Microbunching Instability in a Chicane: Two-dimensional Mean Field Treatment *

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Collaborators

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5. Microbunching Instability Studies for FERMI@Elettra
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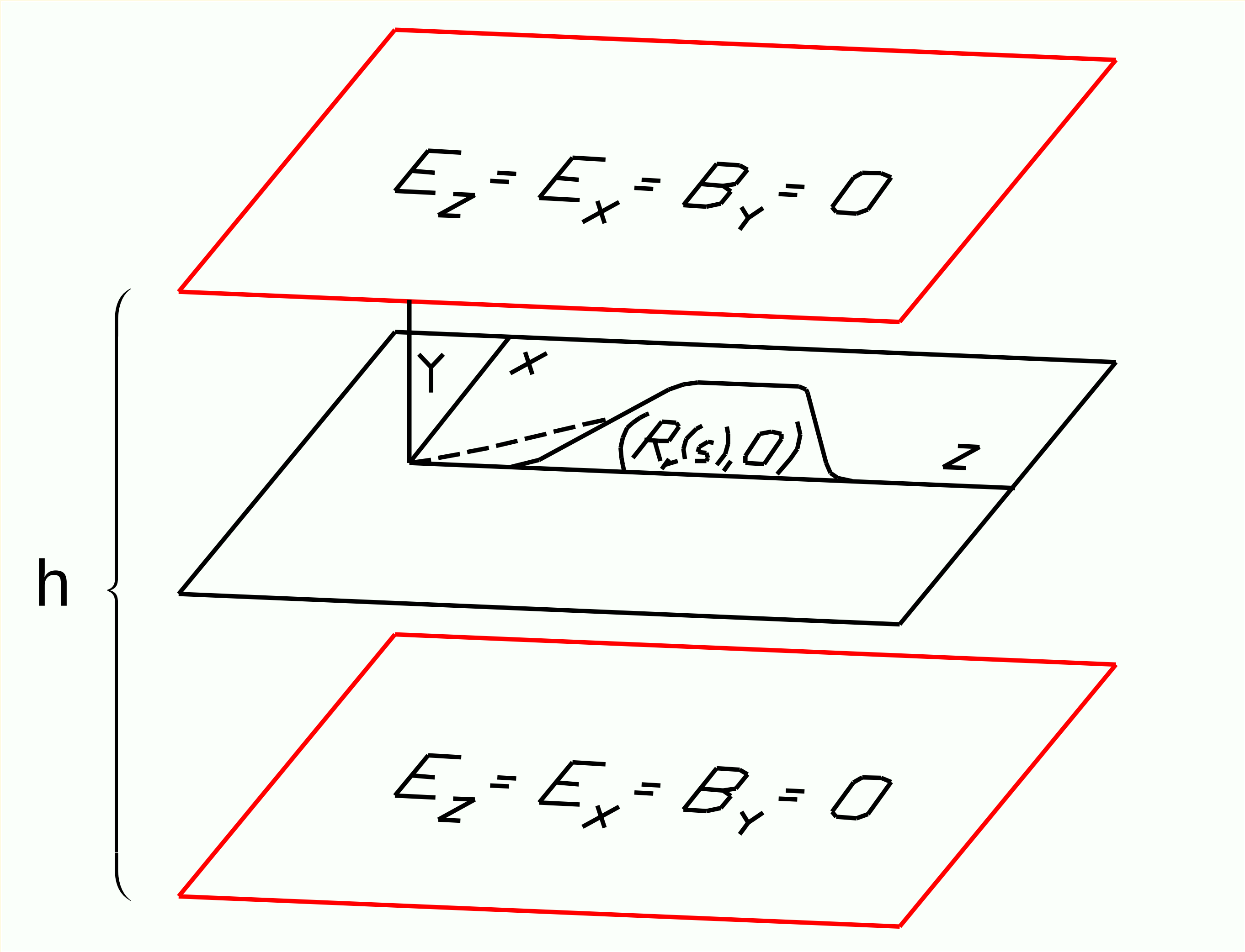
* G. Bassi, J.A. Ellison, K. Heinemann and R. Warnock, PRSTAB **12**, 080704 (2009);

Basic Lab Frame Setup



Self Consistent Vlasov-Maxwell Treatment for Sheet Beam in Lab Frame I

- Sheet Beam model: $f(Z,X,Y,P_Z,P_X,P_Y;u_0)=\delta(Y)\delta(P_Y)f_L(\mathbf{R},\mathbf{P};u_0)$.

  $3D$ Wave equation

$$(\partial_Z^2+\partial_X^2+\partial_Y^2-\partial_u^2)\mathcal{E}=\delta(Y)\mathcal{S}(\mathbf{R},u),\quad \mathcal{E}(\mathbf{R},Y=\pm g,u)=0,$$

where $u=ct$, $\mathcal{E}(\mathbf{R},Y,u)=(E_Z,E_X,B_Y)$, $\mathbf{R}=(Z,X)^T$ and $\dot{}=d/du$.

Vlasov equation

$$\partial_u f_L+\dot{\mathbf{R}}\cdot\partial_{\mathbf{R}}f_L+\dot{\mathbf{P}}\cdot\partial_{\mathbf{P}}f_L=0,\quad f_L(\mathbf{R},\mathbf{P};u_0)=f_{L0}(\mathbf{R},\mathbf{P}),$$

where

$$\dot{\mathbf{R}}=\frac{\mathbf{P}}{m\gamma(P)c},$$
$$\dot{\mathbf{P}}=\frac{q}{c}\Big[\begin{pmatrix}E_Z(\mathbf{R},u)\\E_X(\mathbf{R},u)\end{pmatrix}+[B_{ext}(Z)+B_Y(\mathbf{R},u)]\frac{1}{m\gamma(\mathbf{P})}\begin{pmatrix}P_X\\-P_Z\end{pmatrix}\Big],$$

$\mathbf{P}=(P_Z,P_X)^T$ and $(\mathbf{E}_Z(\mathbf{R},u),\mathbf{E}_X(\mathbf{R},u),\mathbf{B}_Y(\mathbf{R},u))\equiv\mathcal{E}(\mathbf{R},0,u)$.

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Self Consistent Vlasov-Maxwell Treatment for Sheet Beam in Lab Frame II

Field formula:

$$\mathcal{F}_L(\mathbf{R},u) = -\frac{1}{4\pi}\sum_{k=-\infty}^{\infty}(-1)^k \int_{\mathbb{R}^2} d\acute{\mathbf{R}} \frac{\mathbf{S}(\acute{\mathbf{R}}, u-[|\acute{\mathbf{R}}-\mathbf{R}|^2+(kh)^2]^{1/2})}{[|\acute{\mathbf{R}}-\mathbf{R}|^2+(kh)^2]^{1/2}} \, ,$$

where $\mathcal{F}_L(\mathbf{R},u) = (\mathbf{E}_Z(\mathbf{R},u), \mathbf{E}_X(\mathbf{R},u), \mathbf{B}_Y(\mathbf{R},u))$ and the source is

$$\mathbf{S}(\mathbf{R},u) = Z_0 Q H(u-u_0)\begin{pmatrix} c\partial_Z\rho_L + \partial_u J_{L,Z} \\ c\partial_X\rho_L + \partial_u J_{L,X} \\ \partial_X J_{L,Z} - \partial_Z J_{L,X}\end{pmatrix}, \quad \mathbf{J}_L = (J_{L,Z}, J_{L,X})^T,$$

where H is the unit step function.

The Vlasov equation and the self fields are coupled by $Q\rho_L$ and $Q\mathbf{J}_L$

$$\begin{aligned}\rho_L(\mathbf{R},u) &= \int_{\mathbb{R}^2} d\mathbf{P}\, f_L(\mathbf{R},\mathbf{P},u), \\ \mathbf{J}_L(\mathbf{R},u) &= \int_{\mathbb{R}^2} d\mathbf{P}(\mathbf{P}/m\gamma(\mathbf{P}))f_L(\mathbf{R},\mathbf{P},u).\end{aligned}$$

Self Consistent Vlasov-Maxwell Treatment for Sheet Beam in Beam Frame I

Beam frame Frenet-Serret coordinates defined in terms of the reference orbit $\mathbf{R}_r(s) = (Z_r(s), X_r(s))^T$ in the $Y = 0$ plane.

Phase space transformation $(\mathbf{R}, \mathbf{P}; u) \to (s, p_s, x, p_x; u)$

$$\mathbf{R} = \mathbf{R}_r(s) + x\mathbf{n}(s), \quad \mathbf{P} = P_r(p_s\mathbf{t}(s) + p_x\mathbf{n}(s)).$$

Lab to beam transformation steps

$$(\mathbf{R}, \mathbf{P}; u) \to (s, p_s, x, p_x; u) \to (u, p_s, x, p_x; s) \to (z, p_z, x, p_x; s),$$

where $z := s - \beta_r u$ and $p_z := (\gamma - \gamma_r)/\gamma_r$.

Exact relation between lab, f_L, and beam, f_B, phase space densities

$$f_B(\mathbf{r}, \mathbf{p}; s) = \frac{P_r^2}{\beta_r^2} f_L\{\mathbf{R}_r(s) + x\mathbf{n}(s), P_r[p_s(\mathbf{p})\mathbf{t}(s) + p_x\mathbf{n}(s)]; (s - z)/\beta_r\}.$$

Self Consistent Vlasov-Maxwell Treatment for Sheet Beam in Beam Frame II

Approximate beam frame equations of motion

$$\begin{aligned} z' &= -\kappa(s)x, \qquad p_z' = F_{z1}(z,x;s) + p_z F_{z2}(z,x;s), \\ x' &= p_x, \qquad\qquad p_x' = \kappa(s)p_z + F_x(z,x;s), \end{aligned}$$

where the self forces are

$$\begin{aligned} F_{z1} &= \frac{q}{P_r c}\mathbf{E}_\parallel(\mathbf{R}(s,x);\frac{s-z}{\beta_r})\cdot\mathbf{t}(s),\ F_{z2} = \frac{q}{P_r c}\mathbf{E}_\parallel(\mathbf{R}(s,x);\frac{s-z}{\beta_r})\cdot\mathbf{n}(s), \\ F_x &= \frac{q}{P_r c}[\mathbf{E}_\parallel(\mathbf{R}(s,x);\frac{s-z}{\beta_r})\cdot\mathbf{n}(s) - cB_Y(\mathbf{R}(s,x);\frac{s-z}{\beta_r})]. \end{aligned}$$

The associated Vlasov IVP for the evolution of the beam frame phase space density

$$\begin{aligned} &\partial_s f_B + \mathbf{r}'\cdot\nabla_{\mathbf{r}} f_B + \mathbf{p}'\cdot\nabla_{\mathbf{p}} f_B = 0, \\ &f_B(\mathbf{r},\mathbf{p};0) = f_{B0}(\mathbf{r},\mathbf{p}). \end{aligned}$$

Beam to Lab Density Transformations

- To solve Maxwell equations in lab frame must express lab frame charge/current density in terms of beam frame phase space density
- To a very good approximation

$$\rho_B(\mathbf{r}; s) \approx \rho_L(\mathbf{R}_r(s) + x\mathbf{n}(s); (s-z)/\beta_r),$$

thus

$$\rho_L(\mathbf{R}; u) \approx \rho_B(s(\mathbf{R}) - \beta_r u, x(\mathbf{R}); s(\mathbf{R})).$$

Replacing s by $\beta_r u + z$ and expanding in z gives $\rho_L(\mathbf{R}_r(\beta_r u) + M(\beta_r u)\mathbf{r}; u) \approx \rho_B(\mathbf{r}; \beta_r u + z)$. Finally, inverting gives (and similarly for $\mathbf{J}_L$)

$$\begin{aligned} \rho_L(\mathbf{R}; u) &\approx \rho_B(\mathbf{r}(\mathbf{R}, u); s(\mathbf{R}, u)), \\ \mathbf{J}_L(\mathbf{R}, u) &\approx \beta_r c\{\rho_B[s(\mathbf{R}) - \beta_r u, x(\mathbf{R}); s(\mathbf{R})]\mathbf{t}(s(\mathbf{R})) \\ &+ \tau_B[s(\mathbf{R}) - \beta_r u, x(\mathbf{R}); s(\mathbf{R})]\mathbf{n}(s(\mathbf{R}))\}, \end{aligned}$$

where $\tau_B(\mathbf{r}, s) = \beta_r \int_{\mathbb{R}^2} p_x f_B(\mathbf{r}, \mathbf{p}; s) d\mathbf{p}$.

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Field in Terms of Beam Frame Density and Causality Issue

- To solve the beam frame equations of motion we need (ignoring shielding)

$$\mathcal{F}_L(\mathbf{R}_r(s)+x\mathbf{n}(s),(s-z)/\beta_r) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\acute{\mathbf{R}}\frac{\mathbf{S}[\acute{\mathbf{R}};(s-z)/\beta_r-|\acute{\mathbf{R}}-\mathbf{R}_r(s)-x\mathbf{n}(s)|]}{|\acute{\mathbf{R}}-\mathbf{R}_r(s)-x\mathbf{n}(s)|}.$$

To compute this we need $\rho_L[\acute{\mathbf{R}};(s-z)/\beta_r-|\acute{\mathbf{R}}-\mathbf{R}_r(s)-x\mathbf{n}(s)|]$, as $\acute{\mathbf{R}}$ varies over the support of ρ_L in $\mathbb{R}^2$, given $\rho_B(\cdot;\acute{s})$ for $0\le\acute{s}\le s$.

There is a causality issue here, since the calculation of ρ_L requires values of ρ_B for $\acute{s}$ slightly outside the range $0\le\acute{s}\le s$.
This issue can be easily resolved with the following slowly varying approximation

$$f_B(\mathbf{r},\mathbf{p};s)\approx f_B(\mathbf{r},\mathbf{p};s+\Delta),$$

where Δ is of the order of the bunch size.

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Field Calculation: Polar Coordinates (at Present)

- Transform to polar coordinates (χ, θ), and then take the temporal argument v in place of the radial coordinate χ: make the transformation $\acute{\mathbf{R}} \to (\theta, v)$ via

$$\acute{\mathbf{R}} - \mathbf{R} = \chi \mathbf{e}(\theta) \ , \quad \mathbf{e}(\theta) = (\cos\theta, \sin\theta)^T \ , \quad v = u - [\chi^2 + (kh)^2]^{1/2} \ .$$

This conveniently gets rid of the integrable singularity, giving the field simply as an integral over the source (ignoring shielding)

$$\mathcal{F}_L(\mathbf{R}_r(s) + M(s)\mathbf{r}, s/\beta_r) = -\frac{1}{4\pi} \int_{u_i}^{s/\beta_r} dv \int_{\theta_m}^{\theta_M} d\theta \ \mathbf{S}[\tilde{\mathbf{R}}(\theta, v; \mathbf{r}, s), v],$$

where $\tilde{\mathbf{R}}(\theta, v; \mathbf{r}, s) = \mathbf{R}_r(s) + M(s)\mathbf{r} + (s/\beta_r - v)\mathbf{e}(\theta)$

- θ integration: superconvergent trapezoidal rule (localization in θ for $v \ll s/\beta_r$)
- v integration: adaptive Gauss-Kronrod rule (non uniform behaviour in v)

The computational effort is $O(N_z N_x N_v N_\theta)$, where N_z and N_x are the number of grid points in z ans x respectively, N_v is the number of evaluations for the $v-$integration, and N_θ is the number of evaluations for the $\theta-$integration.
For $N_z = 1000$, $N_x = 128$, $N_v = N_\theta = 1000$, $O(N_z N_x N_v N_\theta) = O(10^{12})$.

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Field Calculation: Search for Improvement

- Polar coordinates

  Optimize subroutine to find support of θ integration

  For v close to s/β_r, make a change of variable to avoid an adaptive integrator

- Beam frame coordinates (a natural coordinate system to consider)

  Let $\mathbf{R}(\tau) := \mathbf{R}_r(s) + x\mathbf{n}(s)$, where $\tau = (x, s)^T$. Then $\acute{\mathbf{R}} \to \acute{\tau}$ via $\acute{\mathbf{R}} = \mathbf{R}(\acute{\tau})$ gives

$$\mathcal{F}_L(\mathbf{R}_r(s) + x\mathbf{n}(s), (s-z)/\beta_r) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} d\acute{\tau}\frac{1+\acute{x}\kappa(\acute{s})}{|\mathbf{R}(\acute{\tau})-\mathbf{R}(\tau)|}\mathbf{S}(\mathbf{R}(\acute{\tau}); \frac{s-z}{\beta_r} - |\mathbf{R}(\acute{\tau})-\mathbf{R}(\tau)|).$$

Focusing on $\mathbf{E}_{||}$, the nonsingular part of the integrand is

$$\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}(\mathbf{R}(\acute{\tau}); \frac{s-z}{\beta_r} - |\mathbf{R}(\acute{\tau})-\mathbf{R}(\tau)|)(1+\acute{x}\kappa(\acute{s})) = QZ_0c\{[(\frac{1}{\gamma_r^2} -\beta_r^2\acute{x}\kappa(\acute{s}))D_1\rho_B() + D_3\rho_B()]\mathbf{t}(\acute{s}) + (1+\acute{x}\kappa(\acute{s}))[D_2\rho_B() + D_1\tau_B()]\mathbf{n}(\acute{s})\},$$

where $() = (\acute{z}, \acute{x}; \acute{s})$ and $\acute{z} = \acute{s} - s + z - \beta_r|\mathbf{R}(\acute{\tau}) - \mathbf{R}(\tau)|$.

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Density Estimation: Orthogonal Series Method (at Present)

- From scattered beam frame points at s $\longrightarrow$ smooth/global lab frame charge/current density via a 2D Fourier method.

  1D Example: 1D orthogonal series estimator of $f(x)$, $x \in [0,1]$

$$f_J(x) := \sum_{j=0}^{J} \theta_j \phi_j(x), \quad \theta_j = \int_0^1 \phi_j(x) f(x) dx, \quad \phi_0(x) = 1, \phi_j(x) = \sqrt{2}\cos(j\pi x), j = 1, 2, ...$$

  Since $f(x)$ is a probability density (X,X_n random variables distributed via f)

$$\theta_j = E\{\phi_j(X)\}, \quad \text{thus from Monte Carlo a natural estimate is} \quad \hat{\theta}_j := \frac{1}{N}\sum_{n=1}^{N} \phi_j(X_n).$$

- The computational effort is $O(\mathcal{N} J_z J_x)$, where $\mathcal{N}$ is the number of simulated particles, J_z and J_x the number of Fourier coefficients in z and x respectively. For $\mathcal{N} = 5 \times 10^8$, $J_z = 150$ and $J_x = 50$, $O(\mathcal{N} J_z J_x) = O(10^{12})$.

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Density Estimation: Search for Improvement

- Cloud in cell charge deposition followed by computation of the Fourier coefficients of the truncated Fourier series by a simple quadrature (implemented).

  The computational effort is $O(\mathcal{N}) + O(N_z N_x J_z J_x)$, where $\mathcal{N}$ is the number of simulated particles, N_z and N_x are the number of grid points in z ans x respectively, and J_z and J_x the number of Fourier coefficients in z and x respectively.
  For $Nz = 1000$, $Nx = 128$, $J_z = 150$ and $J_x = 50$, $O(N_z N_x J_z J_x) = O(10^9)$.

- Kernel density estimation using standard kernels like bivariate Gaussians or bivariate compact support kernels (e.g. Epanechnikov kernels).
  The computational effort is $O(\mathcal{N}\tilde{N}_z\tilde{N}_x)$, where $\mathcal{N}$ is the number of simulated particles and $\tilde{N}_z\tilde{N}_x$ is the number of grid points inside the circle of radius h (band width) centered at the scattered particle position z, x.
  For $\mathcal{N} = 5 \times 10^8$ and $\tilde{N}_z = \tilde{N}_x = 4$, $O(\mathcal{N}\tilde{N}_z\tilde{N}_x) = O(10^{10})$.

- Wavelets-denoising (G. Bassi, B. Terzić, PAC09), (implemented).
  The computational effort $O(MN_zN_x)$, where M is the with of the wavelet family, is comparable to the computational effort for charge deposition + Fourier method.

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Interaction Picture

- Interaction picture to isolate CSR dynamics.

From $F_z = F_x = 0 \implies \zeta = \Phi(s|0)\zeta_0$
$\therefore \zeta_0' = \Phi(0|s)F, \qquad F = (0, F_z, 0, F_x),$
where $F_z = F_{z1} + F_{z2}$.

- In component form

$$\begin{aligned} z_0' &= -R_{56}(s)F_z - D(s)F_x, & p_{z0}' &= F_z, \\ x_0' &= (sD'(s) - D(s))F_z - sF_x, & p_{x0}' &= -D'(s)F_z + F_x, \end{aligned}$$

where D and R_{56} are standard lattice functions.

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Numerical Results: FERMI@Elettra First Bunch Compressor *

Table 1: Chicane parameters and beam parameters at first dipole

Parameter	Symbol	Value	Unit
Energy reference particle	E_r	233	MeV
Peak current	I	120	A
Bunch charge	Q	1	nC
Norm. transverse emittance	$\gamma\epsilon_0$	1	μm
Alpha function	α_0	0	
Beta function	β_0	10	m
Linear energy chirp	h	-12.6	1/m
Uncorrelated energy spread	σ_E	2	KeV
Momentum compaction	R_{56}	0.057	m
Radius of curvature	ρ_0	5	m
Magnetic length	L_b	0.5	m
Distance 1st-2nd, 3rd-4th bend	L_1	2.5	m
Distance 2rd-3nd bend	L_2	1	m

* G. Bassi, J.A. Ellison, K. Heinemann and R. Warnock, PRSTAB **12**, 080704 (2009)

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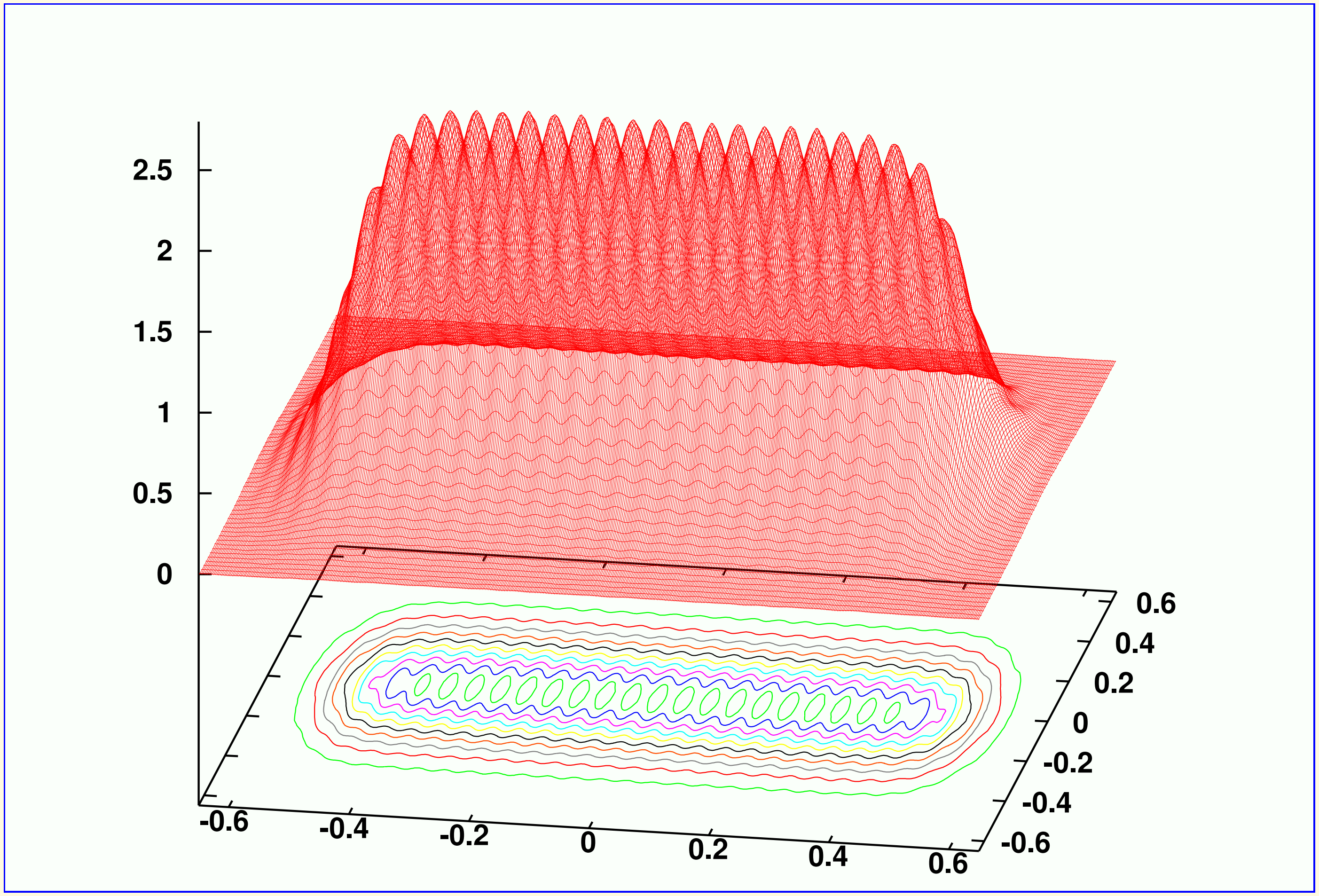
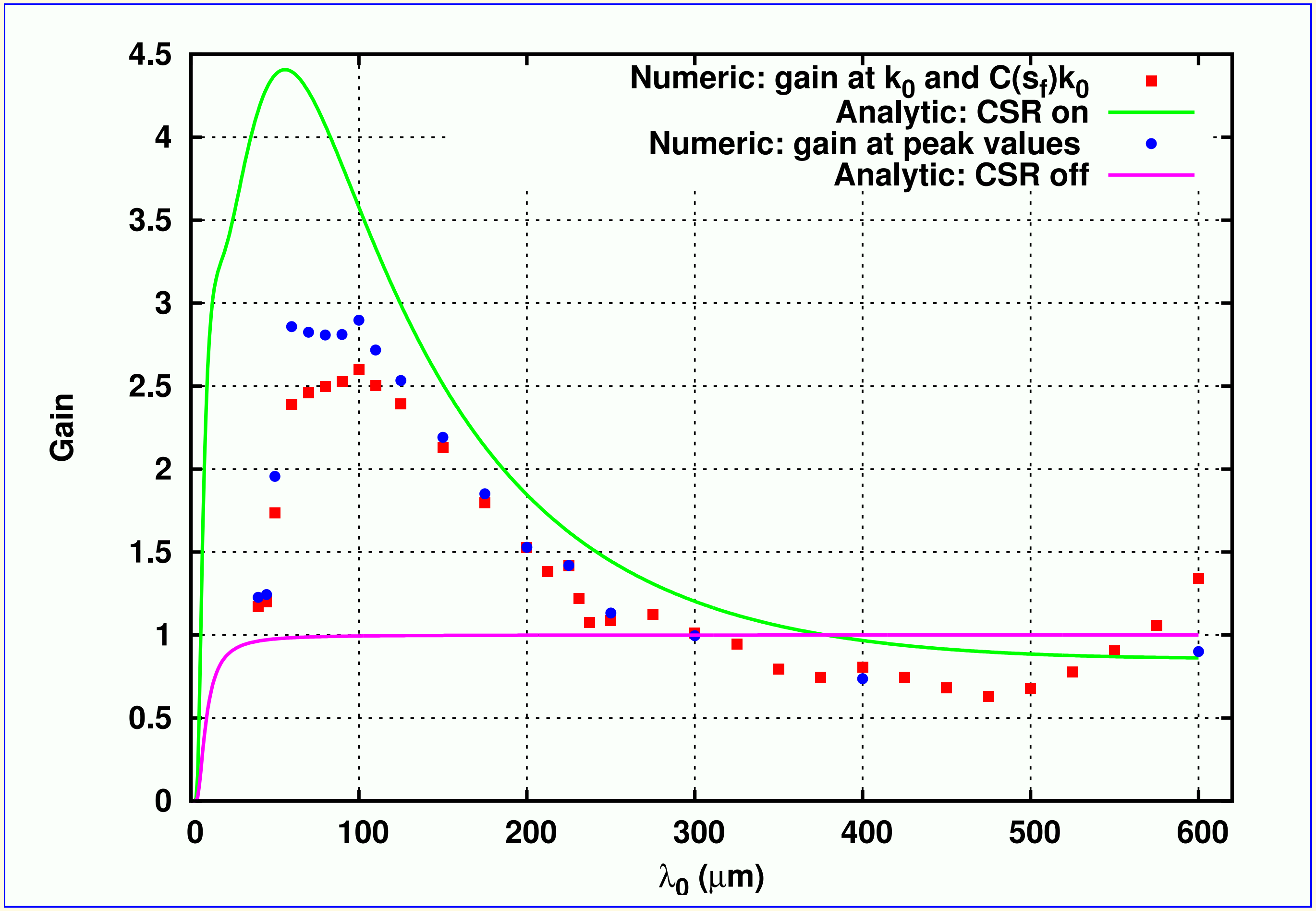
Initial 2D Spatial Density



Initial spatial density in grid coordinates for A=0.05, $\lambda_0 = 100\mu$m.

Init. phase space density $= (1 + A\cos(2\pi z/\lambda_0))\mu(z)\rho_c(p_z - hz)g(x, p_x)$.

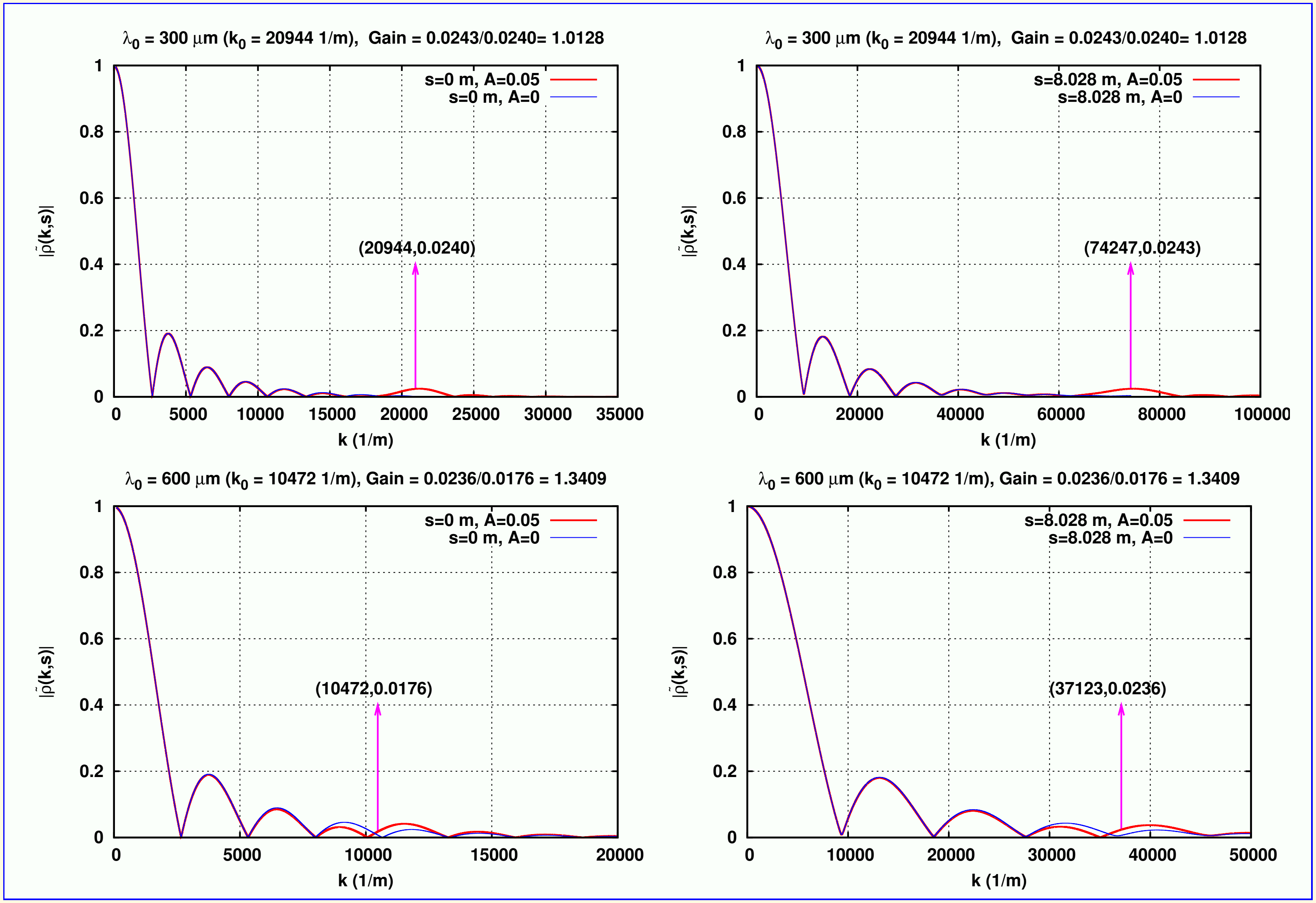
Gain factor



Gain $:= |\tilde{\rho}(k_f, s_f)/\tilde{\rho}(k_0, 0)|$, $\tilde{\rho}(k,s) = \int dz \exp(-ikz)\rho(z,s)$ and $k_f = C(s_f)k_0$ for $\lambda_0 = 2\pi/k_0$. Here $C(s_f) = 1/(1 + hR_{56}(s_f)) = 3.54$, $s_f = 8.029$m.

H. Huang and K.Kim, PRSTAB **5**, 074401, 129903 (2002); S.Heifets, G.Stupakov and S.Krinsky, PRSTAB **5**, 064401 (2002), G. Bassi, J.A. Ellison, K. Heinemann and R. Warnock, PRSTAB **12**, 080704 (2009).

Spectra Longitudinal Density I



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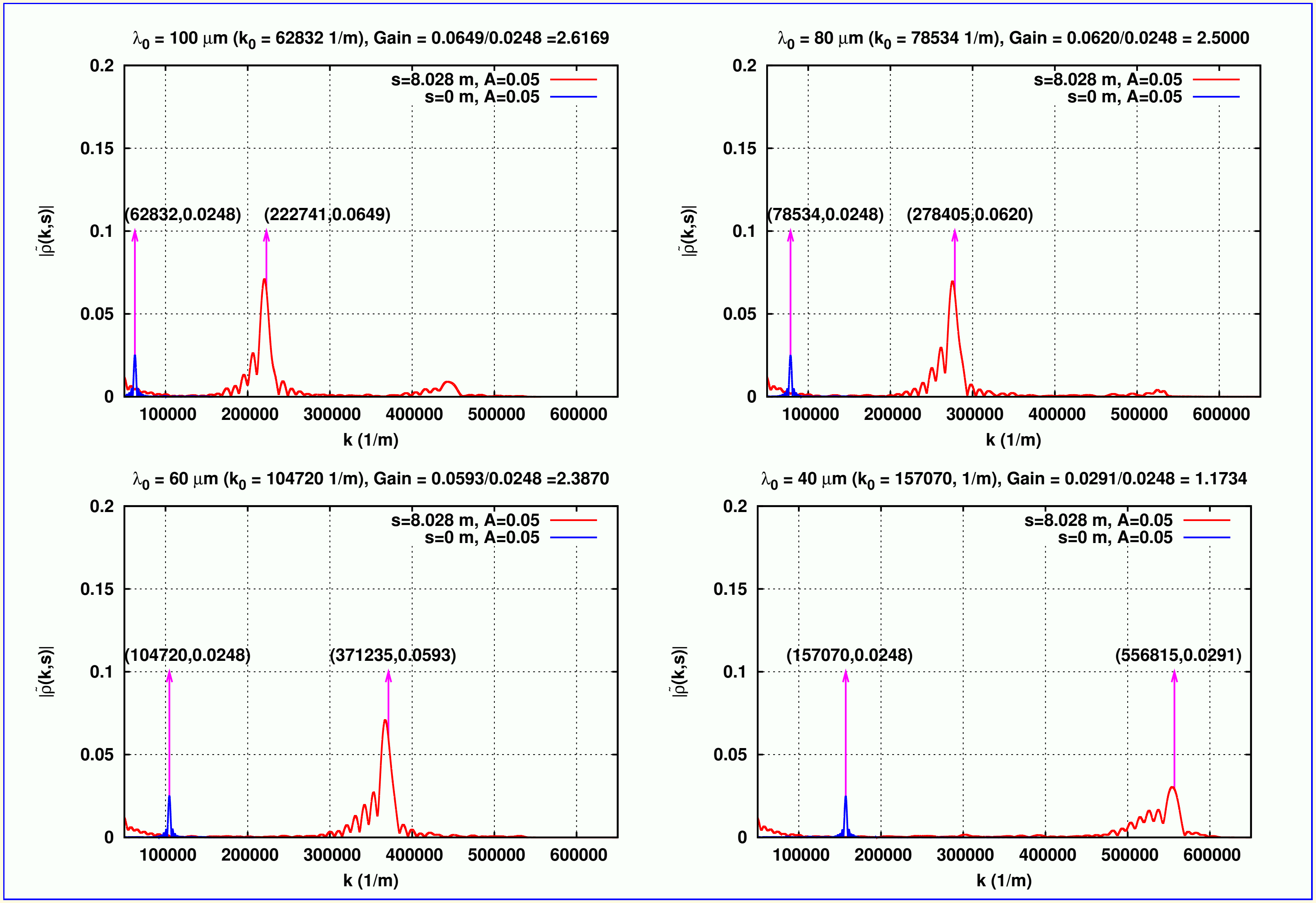
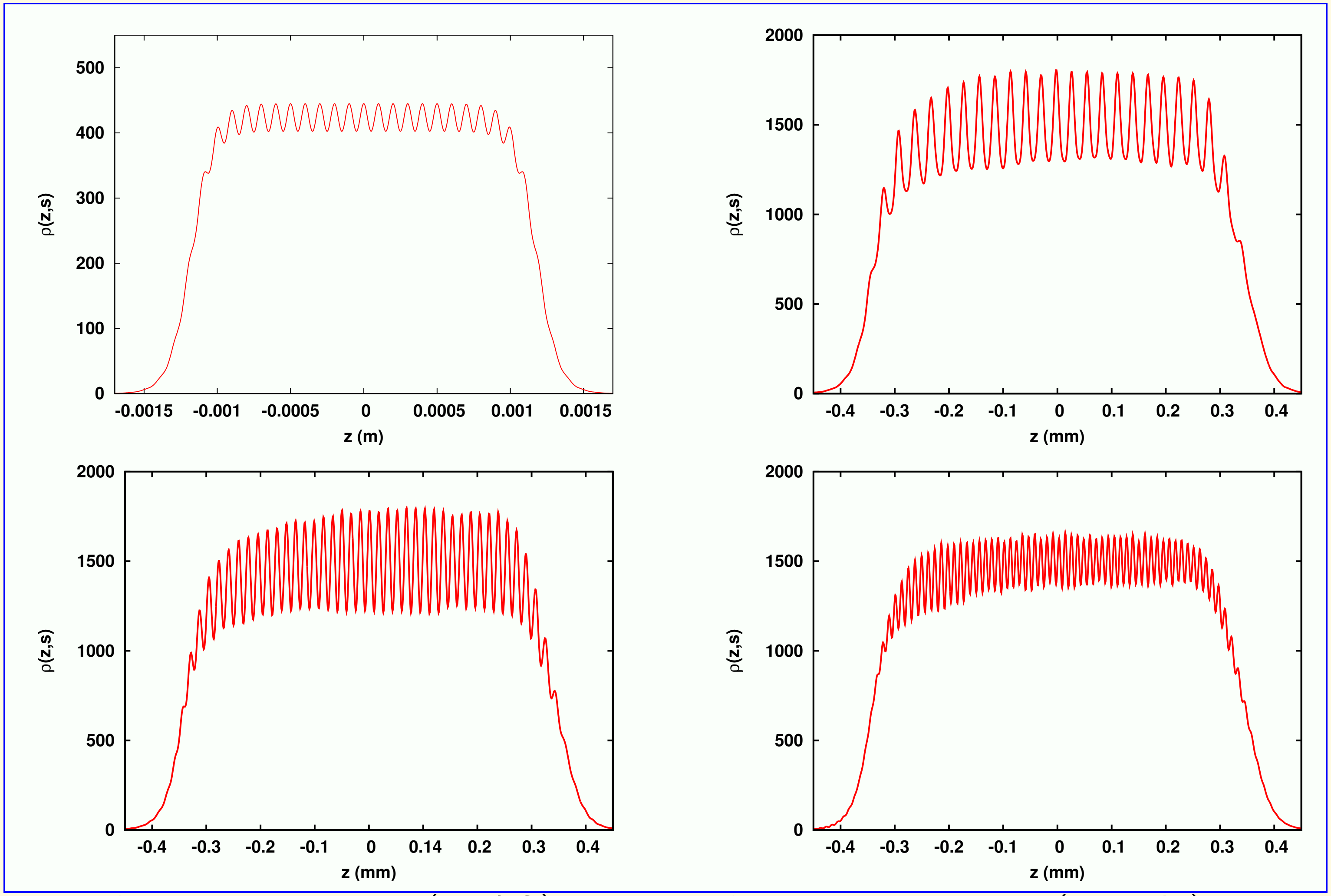
Spectra Longitudinal Density II

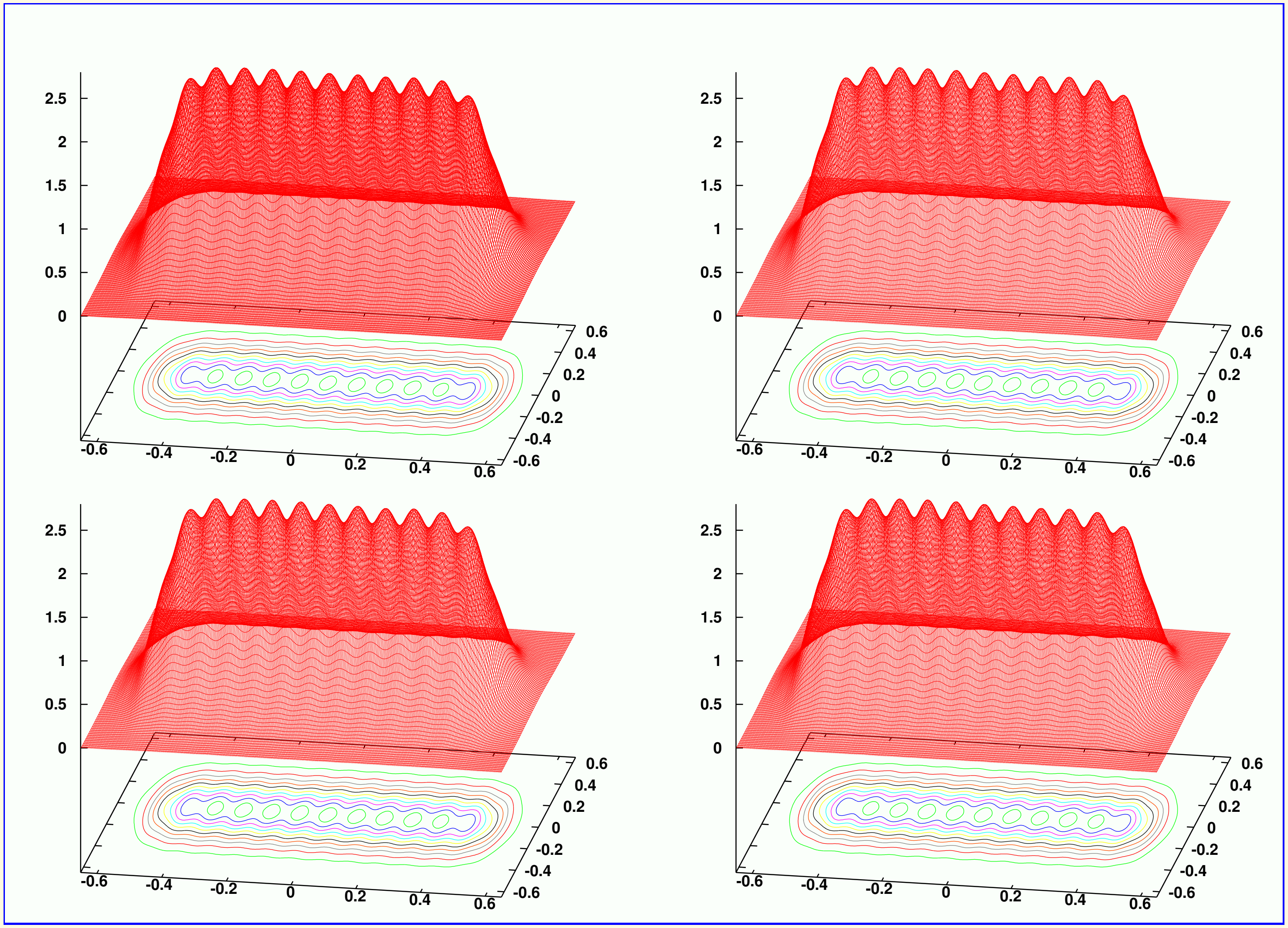


Longitudinal Density



$\lambda_0 = 100\mu$m at $s = 0$ (top left), $\lambda_0 = 100\mu$m at $s = s_f$ (top right),
$\lambda_0 = 60\mu$m at $s = s_f$ (bottom left), $\lambda_0 = 40\mu$m at $s = s_f$ (bottom right).

Stationary Grid



λ_0=200μm. s=0.25s_f (top left), s=0.5s_f (top right), s=0.75s_f (bottom left), s=s_f (bottom right).

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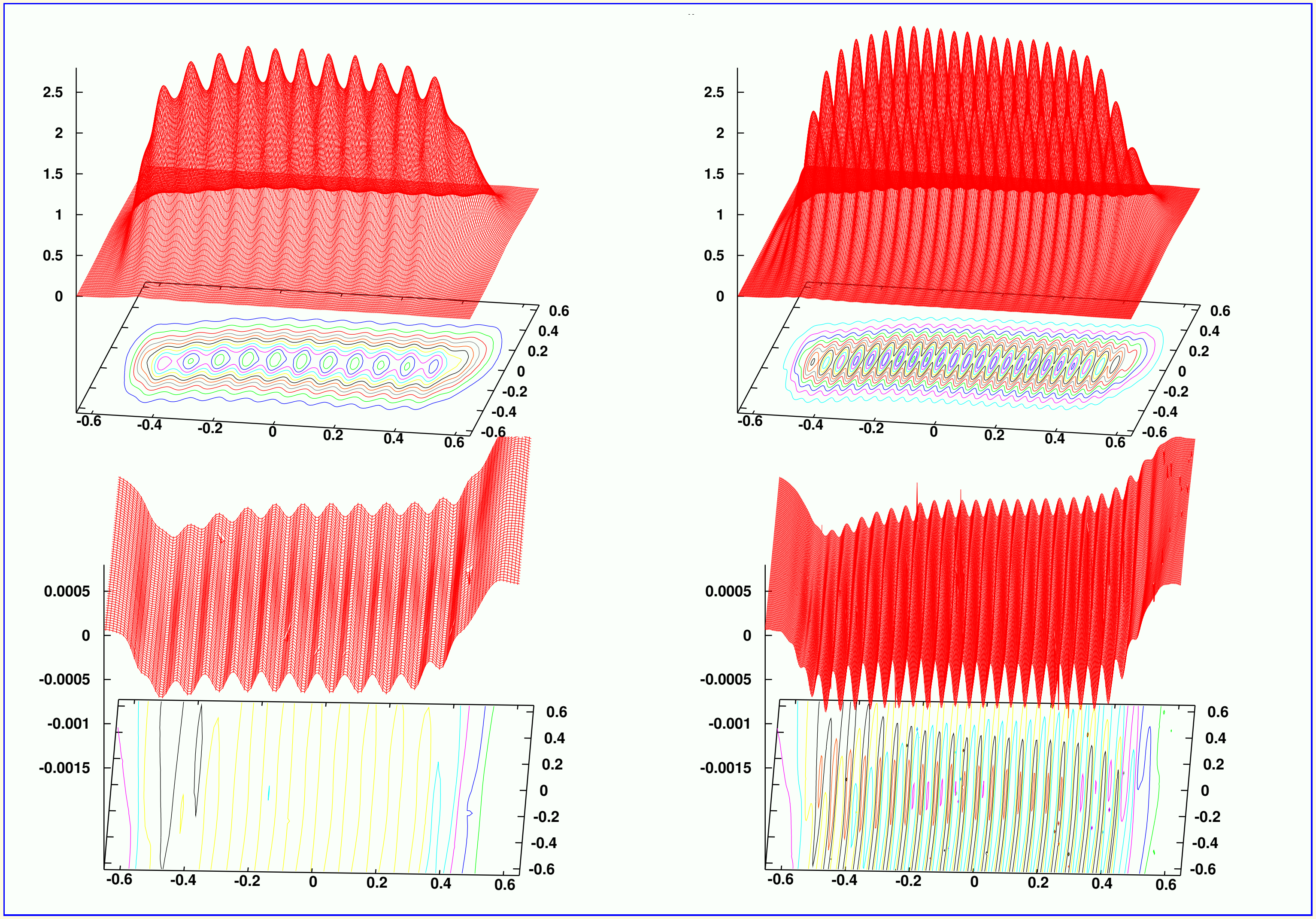
2D spatial density and longitudinal force at $s = s_f$



$\lambda_0 = 200\mu$m (top left), $\lambda_0 = 100\mu$m (top right), $\lambda_0 = 200\mu$m (bottom left), $\lambda_0 = 100\mu$m (bottom right)

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Discussion

- FERMI@Elettra microbunching studies at $\lambda_0 \geq 40\mu$m:
  - Very small effect of μBI on mean power and transverse emittance
  - Gain factor at long wavelengths shows breakdown coasting beam assumption
  - Gain factor at short wavelengths indicates deviations from analytical gain formula
- Work in progress and future work:
  - Study wavelengths shorter than $\lambda_0 = 40\mu$m
  - Study dependence on the amplitude of the initial modulation and on the uncorrelated energy spread
  - Study initial perturbation with more than one frequency
  - Study energy modulations

Computational Issues

- Intensive memory requirement and expensive computational cost:

  Typical simulations done on the parallel clusters ENCANTO in New Mexico and NERSC at LBNL: N procs = 200-1000, N particles = 2×10^7-5×10^8, few hours of CPU time

  Memory requirement: for $\lambda_0 = 50\mu$m store 3D array of dimension $1500 \times 128 \times 200$ on master processor (to avoid massive communications between slave processors)

- To reduce storage/computational cost:

  Analytical work + state of the art numerical techniques:
  - integration
  - interpolation
  - density estimation

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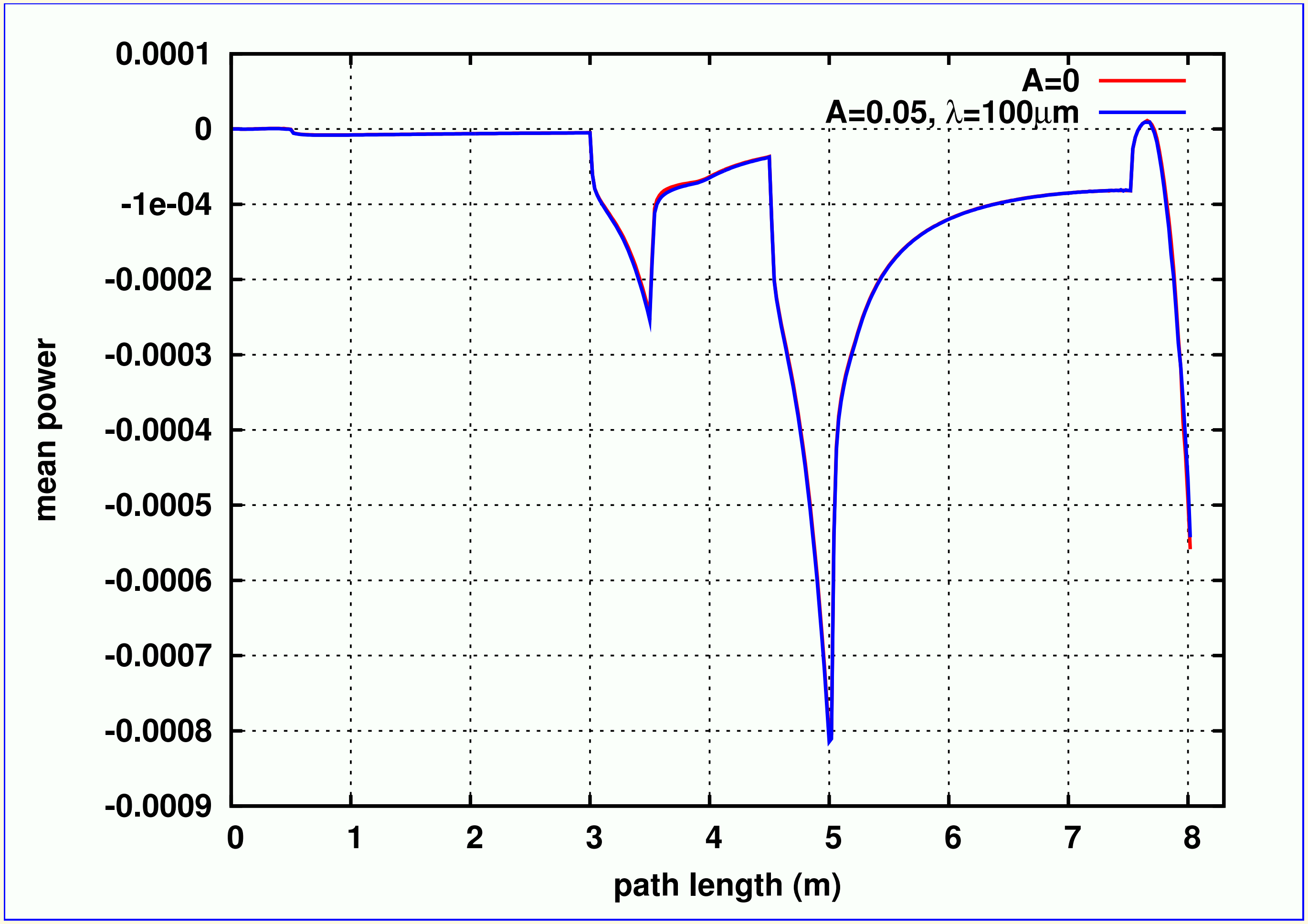
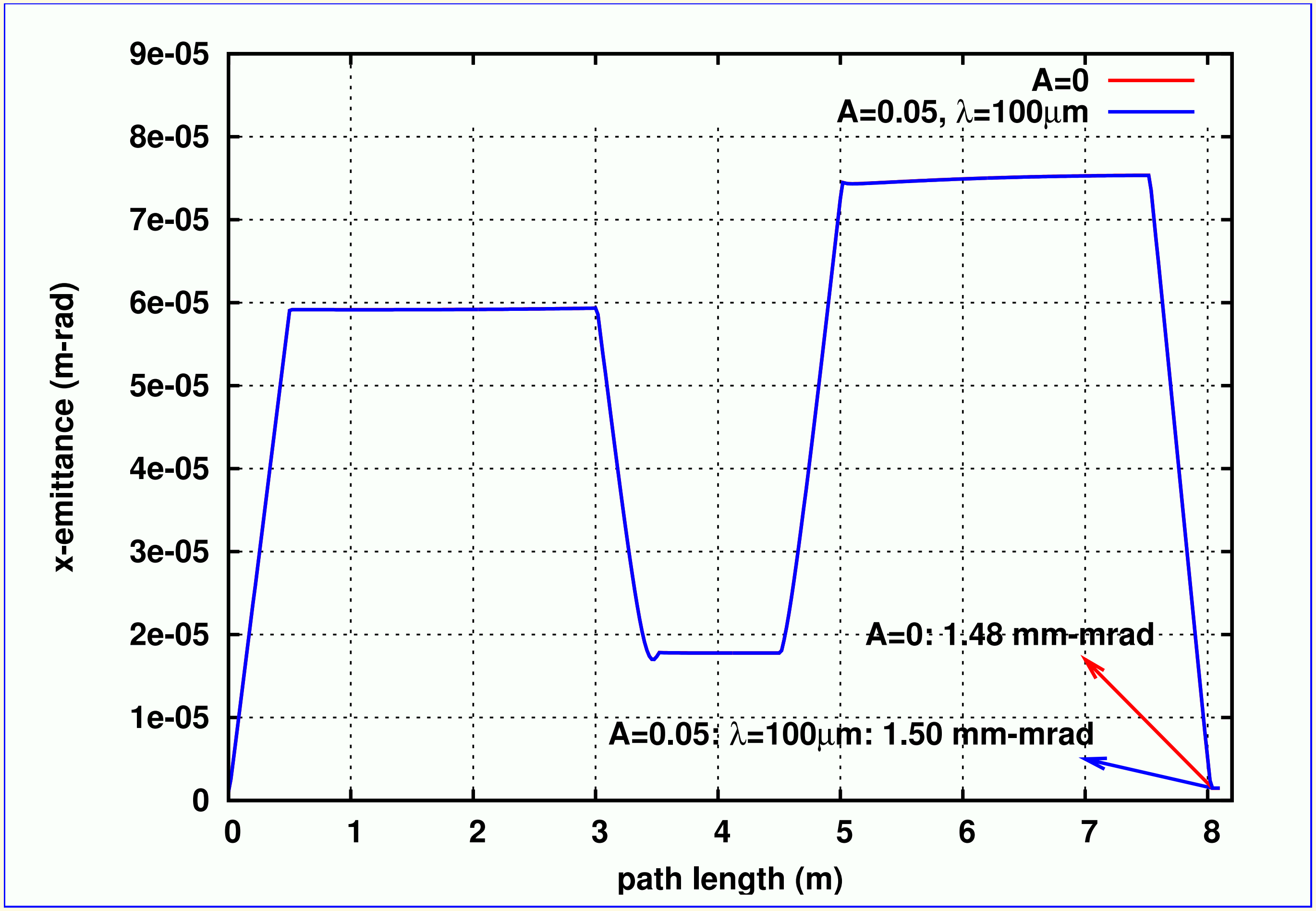
FERMI@Elettra First Bunch Compressor II



Mean power

FERMI@Elettra First Bunch Compressor III



x-emittance

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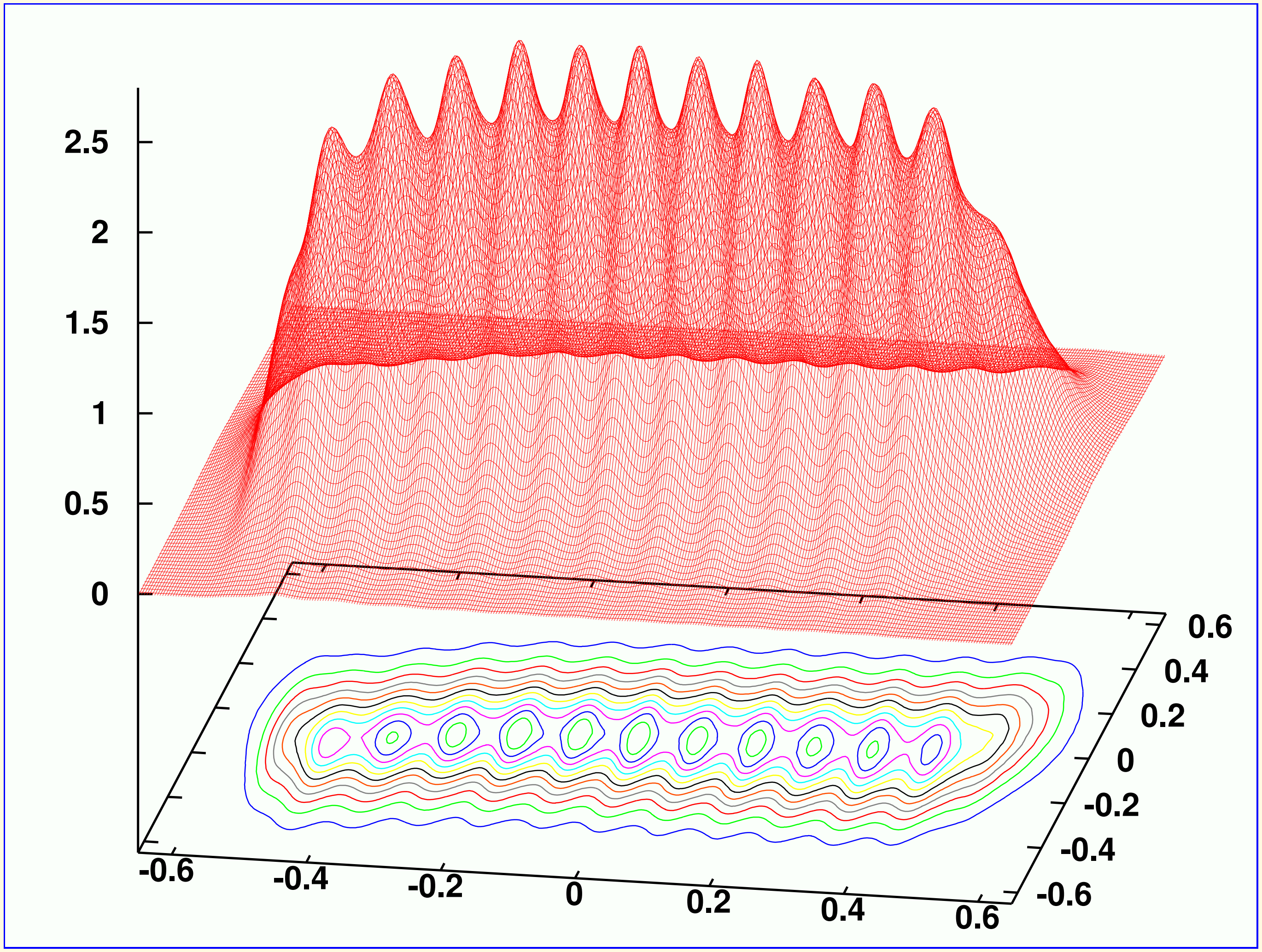
2D spatial density in grid coordinates at $s = s_f$ for $\lambda_0 = 200\mu$m



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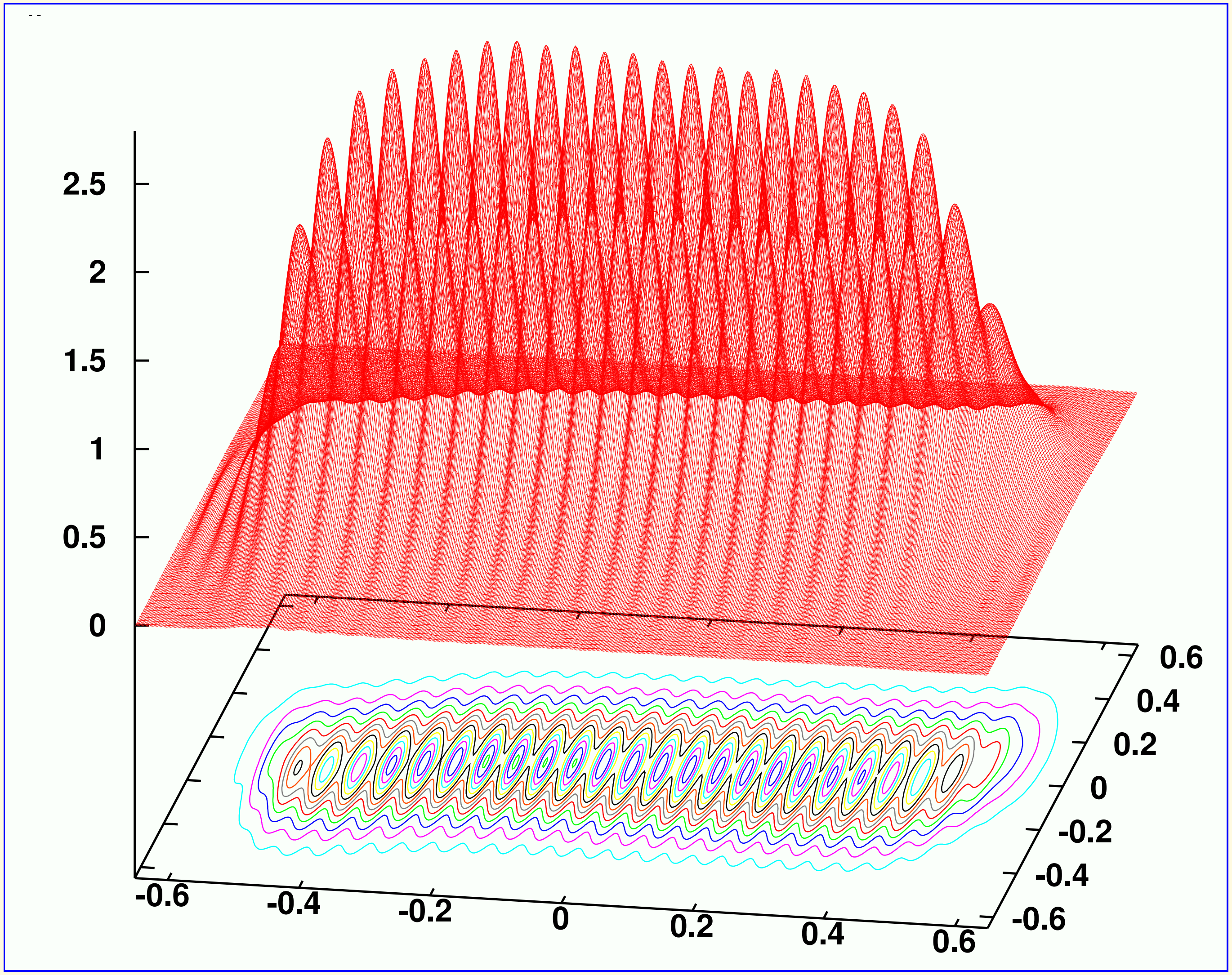
2D spatial density in grid coordinates at $s = s_f$ for $\lambda_0 = 100\mu$m



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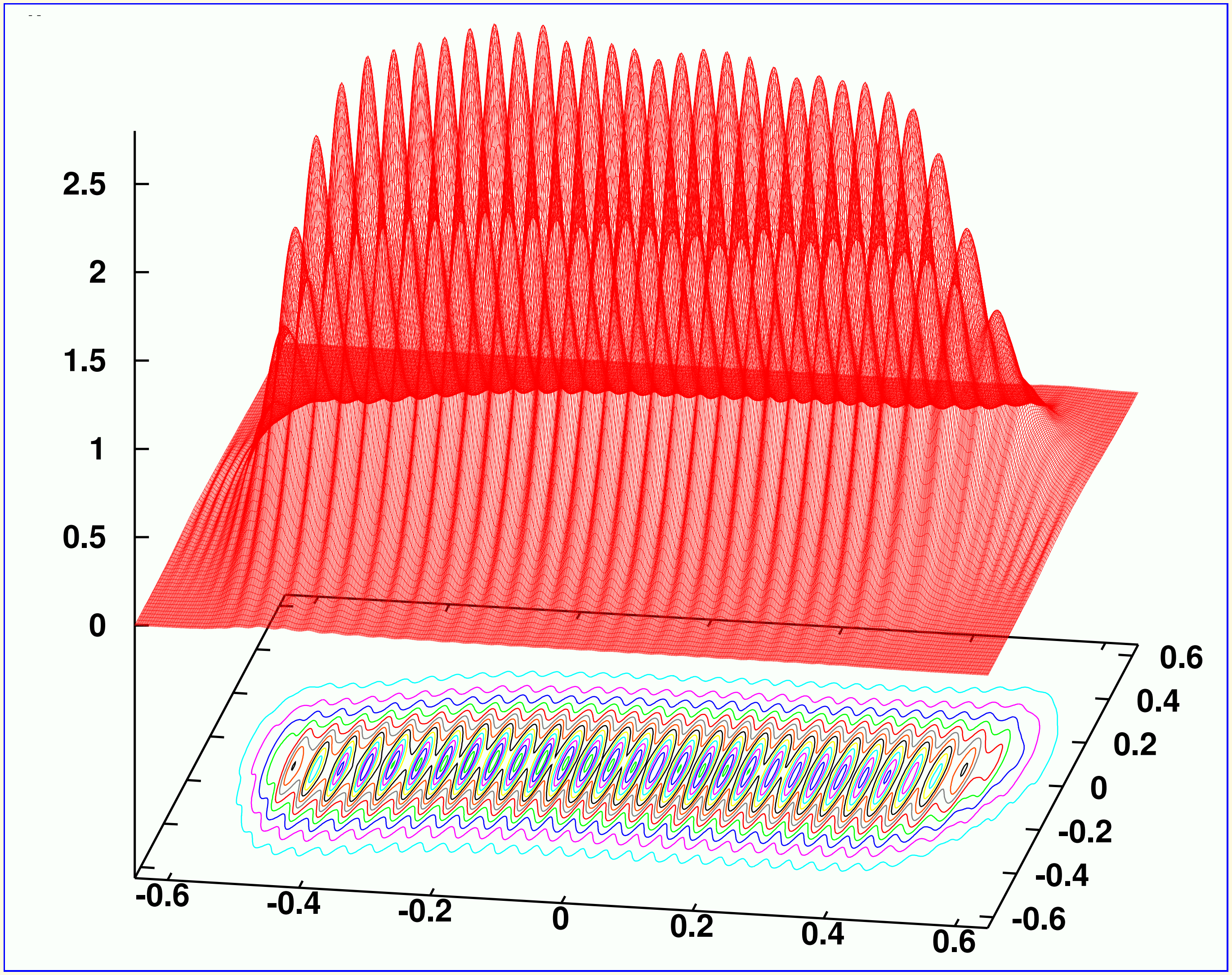
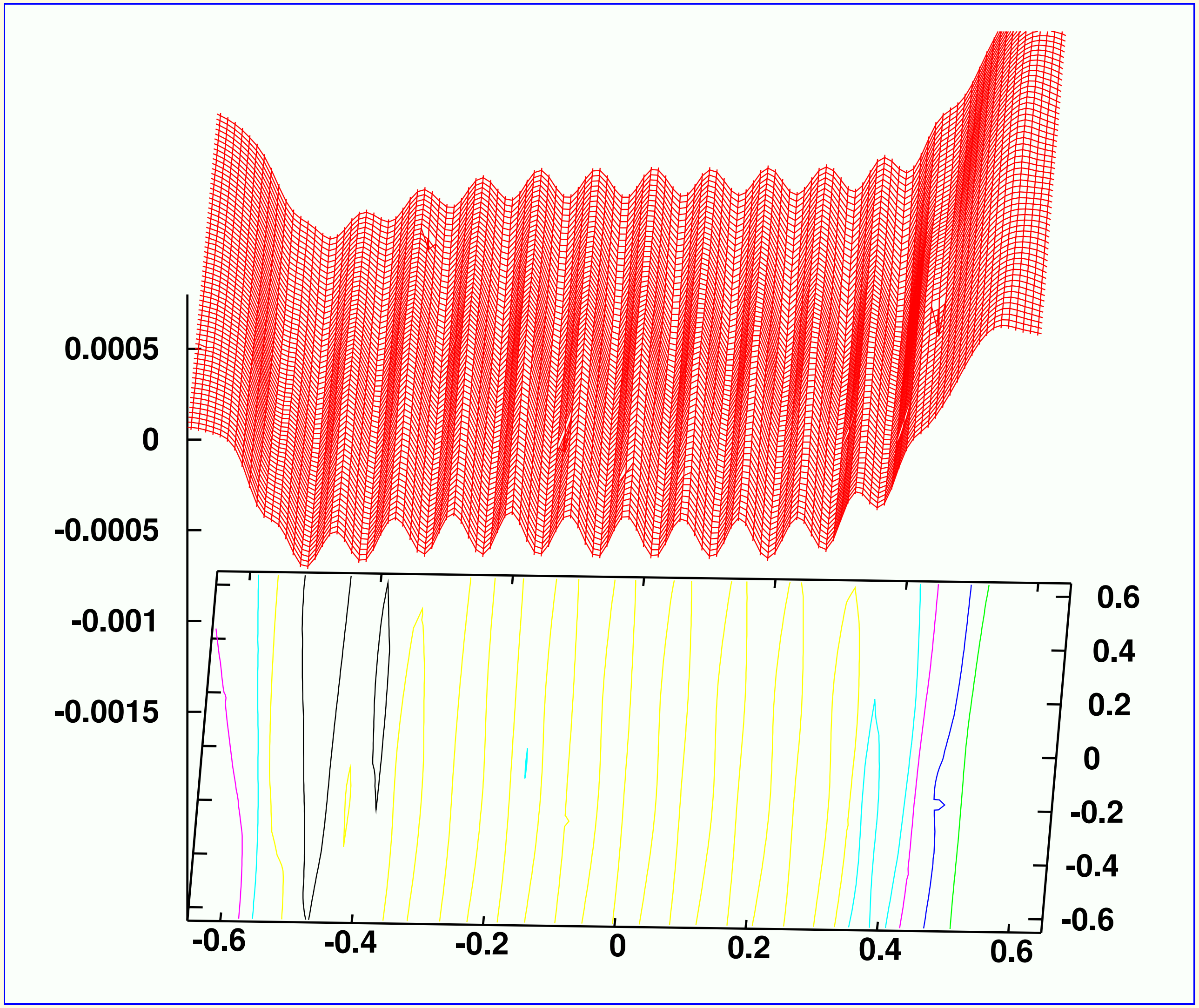
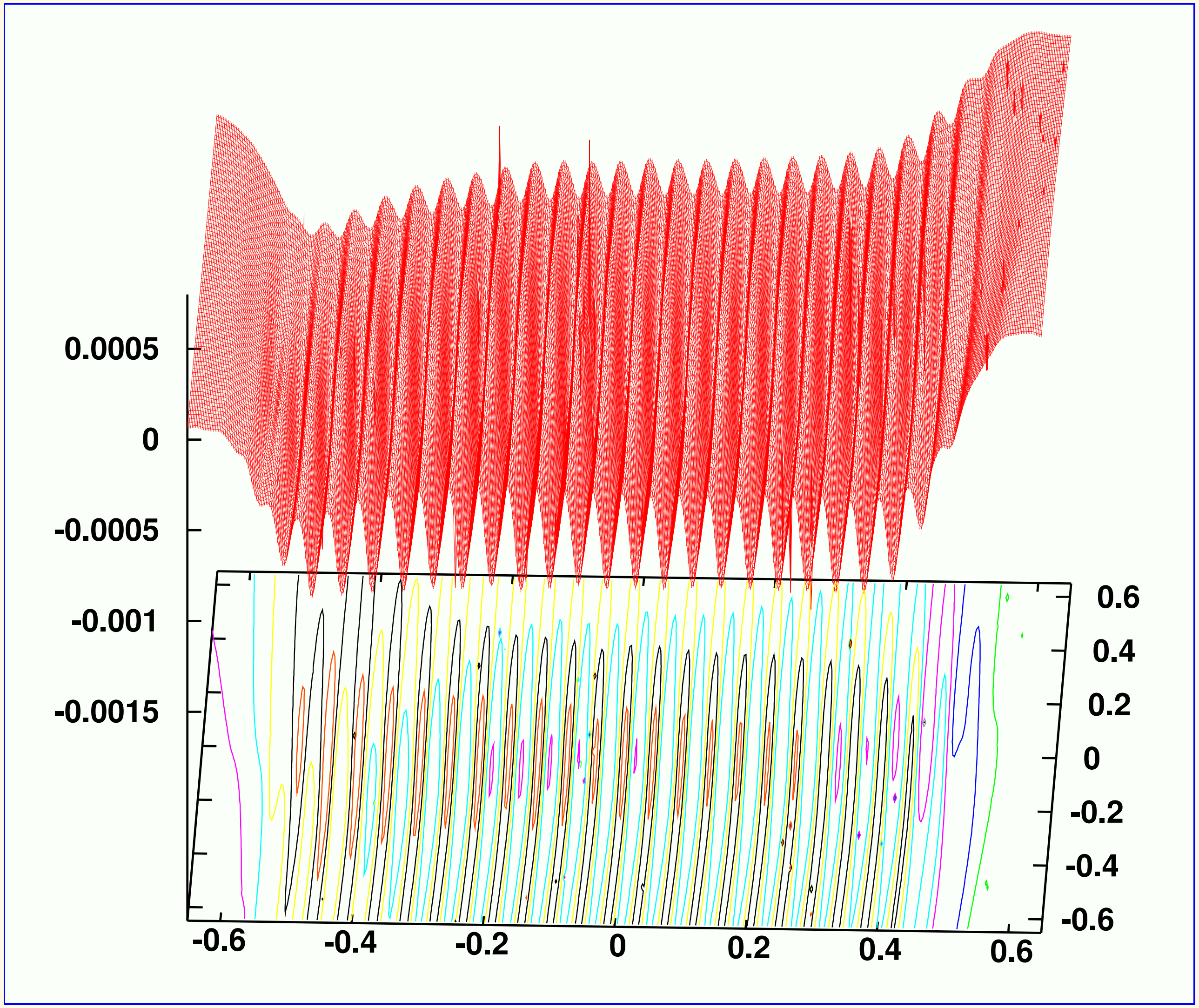
2D spatial density in grid coordinates at $s = s_f$ for $\lambda_0 = 80\mu$m



Longitudinal force in grid coordinates at $s = s_f$ for $\lambda_0 = 200\mu$m



Longitudinal force in grid coordinates at $s = s_f$ for $\lambda_0 = 100\mu$m



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Longitudinal force in grid coordinates at $s = s_f$ for $\lambda_0 = 80\mu$m