Simulations of Coherent Synchrotron Radiation and Wavelet Methodology

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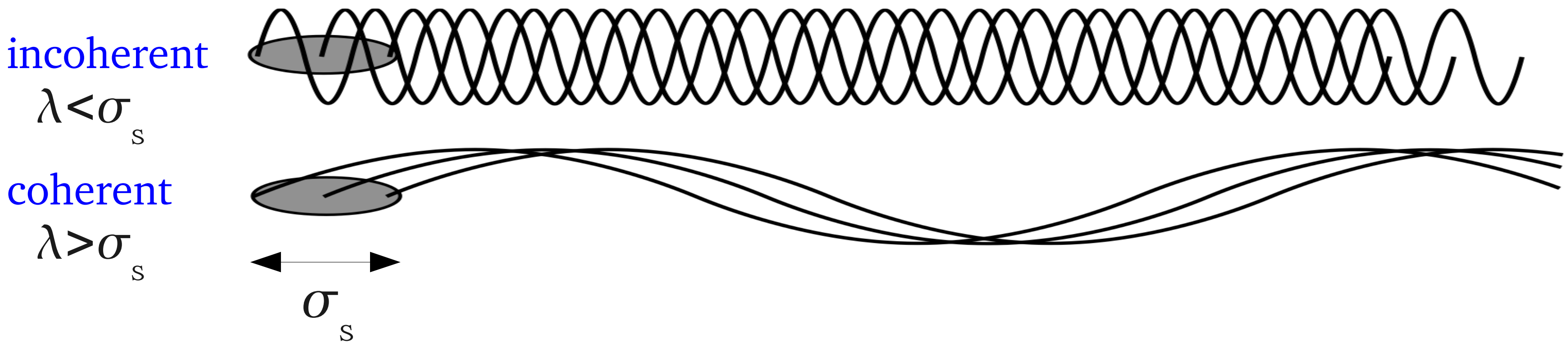
Outline of the Talk

- Coherent Synchrotron Radiation (CSR):
  - Physical problem
  - Mathematical problem
  - Computational problem
    - Two approaches: point-to-point (P2P) and mean field (MF)
    - We present reasons why we choose do develop a MF code from an existing P2P code designed by Rui Li
    - Demand for increased sensitivity necessitates numerical noise removal
- Wavelet Methodology
  - Brief outline of wavelets
  - Wavelet denoising: examples and applications
  - Harnessing the power of wavelets: past, present and the future
- Summary

Coherent Synchrotron Radiation: A Physical Problem

- When a charged particle beam travels along a curved trajectory (bending magnet), beam emits synchrotron radiation

- If the wavelength λ of synchrotron radiation is longer than the bunch length σ_s, the resulting radiation is coherent synchrotron radiation (CSR)

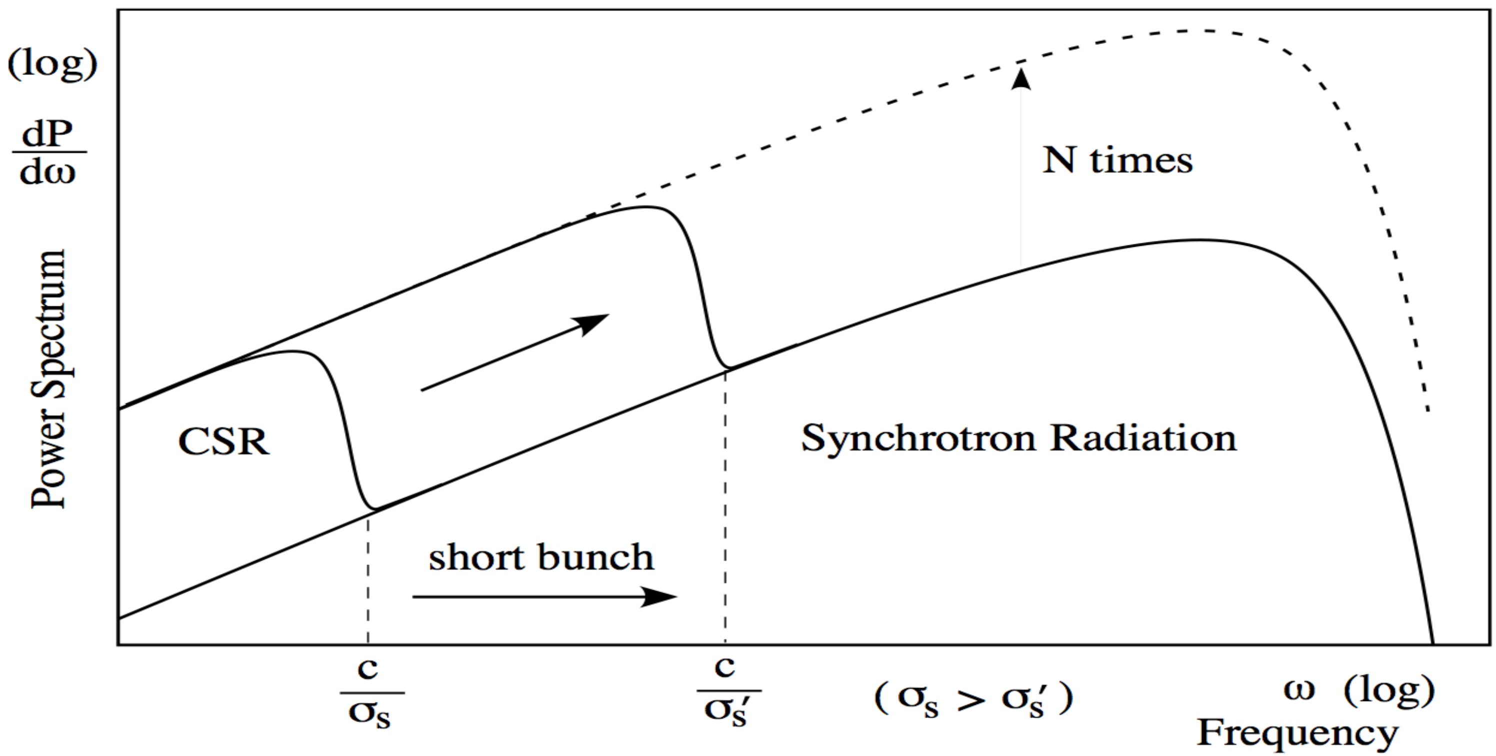


- Incoherent synchrotron radiation: largely cancels out

- Coherent synchrotron radiation: has systematic effects

Coherent Synchrotron Radiation: A Physical Problem

- CSR is the low frequency (long wavelength) part of the power spectrum



- *N* particles in the bunch act in phase and enhance intensity by a factor *N* (typically $N = 10^9$-10^{11})
- Therefore for shorter bunch (σ_s small), CSR is more pronounced

Coherent Synchrotron Radiation: A Physical Problem

- Short bunch lengths are desirable in many different contexts:
  - FEL require high peak current for a given bunch charge
  - ERL often require a short duration of radiation
  - B-factories and linear colliders require short bunch to achieve higher luminosities
- The demand for short bunches is expected to increase in the future
- This presents a problem:

  short beam bunch $\Rightarrow$ CSR is dominant $\Rightarrow$

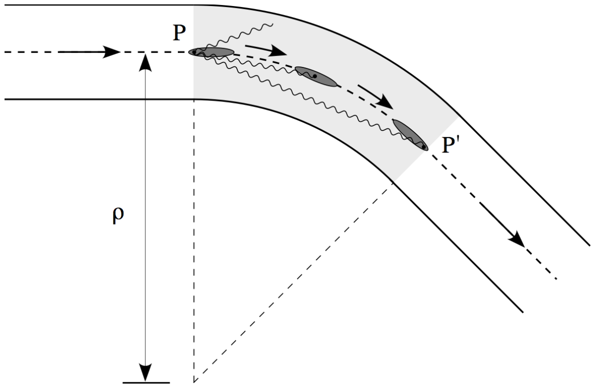
  $\Rightarrow$ beam is a subject to adverse CSR effects
- Adverse CSR effects, which can seriously impair beam quality:

  Energy change $\Rightarrow$ energy spread $\Rightarrow$ longitudinal instability (microbunching)

  $\Rightarrow$ emittance degradation
- Having a trustworthy code to simulate CSR is of great importance

Coherent Synchrotron Radiation: A Mathematical Problem

- Dynamics of an electron bunch is governed by

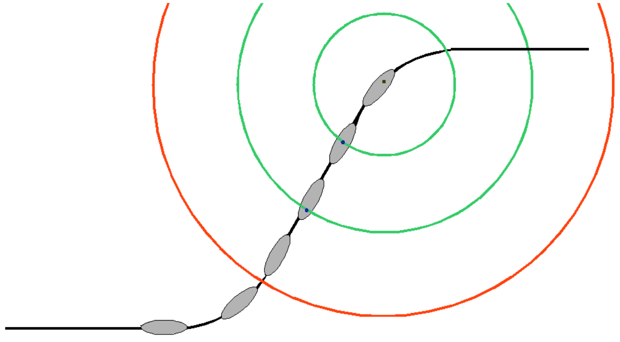


$$\frac{d}{dt}(\gamma m_e \vec{v}) = e(\vec{E} + \vec{\beta} x \vec{B})$$

$$\vec{\beta} = \vec{v}/c$$
$$\vec{E} = \vec{E}^{ext} + \vec{E}^{self}$$
$$\vec{B} = \vec{B}^{ext} + \vec{B}^{self}$$

  - $\vec{E}^{ext}, \vec{B}^{ext}$: external EM fields
  - $\vec{E}^{self}, \vec{B}^{self}$: self-interaction (CSR)

$$\vec{E}^{self} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$
$$\vec{B}^{self} = \vec{\nabla} \times \vec{A}$$



where $\phi(\vec{r},t) = \int \frac{d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\rho(\vec{r}\,',t')$, $\vec{A} = \frac{1}{c}\int \frac{d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\vec{J}(\vec{r}\,',t')$ } retarded potentials $t' = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$

charge density: $\rho(\vec{r},t) = \int f(\vec{r},\vec{v},t)\,d\vec{v}$
current density: $\vec{J}(\vec{r},t) = \int \vec{v} f(\vec{r},\vec{v},t)\,d\vec{v}$ } Need to know the history of the bunch

beam distribution function (DF): $f(\vec{r},\vec{v},t)$

Coherent Synchrotron Radiation: A Computational Problem

- Storing and computing with a 4D (3 positions, 1 time) charge and current densities is prohibitively expensive
  - $\Rightarrow$ Need simplifications/approximations

- Possible simplifications to full dimensional CSR modeling:
  - 1D line approximation (IMPACT, ELEGANT): probably too simplistic
  - 2D approximation (vertically flat beam):
    - codes by Li 1998, Bassi *et al*. 2006

- Based on how the DF (and, consequently, charge and current densities) are represented, two approaches emerge:
  - *Point-to-point (tracking) methods*: solving microscopic Maxwell's equation using retarded potentials
  - *Mean field (PIC, grid, mesh) methods*: solving Maxwell equation using finite difference, finite element, Green's function, retarded potentials...

Coherent Synchrotron Radiation: A Computational Problem

- *Point-to-point approach (2D)*: Li 1998

$$f(\vec{r},\vec{v},t)=q\sum_{i=1}^{N} n_m(\vec{r}-\vec{r}_0^{(i)}(t))\,\delta(\vec{v}-\frac{\vec{v}_0^{(i)}(t)}{c})$$ DF

$$\rho(\vec{r},t)=q\sum_{i=1}^{N} n_m(\vec{r}-\vec{r}_0^{(i)}(t))$$ charge density

$$\vec{J}(\vec{r},t)=q\sum_{i=1}^{N} \vec{\beta}_0^{(i)}(t)\,n_m(\vec{r}-\vec{r}_0^{(i)}(t))$$ current density

$$n_m(\vec{r}-\vec{r}_0^{(i)}(t))=\frac{1}{2\pi\sigma_m^2}e^{-\frac{(x-x_0(t))^2+(y-y_0(t))^2}{2\sigma_m^2}}$$ Gaussian macroparticle

- Charge density is sampled with *N* Gaussian-shaped 2D macroparticles (2D distribution without vertical spread)
- Each macroparticle interact with each other one throughout history
- Expensive: computation of retarded potentials and self fields ~ O(N^2)
  - ⇒ small number *N* ⇒ poor spatial resolution
  - ⇒ difficult to see small-scale structure
- While useful in obtaining low-order moments of the beam, *point-to-point approach is not optimal for studying CSR*

Coherent Synchrotron Radiation: A Computational Problem

- *Mean field approach with retarded potentials (2D)*: Terzić & Li, *in preparation*

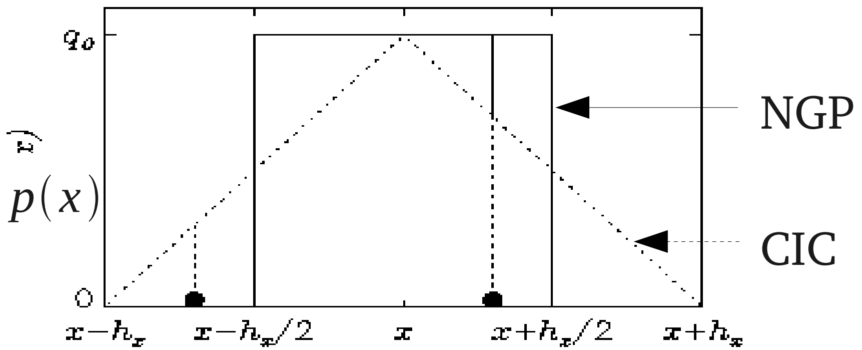
$$f(\vec{x},\vec{v},t)=q\sum_{i=1}^{N}\delta(\vec{r}-\vec{r}_0^{(i)}(t))\,\delta\left(\vec{v}-\frac{\vec{v}_0^{(i)}(t)}{c}\right)$$ DF (Klimontovich)

$$\rho(\vec{x}_{\vec{k}},t)=q\sum_{i=1}^{N}\int_{-h}^{h}\delta(\vec{x}_{\vec{k}}-\vec{x}_0^{(i)}(t)+\vec{X})\,p(\vec{X})\,d\vec{X}$$ charge density

$$\vec{J}(\vec{x}_{\vec{k}},t)=q\sum_{i=1}^{N}\vec{\beta}_0^{(i)}(t)\int_{-h}^{h}\delta(\vec{x}_{\vec{k}}-\vec{x}_0^{(i)}(t)+\vec{X})\,p(\vec{X})\,d\vec{X}$$ current density

- Charge and current densities are sampled with N point-charges (δ-functions) & deposited on a finite grid $\vec{x}_{\vec{k}}$ using a deposition scheme $p(\vec{X})$
  - Two main deposition schemes:
    - Nearest Grid Point (NGP) (constant: deposits to 1^D points)
    - Cloud-In-Cell (CIC) (linear: deposits to 2^D points)

    There exist higher-order schemes



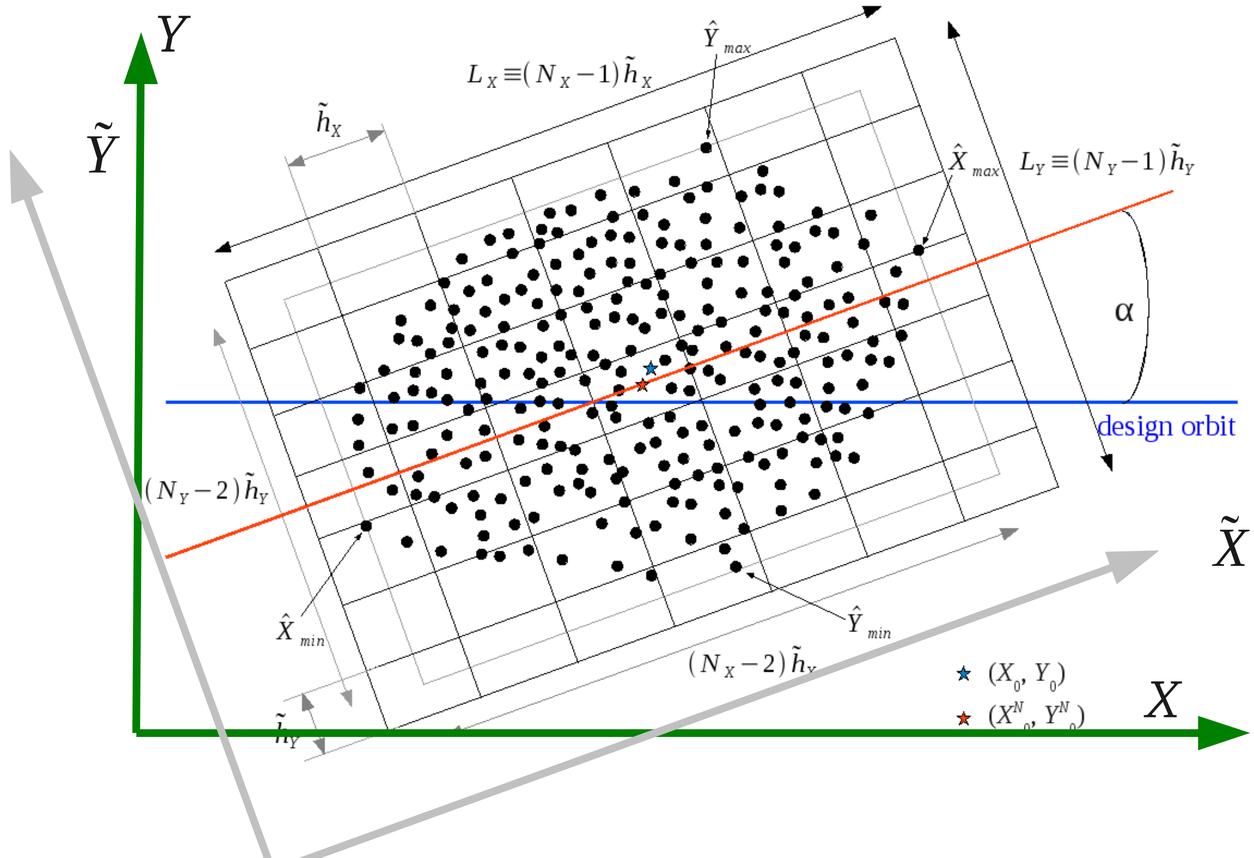
x – macroparticle location
● – gridpoint location

- Particles do not directly interact with each other, but only through a mean-field of the gridded representation

Coherent Synchrotron Radiation: A Computational Problem

- *Mean field approach with retarded potentials (2D)*: Terzić & Li, *in preparation* (continued)

  - Grid resolution is specified *a priori* (fixed grid) or changes as necessary (adaptive grid)
  - N_X : # of gridpoints in X
  - N_Y : # of gridpoints in Y
  - $N_{grid} = N_X N_Y$ total gridpts
  - Grid: $\vec{x}_{\vec{k}} = [\tilde{X}_{ij}, \tilde{Y}_{ij}]$
    $i = 1,.., N_X \quad j = 1,.., N_Y$
  - Inclination angle α



  - Grid is determined so as to tightly envelope all particles
    Minimizing number of empty cells $\Rightarrow$ optimizing spatial resolution

Coherent Synchrotron Radiation: A Computational Problem

- *Mean field approach with retarded potentials (2D)*: Terzić & Li, *in preparation* (continued)

  - Computational cost:
    - Particle deposition (yields charge and current densities on the grid):
      - $O(N)$ operations
    - Integration over history (yields retarded potentials):
      - $O(N_{grid}^2)$ operations
    - Finite difference (yields self-forces on the grid):
      - $O(N_{grid})$ operations
    - Interpolation (yields self-forces acting on N individual particles)
      - $O(N)$ operations
    - Total cost ~ $O(N_{grid}^2)+O(N)$ operations (in realistic sim.: $N_{grid}^2 >> N$)
  - N_{grid} and N should be chosen *judiciously*
  - Favorable scaling allows for larger N, and reasonable grid resolution

    $\Rightarrow$ improved spatial resolution

Coherent Synchrotron Radiation: A Computational Problem

- *Point-to-point (P2P) Vs. Mean field (MF):*
  - Computational cost: $O(N^2)$ Vs. $O(N_{grid}{}^2)+O(N)$

    Fair comparison: P2P with N macroparticles and MF with $N_{grid}=N$

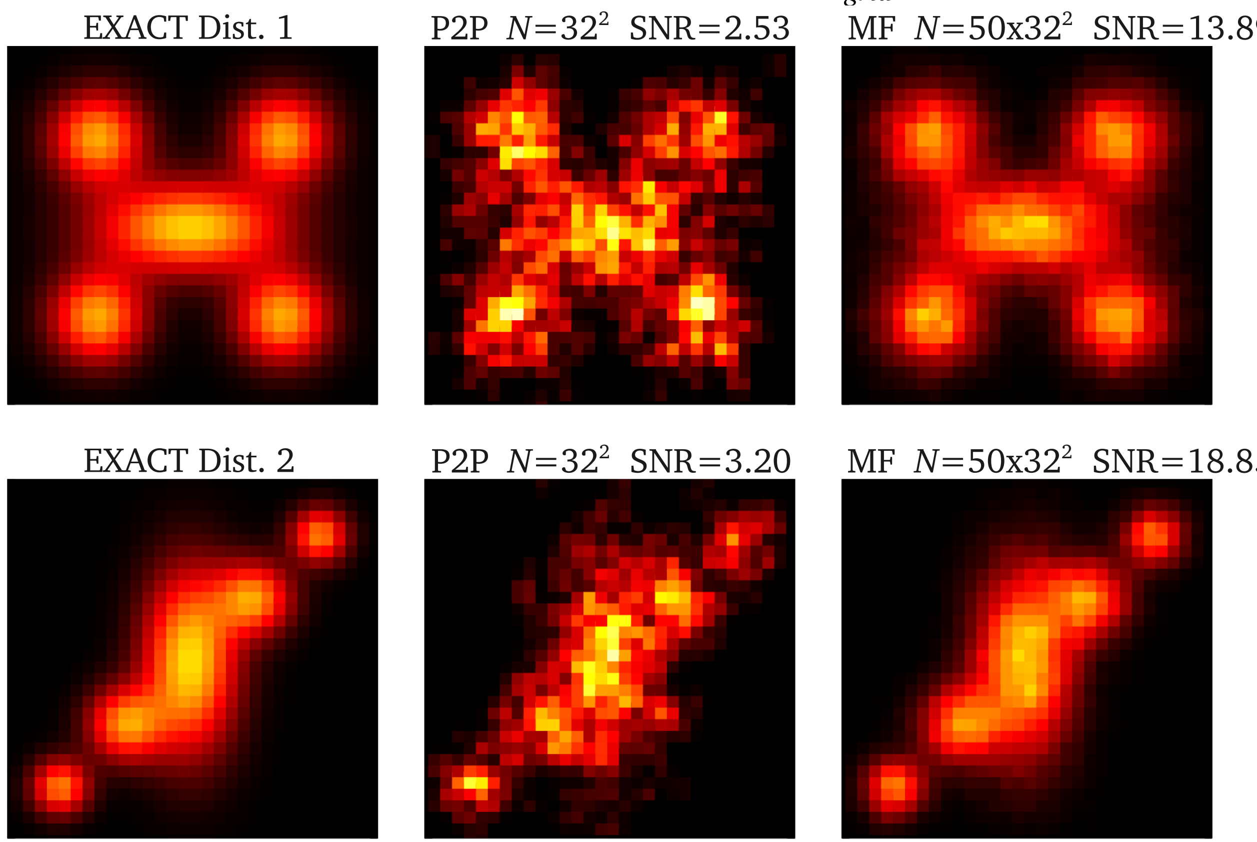
  - 2D grid: $N_X=N_Y=32$

Signal-to-Noise Ratio

$$SNR=\sqrt{\frac{\sum_i \bar{q}_i^2}{\sum_i (q_i-\bar{q}_i)^2}}$$

$\bar{q}_i = exact$

$q_i = approx.$



- MF approach provides superior spatial resolution to P2P approach

  $\Rightarrow$ Modify Rui Li's P2P CSR code into a MF

Coherent Synchrotron Radiation: Numerical Noise in the Mean Field Simulations

- There are the two major sources of numerical noise in MF simulations:
  - *graininess of the distribution function*: $N_{\text{simulation}} << N_{\text{physical}}$
  - *discreteness of the computational domain*: quantities defined on a finite grid
- One must first understand the profile of the numerical noise associated with the discreteness of the computational in order to be able to remove it
- Systematic removal of numerical noise from the MF simulations leads to physically more reliable results, equivalent to simulations with many more particles

Coherent Synchrotron Radiation: Numerical Noise in the Mean Field Simulations

- If many random realizations of a given particle distribution have are deposited onto a grid, the number of particles in each gridpoint is Poisson-distributed (variance = mean) $\Rightarrow$ noise is *signal-dependent*
- Wavelet denoising is at its most powerful (and mathematically strongest ground) when the noise is Gaussian-distributed (*signal-independent, white*)
- Signal contaminated with Poissonian noise can be transformed to signal with Gaussian noise by a variance-stabilizing *Anscombe transform* (1948):

$$Y_G = 2\sqrt{Y_P + \frac{3}{8}}$$

Y_P = signal with Poissonian noise
Y_G = signal with Gaussian noise

- After the transformation the noise in each gridpoint is (nearly) Gaussian-distributed with variance $\sigma = 1$
- Essentially, we have pre-processed the signal before denoising it
- This error/noise estimate σ is crucial for optimal wavelet noise removal

[For more details see Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201]

Coherent Synchrotron Radiation: Removing Numerical Noise from Mean Field Simulations

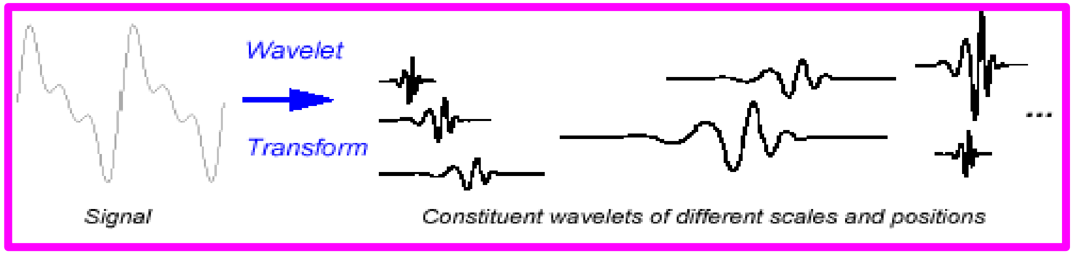
- It is desirable to remove noise from the MF simulations
  less numerical noise $\Leftrightarrow$ running simulations with more particles
  $\Rightarrow$ increased sensitivity to physical small-scale structure

- Noise removal from the MF simulations can be done in several ways:
  - Particle deposition schemes:
    - Higher order deposition schemes serve as smoothing filters
  - Filtering:
    - Savitzky-Golay smoothing filter (local polynomial regression)
  - In Fourier space:
    - Truncating the highest Fourier frequencies
  - In wavelet space:
    - Wavelet coefficient thresholding

- Wavelets provide a natural setting for *judicious* noise removal
  (other methods indiscriminantly smooth over/truncate small scale structures)

Brief Overview of Wavelets

- **Wavelets:** orthogonal basis composed of scaled and translated versions of the same localized *wavelet* $\psi(x)$:

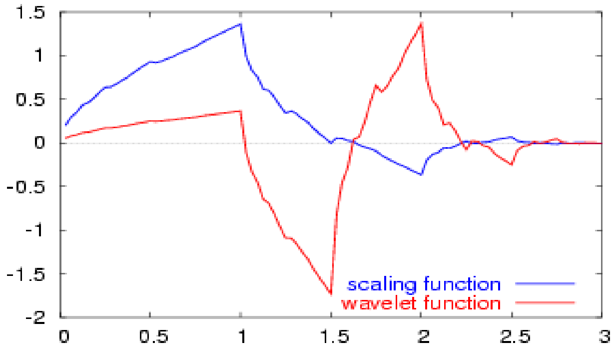


$$\psi_i^k(x) = 2^{k/2}\psi(2^k x - i) \quad k, i \in Z$$

$$f(x) \approx \Sigma_k \Sigma_i d_i^k \psi_i^k(x)$$

- Each new resolution level k is orthogonal to the previous levels

Daubachies 4th order wavelet



- Wavelets are derived from the *scaling function* $\phi(x)$ which satisfies

$$\phi(x) = \sqrt{2}\,\Sigma_j h_j \phi(2x - j)$$

$$\psi(x) = \sqrt{2}\,\Sigma_j g_j \phi(2x - j)$$

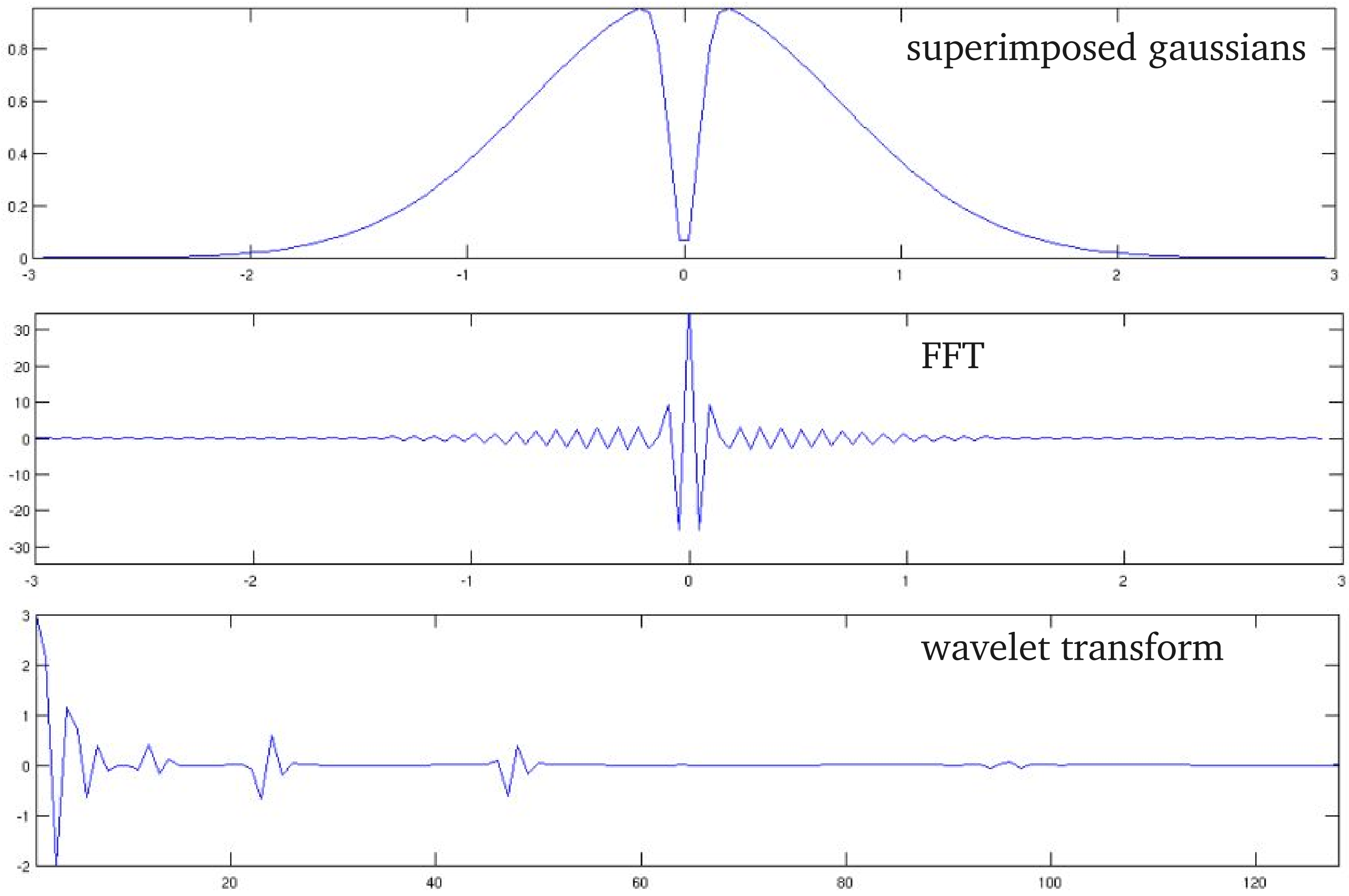
(only finite number of filter coefficients h_j and g_j are non-zero: *compact support*)

- In order to attain orthogonality of different scales, their shapes are strange
  - Makes them suitable to represent irregularly shaped functions

- For discrete signals (gridded quantities), fast Discrete Wavelet Transform (DFT) is an $O(MN)$ operation, M size of the wavelet filter, N signal size

Brief Overview of Wavelets

- Wavelet transform separates scales



Brief Overview of Wavelets

- Advantages of wavelet formulation:
  - Wavelet basis functions have compact support $\Rightarrow$ signal localized in space
    Wavelet basis functions have increasing resolution levels
    $\Rightarrow$ signal localized in frequency
    $\Rightarrow$ *simultaneous localization in space and frequency* (FFT only frequency)
  - Wavelet basis functions correlate well with various signal types (including signals with singularities, cusps and other irregularities)
    $\Rightarrow$ *compact and accurate representation of data (compression)*
  - Wavelet transform *preserves hierarchy of scales*
  - In wavelet space, discretized operators (Laplacian) are also sparse and have an efficient preconditioner $\Rightarrow$ *solving some PDEs is faster and more accurate*
  - Wavelets provide a natural setting for noise removal $\Rightarrow$ *wavelet denoising*

Wavelet Denoising

- In wavelet space:

  signal $\longrightarrow$ few large wavelet coefficients c_{ij}

  noise $\longrightarrow$ many small wavelet coefficients c_{ij}

- Denoising by wavelet thresholding:

  if $|c_{ij}| < T$, set to $c_{ij} = 0$

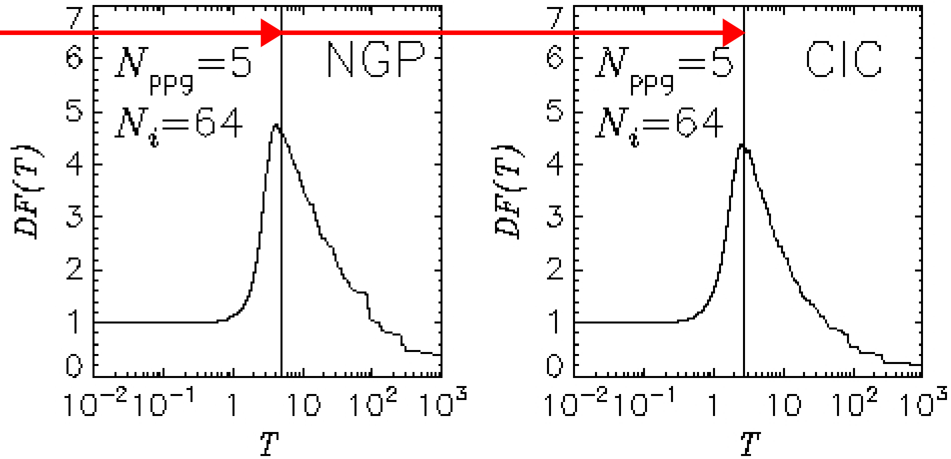
- A great deal of study has been devoted to estimating optimal T

$$T = \sqrt{2 \log N_{grid}}\, \sigma$$

($\sigma = 1$ after Anscombe transform)

Denoising factor (DF):

$$DF = \frac{Error_{original}}{Error_{denoised}}$$



[Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201]

Wavelet Denoising and Compression

- When the signal is known, one can compute *Signal-to-Noise Ratio* (*SNR*):

$$SNR = \sqrt{\frac{\sum_i \bar{q}_i^2}{\sum_i (q_i - \bar{q}_i)^2}}$$

$\bar{q}_i = exact$

$q_i = approx.$

- $SNR \sim \sqrt{N_{ppc}}$ $\quad N_{ppc}$: avg. # of particles per cell $\quad N_{ppc} = N/N_{cells}$

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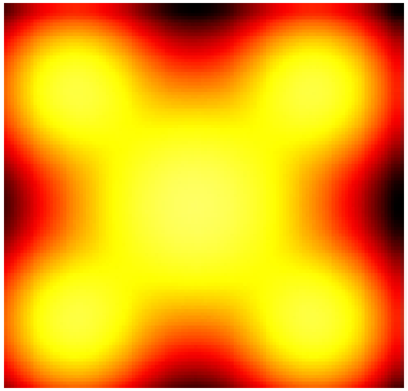
[2D superimposed Gaussians on 256×256 grid]

ANALYTICAL



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2D superimposed Gaussians on 256×256 grid

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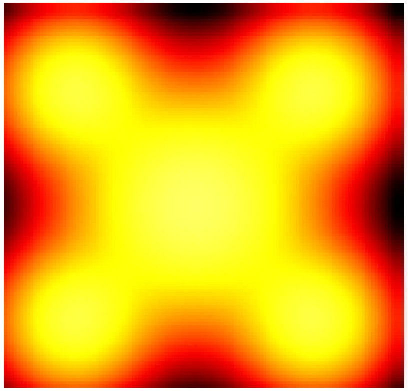
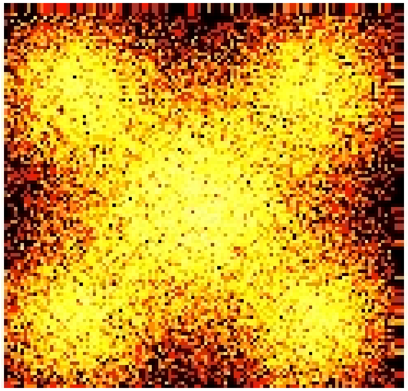


$N_{\rm ppc} = 3$ $SNR = 2.02$



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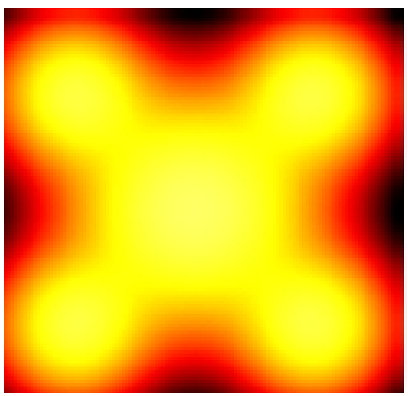
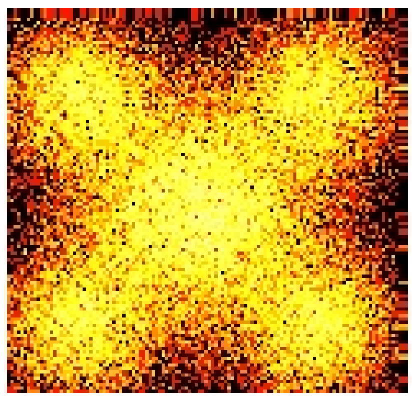
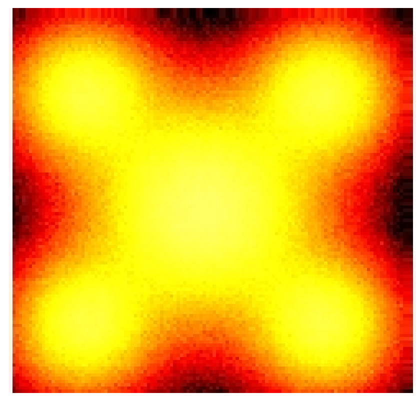


$N_{ppc}=3$ $SNR=2.02$



$N_{ppc}=205$ $SNR=16.89$



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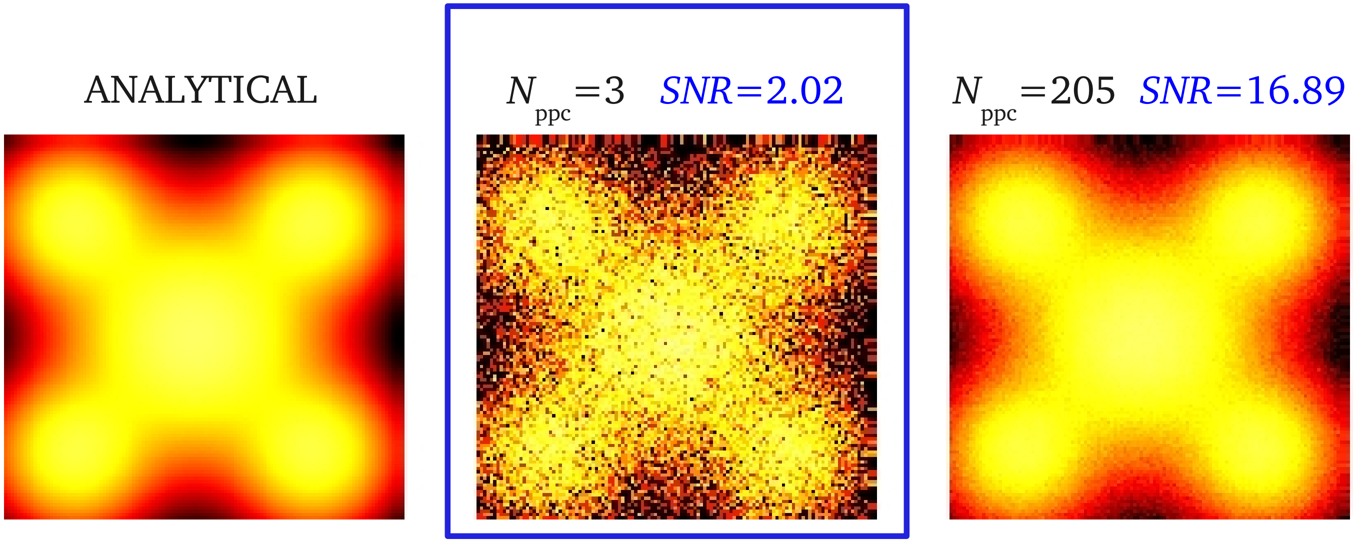

- denoising by wavelet thresholding: if $|c_{ij}| < T$, set to 0

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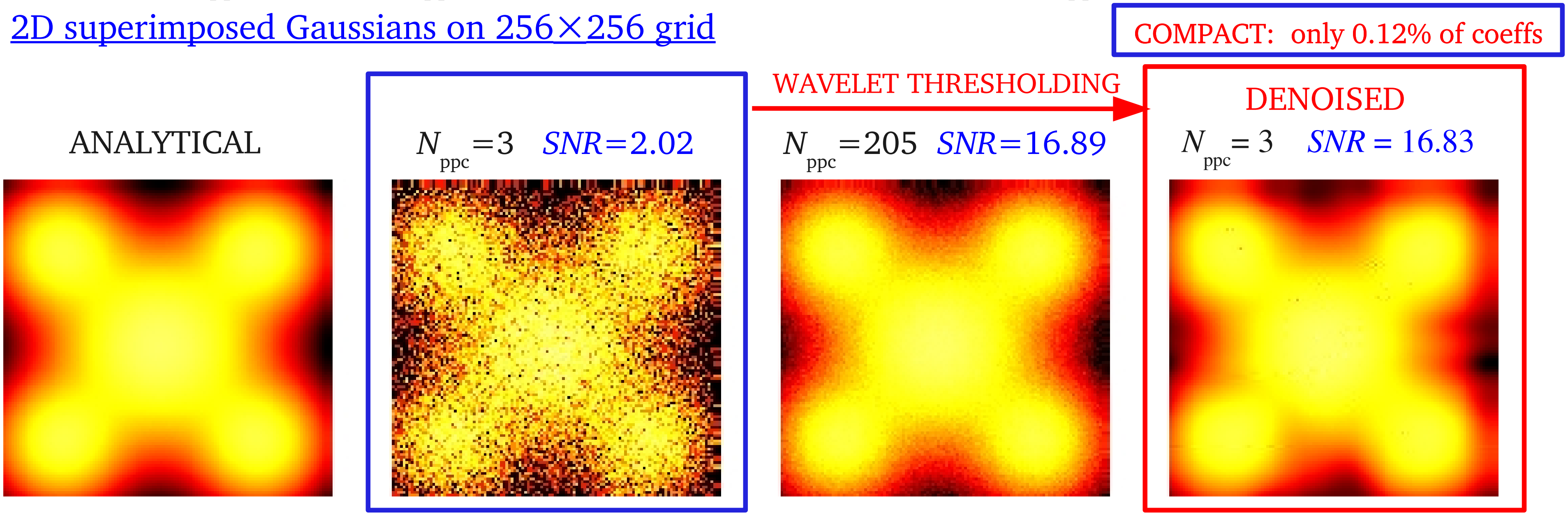


- Wavelet denoising yields a representation which is:
  - Appreciably more accurate than non-denoised representation
  - Sparse (if clever, we can translate this sparsity in computational efficiency)

Harnessing the Power of Wavelets: The Past

- We have already used wavelets in mean field solvers and will greatly benefit from it in the current project:
  - Terzić, Pogorelov & Bohn 2007:
    - Designed a new 3D wavelet-based Poisson equation solver and optimized it for use in PIC beam simulations
    - Integrated the Poisson solver in beam code (IMPACT), benchmarked it and used to model Fermilab/NICADD photoinjector
      - First application of wavelets to 3D beam simulations
    - We provide a detailed treatment of noise in PIC simulations and implemented wavelet denoising
      - Roadmap to follow in the current project
  - Sprague 2008, Sprague & Terzić *in preparation*:
    - Tutorial of for wavelet use in solving PDEs
    - Enhanced the original solver by implementing adaptive grid
      - Will use this to further improve spatial resolution in our MF code

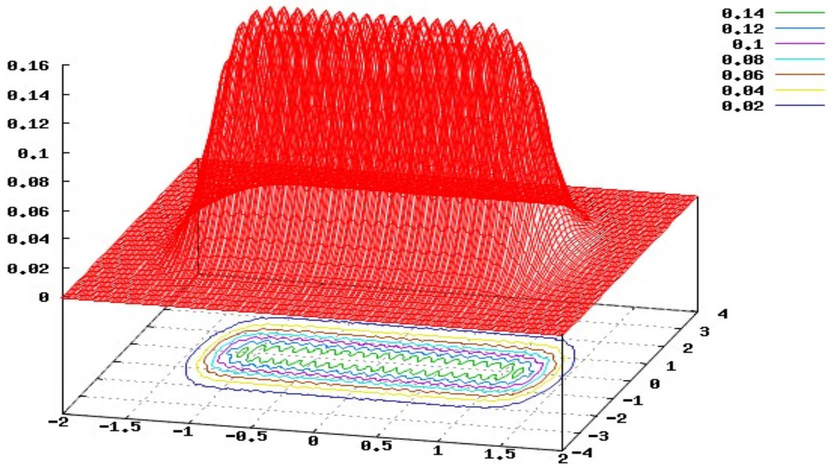
Harnessing the Power of Wavelets: The Present

- I am currently involved in two projects which bring CSR and wavelets together:
  - Collaboration with Rui Li on modifying her 2D CSR P2P code into a MF code:
    - Wavelet denoising of the representation is already implemented (can be turned on and off, enabling a clear comparison)
    - We already ascertained that only a small fraction of coefficients on the grid ($<1\%$ or so) is needed to accurately represent densities
      - Can this translate into a more efficient code?
    - Once the code is completed and tested, we will conduct a comprehensive comparison of the effects of denoising:
      - How much does wavelet denoising improve spatial resolution?
      - How accurate is the wavelet denoised representation?

Harnessing the Power of Wavelets: The Present

- Bassi & Terzić 2009:
  - Improved particle representation in Bassi's 2D CSR code by replacing analytic cosine expansion with a wavelet approximation
    - Better spatial resolution (needed to study microbunching)
    - Appreciably more accurate (after wavelet thresholding)
    - Orders of magnitude faster
  - How accurately can small-scale structures be represented by an approximation?
    - Analytic Monte Carlo cosine
    - Simple grid
    - Thresholded FFT (grid)
    - Thresholded wavelet (grid)

Flat-top with sinusoidally modulated frequency (FERMI@ELETTRA first bunch compressor)



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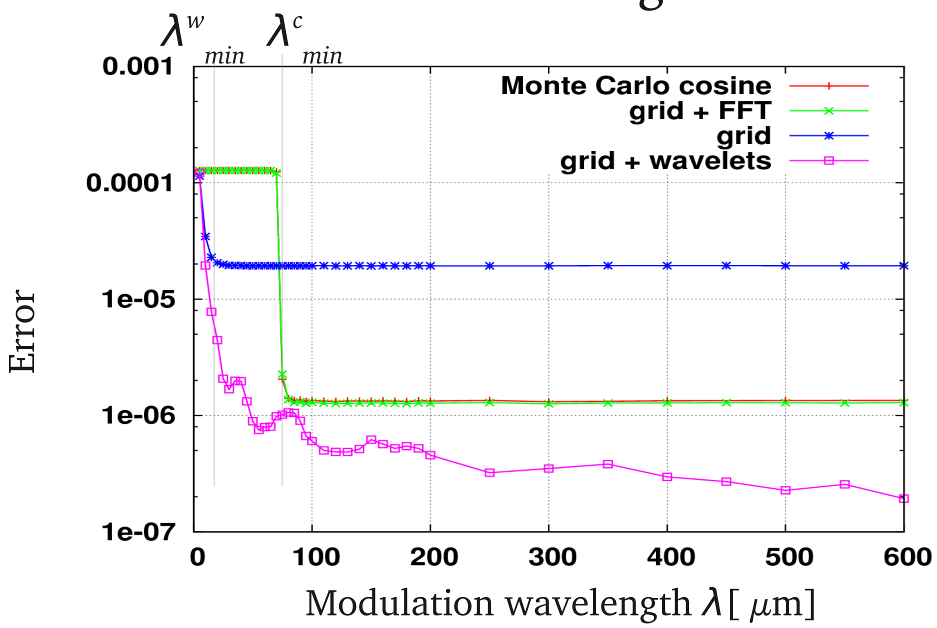

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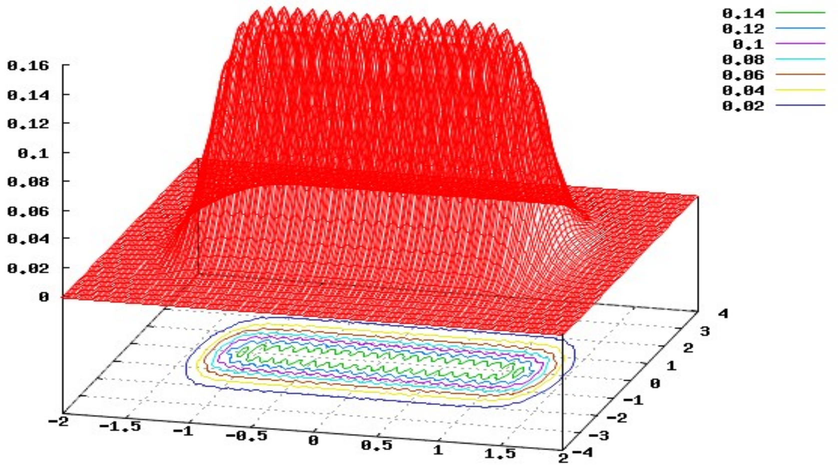

$N=10^8$ cosine expansion: $N_c=40$, $M_c=100$
grid resolution: $N_x=128$, $N_z=1024$

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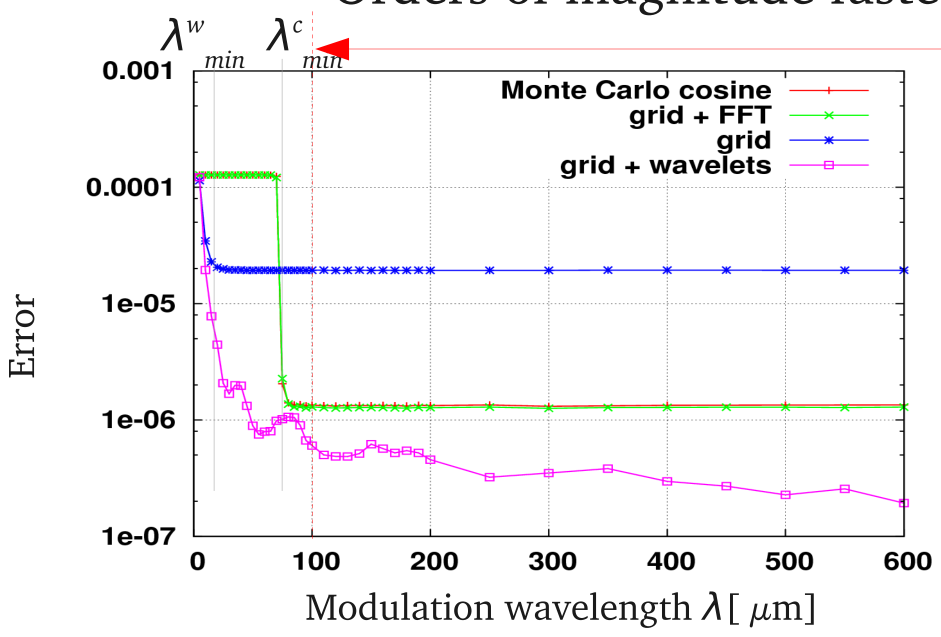
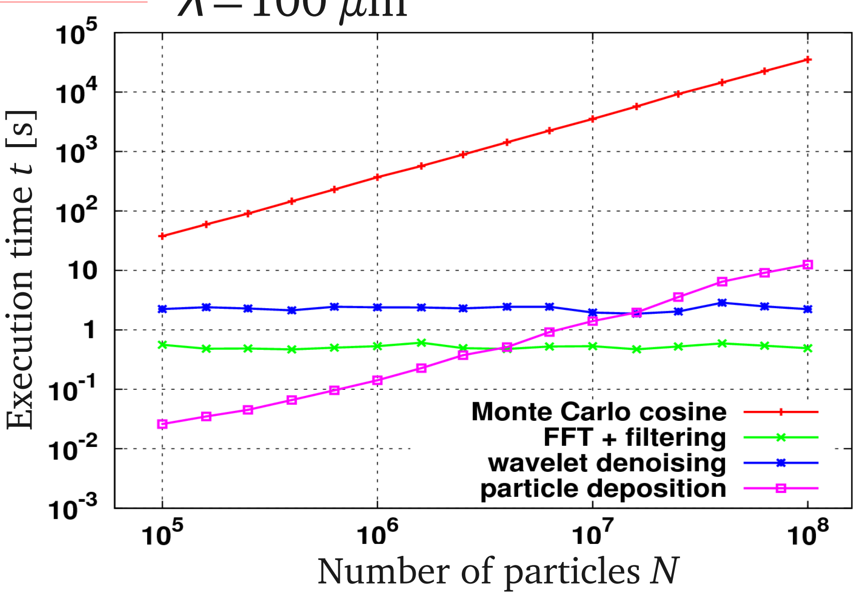


$\lambda = 100\ \mu$m



$N = 10^8$ cosine expansion: $N_c = 40$, $M_c = 100$
grid resolution: $N_x = 128$, $N_z = 1024$

Harnessing the Power of Wavelets: The Future

- In the future, we plan to further harness the power of wavelets:
  - Translate sparsity of operators and datasets in wavelet space to computational efficiency
    - Fast application of discretized operators
    - Efficient preconditioners for other operators?
    - Fast interpolation of discrete data from sparse wavelet representation
  - Use adaptive grid in wavelet-based methods to increase spatial resolution
  - Explore applicability of what we have learned about wavelets to other PDEs

Summary

- We presented two computational approaches to simulating CSR: P2P and MF
  - Demonstrated that the MF approach is better because of:
    - Better spatial resolution (a “must” for small-scale instabilities)
    - Better scaling with the number of particles *N*
  - We are now working on converting Rui Li's P2P code into a MF code (We hope to start benchmarking it within the next few months)
- Compare with Bassi's 2D CSR code for consistency
- Closing in on our intermediate goal: having an accurate, efficient and trustworthy code which faithfully simulates CSR
- Long-term goal: being able to quantitatively simulate CSR in real machines, as a first step toward controlling its adverse effects

Auxiliary Slides

Multi-Resolution Analysis and Wavelets

- Multi-Resolution Analysis (MRA) is a decomposition of Hilbert space $L^2(\boldsymbol{R})$ into a chain of closed subspaces V: $0 \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset .. \subset L^2(R)$

- Define an associated sequence of subspaces W as an orthogonal complement of $V_{j\text{-}1}$ in V_j: $V_j = V_{j-1} + W_j$ Also: $V_j = \Sigma_{j'<j} W_{j'}$

- A set of dilations and translations of the *scaling function* $\phi(x)$:

  $$\{\phi_k^j(x) = 2^{j/2}\phi(2^j x - k)\}_{k\in Z}$$

  forms an orthonormal basis of V_j.

- A set of dilations and translations of the *wavelet function* $\psi(x)$:

  $$\{\psi_k^j(x) = 2^{j/2}\psi(2^j x - k)\}_{k\in Z}$$

  forms an orthonormal basis of W_j.

Quadrature Mirror Filters $H=\{h_k\}$, $G=\{g_k\}$
used in the Discrete Wavelet Transform
(only few of them are non-zero: *compact support*)

- They satisfy *refinement relations*:

  $$\phi(x) = \sqrt{2}\,\Sigma_k h_k \phi(2x - k)$$
  $$\psi(x) = \sqrt{2}\,\Sigma_k g_k \phi(2x - k)$$
  $$g_i = (-1)^i h_{1-i}$$

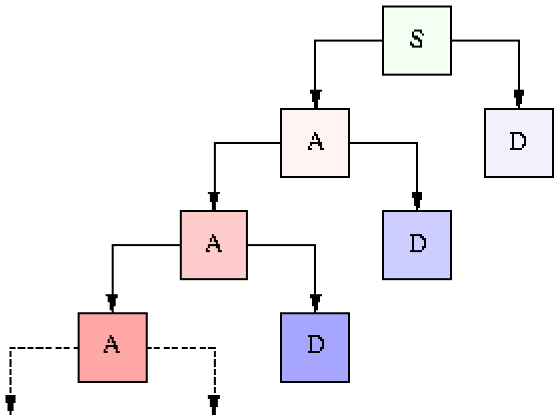
- Projection of function $f(x)$ onto V_j :

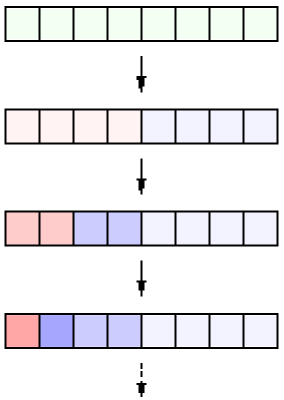
  $$(P_j f)(x) = \Sigma_{k\in Z} s_j^k \psi_k^j(x) = \Sigma_{j'<j} \Sigma_{k\in Z} d_j^k \phi_k^j(x)$$

  $$s_k = \int_{\infty}^{\infty} f(x)\phi_k^j(x)\,dx$$
  $$d_k = \int_{\infty}^{\infty} f(x)\psi_k^j(x)\,dx$$

How Do Wavelets Work?

Wavelet analysis (wavelet transform):





S - signal
A - approximation
D - detail

- Approximation – apply low-pass filter to Signal and down-sample
- Detail – apply high-pass filter to Signal and down-sample
- Wavelet synthesis (inverse wavelet transform): up-sampling & filtering
- Complexity: *4MN*, *M* the size of the wavelet, *N* number of cells
  - Recall: FFT complexity $4N \log_2 N$

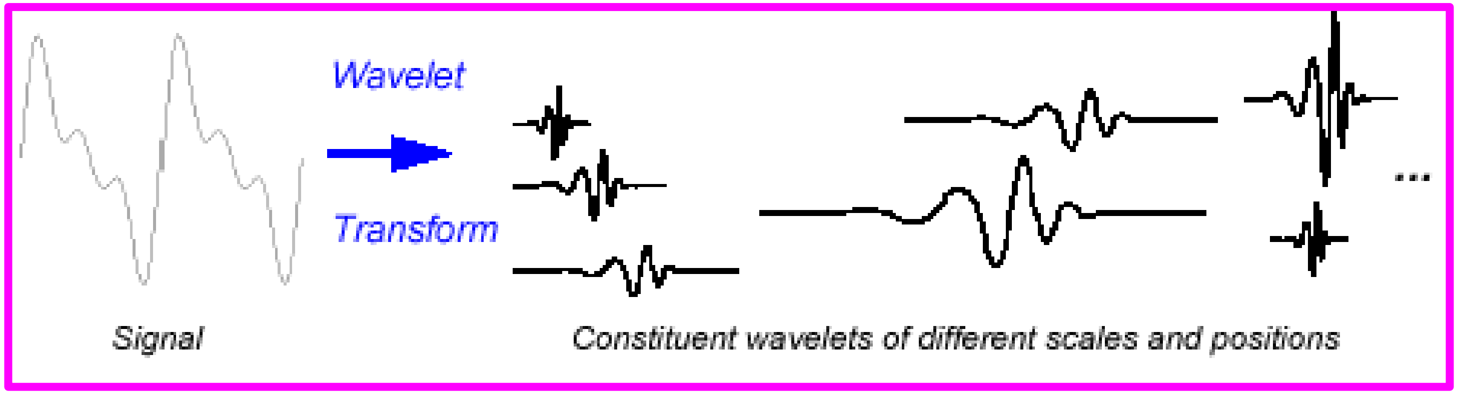
Wavelet Decomposition

The continuous wavelet transform of a function $f(t)$ is

$$\gamma(s,\tau)=\int_{-\infty}^{\infty} f(t)\,\psi_{s,\tau}(t)\,dt$$

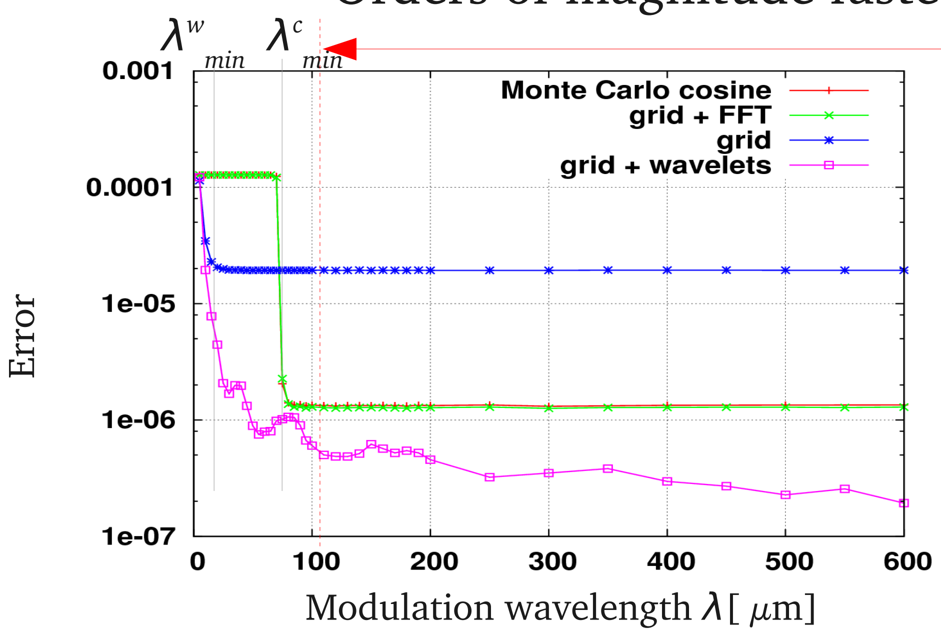
$$\psi_{s,\tau}(t)=\frac{1}{\sqrt{s}}\,\psi\left(t-\frac{\tau}{s}\right)$$

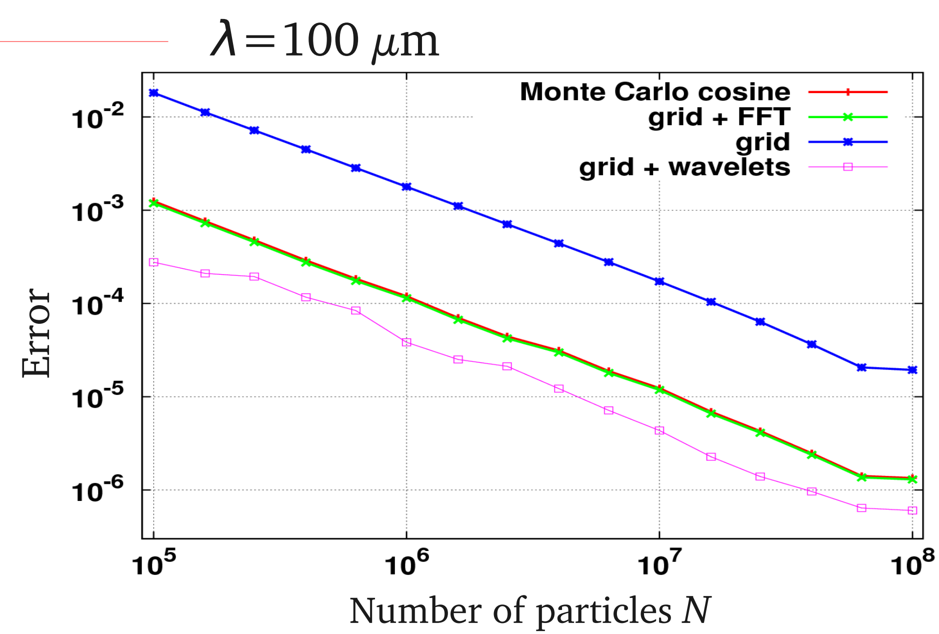
$\psi(t)$ *mother wavelet* with scale and translation dimensions *s* and τ respectively



Harnessing the Power of Wavelets: The Present

- Bassi & Terzić 2009:
  - Improved particle representation in Bassi's 2D CSR code by replacing analytic cosine expansion with a wavelet approximation
    - Better spatial resolution (needed to study microbunching)
    - Appreciably more accurate (after wavelet thresholding)
    - Orders of magnitude faster





$N = 10^8$ cosine expansion: $N_c = 40$, $M_c = 100$

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Numerical Noise in PIC Simulations

- In wavelet space:

  signal → few large wavelet coefficients c_{ij}

  noise → many small wavelet coefficients c_{ij}

- Poissonian noise —(Anscombe transformation)→ Gaussian noise

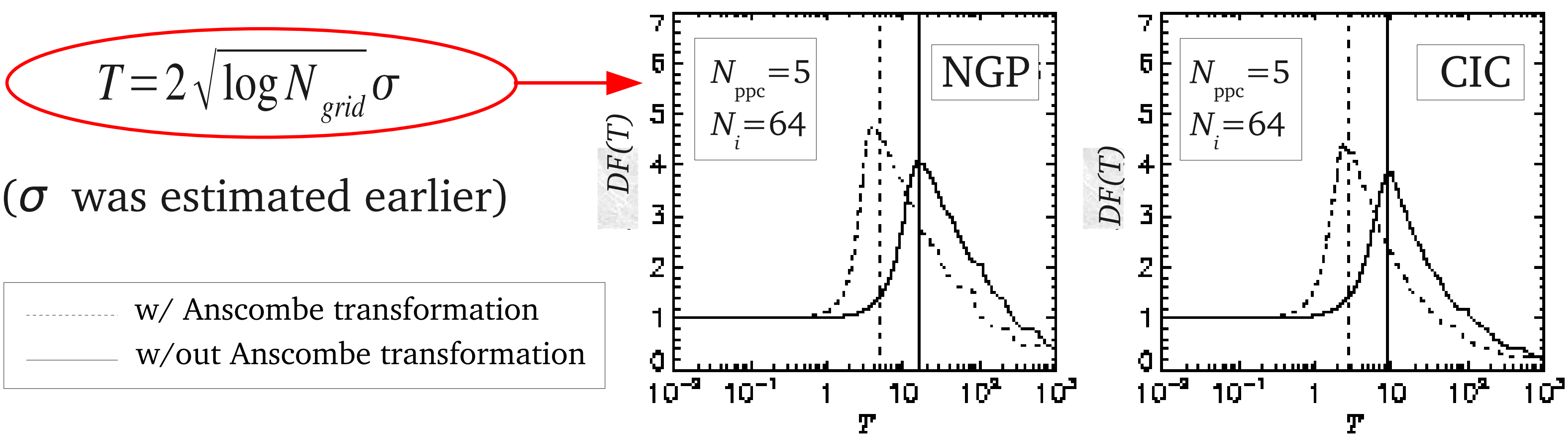
- Denoising by wavelet thresholding:

  if $|c_{ij}| < T$, set to $c_{ij} = 0$ (choose threshold T carefully!)

- A great deal of study has been devoted to estimating optimal T

$$T = 2\sqrt{\log N_{grid}}\,\sigma$$

(σ was estimated earlier)



Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201

Coherent Synchrotron Radiation: Numerical Noise in the Mean Field Simulations

- For NGP, at each gridpoint, density dist. is Poissonian:

  $P=(n!)^{-1} n_j^n e^{-n_j}$ n_j is the expected number in j^{th} cell; n integer

- For CIC, at each gridpoint, density dist. is *contracted* Poissonian:

  $P=(n!)^{-1}(an_j)^n e^{-an_j}$ $a=(2/3)^{(D/2)} \sim 0.54\,(3D),\, 0.67\,(2D),\, 0.82\,(1D)$

  [For more details see Terzić, Pogorelov & Bohn 2007, PR STAB, 10, 034201]

- Measure of error (noise) in depositing macroparticles onto a grid:

  $$\sigma^2=(N_{grid})^{-1}\sum_{i=1}^{N_{grid}} Var(q_i) \qquad \sigma_{NGP}^2=\frac{Q_{total}^2}{N\,N_{grid}} \qquad \sigma_{CIC}^2=\frac{a^2 Q_{total}^2}{N\,N_{grid}}$$

  where $q_i = (Q_{total}/N)n_i$, Q_{total} total charge

- This error/noise σ estimate is crucial for optimal noise removal

- Signal with Poissonian noise can be transformed to the signal with Gaussian noise by *Anscombe transformation*:

  $$Y_G=2\sqrt{Y_P+\frac{3}{8}}$$

  Y_P = signal with Poissonian (multiplicative) noise

  Y_G = signal with Gaussian (additive) noise