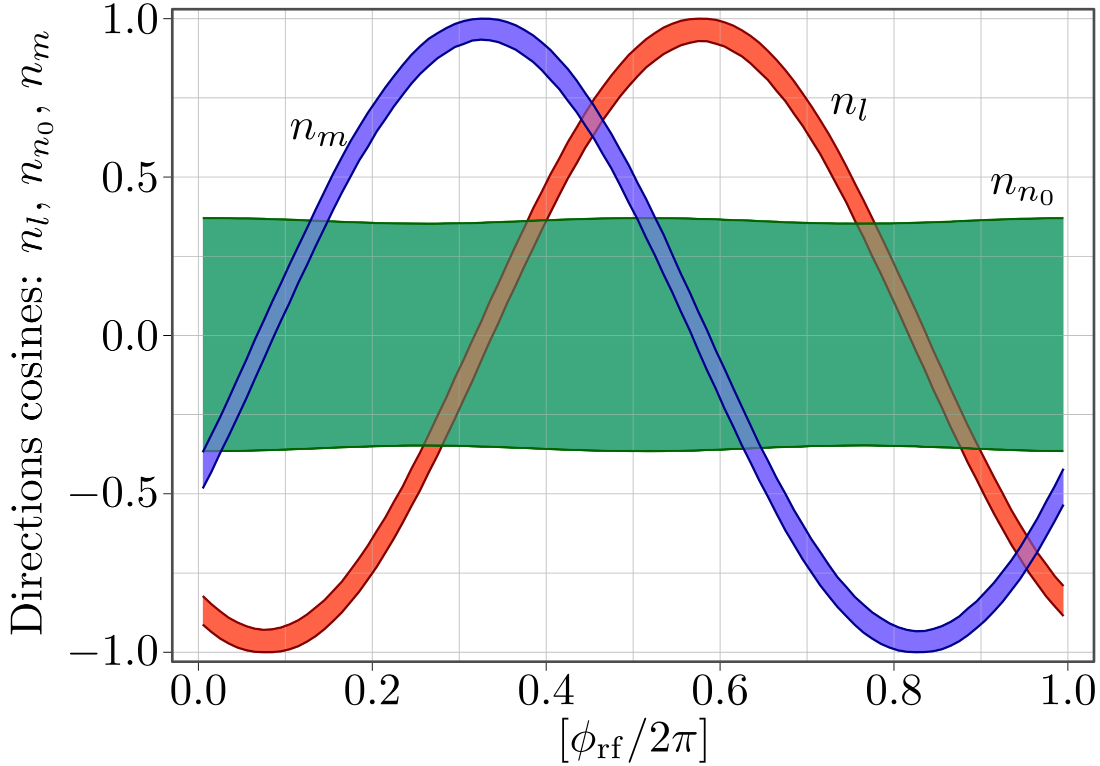
$\nu(J_y) = -\Lambda^{\mathrm{v}} + \kappa^{\mathrm{v}} \Longrightarrow \nu(J_y) - \nu_0 = -\Lambda^{\mathrm{v}} - \delta^{\mathrm{v}} = -0.001779...$ The shift is negative!

- The full $\nu_0 = 4.39498$ at 2.1 GeV/c
- The shift represents about 2.6 kHz in the original scan range of ± 4 kHz!
  $\Longrightarrow$ Chao's "matrix formalism" would not be too relevant.
- **Spectral analysis** of spin motion with **Theorem 9.1.c in BEH (2004)**:
  Lines at $j_\nu \nu(J_y) + j_0 + j_y Q_y : j_\nu \in \{0, 1, -1\}$ The line with $[\nu(J_y)] = 0.393201....$ is prominent
- The Q_{rf} at which spins flip in a Froissart-Stora scan is shifted **upwards** by 0.001779



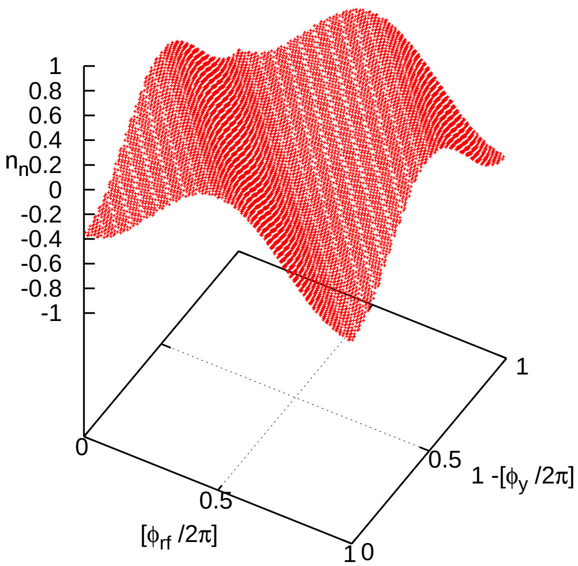
$\Xi = 1$ with $\vec{S}$ initially parallel to $\hat{n}^{\mathrm{v}}$: final average -0.11 instead of -0.264.

Everything is consistent, and with the RFD running such that $\tilde{\delta}^{\mathrm{rf}} := \nu(J_y) - \kappa^{\mathrm{rf}} = 0$ we get



– using Stroboscopic Averaging.

The $\hat{n}_0$ component of the full ISF

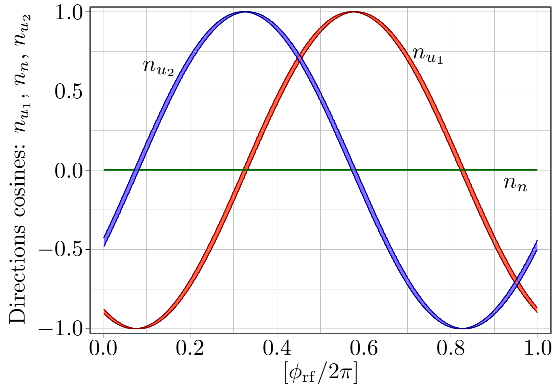


A **surface** so that the ISF is a single valued function of the phases $\phi_{\rm rf}$ and ϕ_y as required.

A very simple form, being very well parametrised by the function

$$\hat{n}_{n_0}(\phi_{\rm rf}, \phi_y) = (|\epsilon_\kappa^{\rm v}|/\Lambda^{\rm v})\ \cos\{[(\phi_{\rm rf} - \phi_y)/2\pi]\}$$

Now view within the u-IFF of the pre-existing vertical betatron motion



An SRM within an SRM!

$$\hat{n}^{\rm full}(\phi_{\rm v},\phi_{\rm rf}) = ({\rm sgn}\,\tilde{\delta}^{\rm rf})\frac{\left(\tilde{\delta}^{\rm rf}\hat{n}^{\rm v} + |\tilde{\epsilon}_{\kappa}^{\rm rf}|(\hat{u}_1^{\rm v}\cos\breve{\phi}_{\rm rf} + \hat{u}_2^{\rm v}\sin\breve{\phi}_{\rm rf})\right)}{\tilde{\Lambda}^{\rm rf}}\,,$$

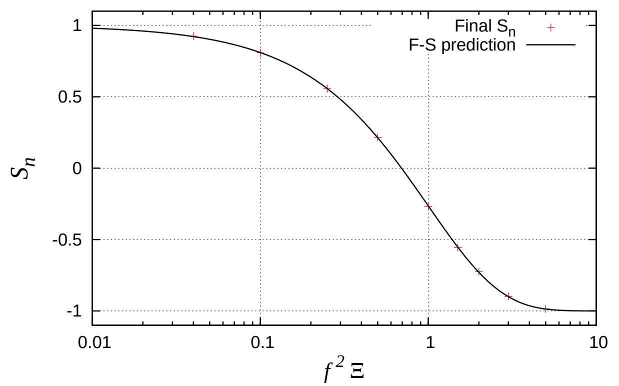
$\tilde{\epsilon}_{\kappa}^{\rm rf}$ is the resonance strength in the u-IFF, and $\tilde{\Lambda}^{\rm rf} = \sqrt{(\tilde{\delta}^{\rm rf})^2 + |\tilde{\epsilon}_{\kappa}^{\rm rf}|^2}$ and $\tilde{\delta}^{\rm rf} = \nu(J) - \kappa^{\rm rf}$

What is $|\tilde{\epsilon}_{\kappa}^{\rm rf}|$???

We naively expect:
$|\tilde{\epsilon}_{\kappa}^{\rm rf}| \leq |\epsilon_{\kappa}^{\rm rf}|$ since $\hat{n}^{\rm v}$ is tilted from the vertical while the perturbing fields are horizontal.

Three tests

- With $|\tilde\delta^{\rm rf}| = |\tilde\epsilon^{\rm rf}_\kappa| \equiv f|\epsilon^{\rm rf}_\kappa| = 0.901\,|\epsilon^{\rm rf}_\kappa| \longrightarrow \hat n^{\rm full}\cdot\hat n^{\rm v} = 1/\sqrt{2}$ etc etc
- The Froissart-Stora formula describes the level of adiabatic invariance of $\vec S\cdot\hat n$
  A scan with $\vec S$ initially parallel to $\hat n^{\rm v}$ and observing the final $\hat n^{\rm full}\cdot\hat n^{\rm v}$ i.e. a F-S scan within the u-IFF. For $f = 0.901$



  A scan with $f = 1.0$ gives a bad fit.
- With the RFD **included** at various $Q_{\rm rf}$ the "preferred member of the ADST" in the **u-IFF** (instead of in $(\hat l,\ \hat n_0,\ \hat m)$ frame) should be:

$$\nu^{\rm full} = ({\rm sgn}\,\tilde\delta^{\rm rf})\tilde\Lambda^{\rm rf} + \kappa^{\rm rf} \quad = \quad ({\rm sgn}\,\tilde\delta^{\rm rf})\tilde\Lambda^{\rm rf} + \nu(J_y) - \tilde\delta^{\rm rf}\,.$$

Spectral analysis of spin motion including the RFD:
With **Theorem 9.1.c in BEH (2004)**, the observed spectra require that $|\tilde\epsilon^{\rm rf}_\kappa| = 0.901\,|\epsilon^{\rm rf}_\kappa|$

Now some fun with spinors

The Pauli matrices and spinors (NOT wavefunctions!):

$$\sigma_l = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{n_0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{S} = \Psi^\dagger \vec{\sigma} \Psi$$

With two dominant harmonics in $\vec{\omega}(u,s) \cdot (\hat{l}(s) - i\hat{m}(s))$:

$$\frac{d\Psi}{d\theta} = -\frac{i}{2} \begin{pmatrix} -\nu_0 & -i(\epsilon_\kappa^{\mathrm{v}})^* \exp(i\kappa^{\mathrm{v}}\theta) - i(\epsilon_\kappa^{\mathrm{rf}})^* \exp(i\kappa^{\mathrm{rf}}\theta) \\ i\epsilon_\kappa^{\mathrm{v}} \exp(-i\kappa^{\mathrm{v}}\theta) + i\epsilon_\kappa^{\mathrm{rf}} \exp(-i\kappa^{\mathrm{rf}}\theta) & \nu_0 \end{pmatrix} \Psi$$

Then with a sequence of unitary transformations into the u-IFF (Maple):

$$\frac{d\Psi_{\mathrm{uiff}}}{d\theta} = -\frac{i}{2}(\mathfrak{M}_{\mathrm{res}} + \mathfrak{D} + \mathfrak{N})\Psi_{\mathrm{uiff}} \,,$$

$$\mathfrak{M}_{\mathrm{res}} = \begin{pmatrix} -\nu(J_y) & -i(\mathcal{E}_\kappa^{\mathrm{rf}})^* \exp(i\kappa^{\mathrm{rf}}\theta) \\ i\mathcal{E}_\kappa^{\mathrm{rf}} \exp(-i\kappa^{\mathrm{rf}}\theta) & \nu(J_y) \end{pmatrix},$$

with $(\mathfrak{D}, \mathfrak{N})$ oscillating but far off resonance and with $(\mathfrak{D}, \mathfrak{N}) \to 0$ as the component $\hat{n}_{n_0}^{\mathrm{v}} \to 1$

$$|\mathcal{E}_\kappa^{\mathrm{rf}}| = \frac{(1 + \hat{n}_{n_0}^{\mathrm{v}})}{2} |\epsilon_\kappa^{\mathrm{rf}}|(\tilde{\delta}=0) \;=\; 0.967 \times |\epsilon_\kappa^{\mathrm{rf}}|(\tilde{\delta}=0) \text{ for } \hat{n}_{n_0}^{\mathrm{v}} = 0.934$$

BUT, $|\epsilon_\kappa^{\mathrm{rf}}|(\tilde{\delta}=0) = 0.930 \times |\epsilon_\kappa^{\mathrm{rf}}|(\delta=0) \implies |\tilde{\epsilon}_\kappa^{\mathrm{rf}}| = |\mathcal{E}_\kappa^{\mathrm{rf}}| = 0.899 \times |\epsilon_\kappa^{\mathrm{rf}}|(\delta=0) \implies f = 0.899$!!!

...and some serious stuff with COSY

- If spin motion for pre-existing vertical betatron motion is described by an SRM and $|\epsilon_\kappa^{\mathrm{v}}|$ is known, we know the preferred member of the ADST.
- At fixed $|\epsilon_\kappa^{\mathrm{v}}|$: $|\nu(J_y) - \nu_0| = |-\Lambda^{\mathrm{v}} - \delta^{\mathrm{v}}|$ **increases** as $|\delta^{\mathrm{v}}|$ **decreases**.
- Measurements of resonance strengths for protons were made at $Q_y = 3.6 \Longrightarrow \delta^{\mathrm{v}} = -0.0050$.
- Then if $|\epsilon_\kappa^{\mathrm{v}}|$ is insensitive to Q_y, $\nu(J_y) - \nu_0 = -0.00158$ ON the ellipse with 3.6 mm.mrad. $\Longrightarrow y^{\max} \approx 10.7$ mm.
- Two parameters: the preferred member of the ADST for the pre-existing vertical betatron motion $\nu(J)$ and the attenuation f.

Suggestion

- Run at $Q_y = 3.6$ and 2.1 GeV/c with polarised protons.
- Cool the beam to get a $3{-}\sigma$ vertical emittance of ≈ 0.3 mm.mrad.
- Use a vertical kicker to put the beam onto ellipses in the range $0 - 3.6$ mm.mrad.
- Use the field of the RF solenoid as a **probe** to measure $\nu(J_y) - \nu_0$ by measuring the RF tune needed to get zero time averaged vertical polarisation in each case and check the dependence of $\nu(J_y) - \nu_0$ on J_y against the expectation.
- Or run at $Q_y = 3.58$ and $a\gamma_0 = 4.4148....$

Why?

(1) Because it's never been done.

(2) Because the literature on spin tune shifts contains too much mumbo-jumbo.

(3) Because COSY is perfectly placed and has a very experienced crew.

(4) Because people wanting to measure beam energies very precisely might want to have a better understanding of the systematics. See (2).

(5) Etc.....

Questions:

- Is it feasible to put the beam on the ellipse? See DESY 09-15 – in preparation.
- Can the energy spread be kept low enough? Orbit length?
- How long does the beam stay on the ellipse (non-linear motion, scattering in the polarimeter, IBS, space charge, image forces....)?
- How can the phase space distribution be measured for checking that the particles are on/off the ellipse?
- If it's only a short time, is it feasible to set up an injection/measurement routine that can deliver enough precision?
- Etc.

$\Longrightarrow$ Homework for the COSY crew.

In simple rings the shifts might be bigger at higher energies – we need large $|\epsilon^{\mathrm{v}}_{\kappa}|$

Summary

- Going beyond the common perception that non-spin-resonant background vertical betatron motion is not too interesting.
- With appropriate computational tools:
  - Stroboscopic Averaging to get the ISF in general situations without models,
  - Efficient spin-orbit tracking
  - Spectral analysis
- and clearly defined mathematical concepts with theorems (rigour). E.g. the proper definition of the ADST based on conventional mathematics!

$\Longrightarrow$ **Efficient analysis.**

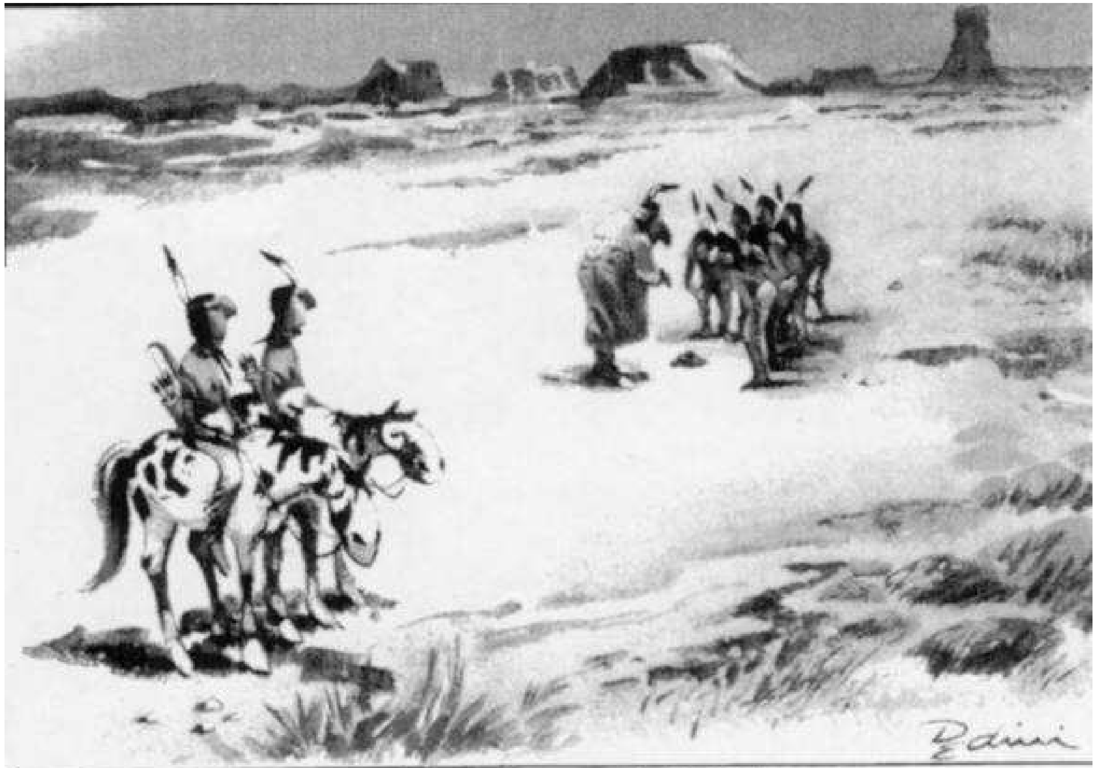
- The resonance strength for the RFD **does** depend on the coordinate system

AND

$\Longrightarrow$ A suggestion for a test using COSY, a perfect facility with its experts.

$\Longrightarrow$ A chance to clean up this business of spin tune shifts.
Confusions, ignorance and laziness about such shifts in the literature.

More information in DESY 09-15 – in preparation.



"When you get too old to hunt, you teach a course in how to read buffalo shit."

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(Eldon Dedini: The Tracker Magazine 1985.)