THEORY AND APPLICATIONS OF INTRABEAM SCATTERING

Sekazi K. Mtingwa
North Carolina A&T State University

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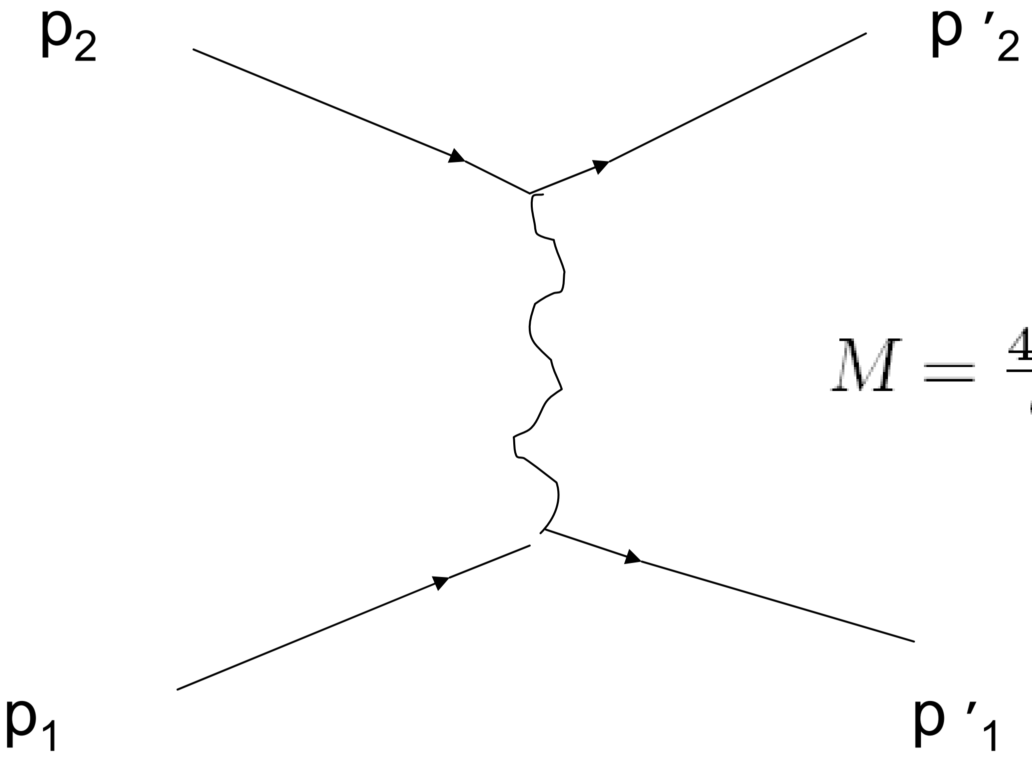
Basic Framework

Fermi's Golden Rule

The relativistic "Golden Rule" for the transition rate due to a 2-body scattering process $p_1+p_2 \rightarrow p_1'+p_2'$ can be written[2]

$$\frac{d\mathcal{P}}{dt} = \frac{1}{2}\int d^3x \, \frac{d^3p_1}{\gamma_1} \frac{d^3p_2}{\gamma_2} \, \rho(x,p_1)\,\rho(x,p_2)\,|\mathcal{M}|^2 \cdot \frac{d^3p_1'}{\gamma_1'} \frac{d^3p_2'}{\gamma_2'} \frac{\delta^{(4)}(p_1'+p_2'-p_1-p_2)}{(2\pi)^2}$$

Invariant Matrix Amplitude
(Evaluate Feynman Diagram)



$$M = \frac{4\pi\alpha}{q^2}$$

Rate of Change of Emittances or other Functions of Momentum

$$\frac{d\langle f(p)\rangle}{dt} = \frac{N}{2\Gamma^2}\int d^3x \, \frac{d^3p_1}{\gamma_1}\,\frac{d^3p_2}{\gamma_2}\; e^{-S(x,p_1)-S(x,p_2)}$$

$$\cdot\; |\mathcal{M}|^2\left[f(p_1') - f(p_1) + f(p_2') - f(p_2)\right]$$

$$\cdot\; \frac{d^3p_1'}{\gamma_1'}\,\frac{d^3p_2'}{\gamma_2'}\;\frac{\delta^{(4)}(p_1' + p_2' - p_1 - p_2)}{(2\pi)^2}$$

Phase Space Distribution

$$\rho(x,p) = \frac{N}{\Gamma} e^{-S(x,p)}$$

$$S(x,p) = \frac{1}{2} A_{ij}\, \delta p_i\, \delta p_j + B_{ij}\, \delta p_i\, \delta x_j + \frac{1}{2} C_{ij}\, \delta x_i\, \delta x_j$$

Total Phase Space Volume

$$\Gamma = \int d^3x \, d^3p \; e^{-S(x,p)}$$

$$S(x,p) = S^{(h)} + S^{(v)} + S^{(\ell)}$$

Horizontal and Vertical Phase Space Components

$$S^{(h)} = \frac{\beta_x}{2 \cdot \varepsilon_x} x_\beta'^2 - \frac{\beta_x'}{2\varepsilon_x} x_\beta x_\beta' + \frac{1}{2\varepsilon_x \beta_x}\left(1 + \frac{\beta_x'^2}{4}\right) x_\beta^2$$

$$S^{(v)} = \frac{\beta_z}{2\varepsilon_z} z'^2 - \frac{\beta_z' z z'}{2\varepsilon_z} + \frac{z^2}{2\varepsilon_z \beta_z}$$

Longitudinal Phase Space Component

$$S^{(\ell)} = \begin{cases} \dfrac{\delta^2}{2\sigma_\eta^2} & \text{(unbunched beam)} \\[2ex] \dfrac{\delta^2}{\sigma_\eta^2} + \dfrac{(s-\bar{s})^2}{2\sigma_s^2} & \text{(bunched beam)} \end{cases}$$

Emittances

$$\varepsilon_x = \frac{\sigma_x^2}{\beta_x}$$

$$\varepsilon_z = \frac{\sigma_z^2}{\beta_z}$$

$$\sigma_l = \frac{\sigma_p}{\bar{p}}$$

Angles, Momentum Spread

$$x' \equiv \frac{\delta p_x}{\bar{p}}$$

$$z' \equiv \frac{\delta p_z}{\bar{p}}$$

$$\delta \equiv \frac{\delta p_y}{\bar{p}}$$

Summary of IBS Theory

K. Bane Proc. of EPAC 2002, Paris

Define

$$\frac{1}{T_p} \equiv \frac{1}{\sigma_p} \frac{d\sigma_p}{dt},$$

$$\frac{1}{T_h} \equiv \frac{1}{\varepsilon_h^{1/2}} \frac{d\varepsilon_h^{1/2}}{dt},$$

$$\frac{1}{T_v} \equiv \frac{1}{\varepsilon_v^{1/2}} \frac{d\varepsilon_v^{1/2}}{dt},$$

$$\frac{1}{T_i} = 4\pi A(\log)\left\langle \int_0^\infty d\lambda \frac{\lambda^{1/2}}{[\det(L+\lambda I)]^{1/2}} \times \left\{ \mathrm{Tr}L^i \, \mathrm{Tr}\left(\frac{1}{L+\lambda I}\right) - 3\,\mathrm{Tr}\left[L^i\left(\frac{1}{L+\lambda I}\right)\right]\right\}\right\rangle$$

where i represents p, h, or v and

DEFINITIONS

$$A = \frac{r_0^2 c N}{64 \pi^2 \beta^3 \gamma^4 \varepsilon_h \varepsilon_v \sigma_s \sigma_p},$$

$$L = L^{(p)} + L^{(h)} + L^{(v)},$$

$$L^{(p)} = \frac{\gamma^2}{\sigma_p^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L^{(h)} = \frac{\beta_h}{\varepsilon_h} \begin{pmatrix} 1 & -\gamma \phi_h & 0 \\ -\gamma \phi_h & \frac{\gamma^2 \mathcal{H}_h}{\beta_h} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L^{(v)} = \frac{\beta_v}{\varepsilon_v} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\gamma^2 \mathcal{H}_v}{\beta_v} & -\gamma \phi_v \\ 0 & -\gamma \phi_v & 1 \end{pmatrix}$$

MORE DEFINITIONS

$$\mathcal{H}_h = [\eta_h^2 + (\beta_h \eta_h' - \tfrac{1}{2}\beta_h' \eta_h)^2]/\beta_h$$

$$\phi_h = \eta_h' - \tfrac{1}{2}\beta_h' \eta_h / \beta_h$$

Same for v

$$\frac{1}{\sigma_H^2} = \frac{1}{\sigma_p^2} + \frac{\mathcal{H}_h}{\varepsilon_h} + \frac{\mathcal{H}_v}{\varepsilon_v},$$

$$a = \frac{\sigma_H}{\gamma}\sqrt{\beta_h/\varepsilon_h},$$

$$b = \frac{\sigma_H}{\gamma}\sqrt{\beta_v/\varepsilon_v},$$

BANE'S HIGH ENERGY APPROX. TO IBS THEORY (Proc. of EPAC 2002, Paris)

$$\frac{1}{T_p} \approx \frac{r_0^2 c N (\log)}{16\gamma^3 \varepsilon_h^{3/4} \varepsilon_v^{3/4} \sigma_s \sigma_p^3} \left\langle \sigma_H g_{\text{bane}}\left(\frac{a}{b}\right) (\beta_h \beta_v)^{-1/4} \right\rangle,$$

$$\frac{1}{T_{h,v}} \approx \frac{\sigma_p^2 \langle \mathcal{H}_{h,v} \rangle}{\varepsilon_{h,v}} \frac{1}{T_p},$$

with

$$g_{\text{bane}}(\alpha) = \frac{2\sqrt{\alpha}}{\pi} \int_0^\infty \frac{du}{\sqrt{1+u^2}\sqrt{\alpha^2+u^2}}.$$

PIWINSKI THEORY

$$\frac{1}{T_p} = A\left\langle \frac{\sigma_h^2}{\sigma_p^2} f(\tilde{a}, \tilde{b}, \tilde{q}) \right\rangle,$$

$$\frac{1}{T_h} = A\left\langle f\left(\frac{1}{\tilde{a}}, \frac{\tilde{b}}{\tilde{a}}, \frac{\tilde{q}}{\tilde{a}}\right) + \frac{\eta_h^2 \sigma_h^2}{\beta_h \varepsilon_h} f(\tilde{a}, \tilde{b}, \tilde{q}) \right\rangle,$$

$$\frac{1}{T_v} = A\left\langle f\left(\frac{1}{\tilde{b}}, \frac{\tilde{a}}{\tilde{b}}, \frac{\tilde{q}}{\tilde{b}}\right) + \frac{\eta_v^2 \sigma_h^2}{\beta_v \varepsilon_v} f(\tilde{a}, \tilde{b}, \tilde{q}) \right\rangle,$$

$$\frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{\eta_h^2}{\beta_h \varepsilon_h} + \frac{\eta_v^2}{\beta_v \varepsilon_v},$$

$$\tilde{a} = \frac{\sigma_h}{\gamma} \sqrt{\beta_h / \varepsilon_h},$$

$$\tilde{b} = \frac{\sigma_h}{\gamma} \sqrt{\beta_v / \varepsilon_v},$$

$$\tilde{q} = \sigma_h \beta \sqrt{2d / r_0}.$$

PIWINSKI-EVANS-ZOTTER SCATTERING INTEGRAL

$$f(\tilde{a}, \tilde{b}, \tilde{q}) = 8\pi \int_0^1 du \frac{(1 - 3u^2)}{PQ} \times \left\{ 2 \ln\left[\frac{\tilde{q}}{2} \left(\frac{1}{P} + \frac{1}{Q} \right) \right] - 0.577 \ldots \right\},$$

with

$$P^2 = \tilde{a}^2 + (1 - \tilde{a}^2) u^2,$$

$$Q^2 = \tilde{b}^2 + (1 - \tilde{b}^2) u^2.$$

where the parameter d functions as a maximum impact parameter

and is normally taken to be the vertical beam size.

MODIFIED PIWINSKI THEORY (BANE'S ANSATZ)

Replace

$$\frac{\eta_h^2}{\beta_h} \longrightarrow \mathcal{H}_h = \left[\eta_h^2 + \left(\beta_h \eta_h' - \frac{1}{2} \beta_h' \eta_h \right)^2 \right] \Big/ \beta_h$$

Same for v

Eg.

$$\frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{\eta_h^2}{\beta_h \varepsilon_h} + \frac{\eta_v^2}{\beta_v \varepsilon_v} \longrightarrow \frac{1}{\sigma_H^2} = \frac{1}{\sigma_p^2} + \frac{\mathcal{H}_h}{\varepsilon_h} + \frac{\mathcal{H}_v}{\varepsilon_v}.$$

With these replacements, Bane was able to show the similarity between Piwinski and strong-focussing theories at high energies.

COMPLETE THE INTEGRATION

There is still one integration to be done.

To completely integrate the Modified Piwinski theory,

return to the original Piwinski scattering function

$$f(a,b,c) = 2\int_0^\infty \int_0^\pi \int_0^{2\pi} \sin\theta \; e^{-r[\cos^2\theta + (a^2\cos^2\phi \; + \; b^2\sin^2\phi)\sin^2\theta]}$$

$$\times \; \ln(c^2 r)(1 - 3\cos^2\theta) d\phi d\theta dr.$$

It satisfies the following relations:

$$f(a,b,c) = f(b,a,c)$$

$$f(a,b,c) \; + \; \frac{1}{a^2} f(\tfrac{1}{a} + \tfrac{b}{a} + \tfrac{c}{a}) \; + \; \frac{1}{b^2} f(\tfrac{1}{b} + \tfrac{a}{b} + \tfrac{c}{b}) = 0.$$

SIMPLIFY MODIFIED PIWINSKI T_p

$$\frac{1}{T_p} = A\left\langle \frac{\sigma_H^2}{\sigma_p^2} f(a, b, q) \right\rangle.$$

where

$$a = \frac{\sigma_H}{\gamma}\sqrt{\beta_h/\varepsilon_h}, \quad \Longrightarrow \quad \text{small}$$

$$b = \frac{\sigma_H}{\gamma}\sqrt{\beta_v/\varepsilon_v}, \quad \Longrightarrow \quad \text{small}$$

$$q = \sigma_H \beta\sqrt{2d/r_0}. \quad \Longrightarrow \quad \text{large} \quad \text{(Relative to } a \text{ and } b\text{)}$$

Thus, we have f(small,small,large).

EVALUATING f(a,b,q)

Rearrange arguments to evaluate only

$$f(\text{large, small, large})$$

This is done by the second Piwinski relation:

$$f(a,b,q) = -\frac{1}{a^2} f\left(\frac{1}{a}, \frac{b}{a}, \frac{q}{a}\right) - \frac{1}{b^2} f\left(\frac{1}{b}, \frac{a}{b}, \frac{q}{b}\right).$$

Thus, we now have

$$f(\text{small, small, large}) = -\frac{1}{a^2} f(\text{large, small, large}) - \frac{1}{b^2} f(\text{large, small, large}).$$

EVALUATE f(α , ω , δ)
(α , δ large while ω much smaller)

Using an old Fermilab result:

SM and Alvin Tollestrup FERMILAB-PUB-89/224 (1987)

After a series of tedious integrations,

$$f(\alpha, \omega, \delta) \approx \frac{-4\pi^{3/2} \ln\delta}{\alpha} g(\omega),$$

where $g(\omega)$ is given by the "Master Equation"

$$g(\omega) = \sqrt{\pi/\omega}\left[P^{0}_{-1/2}\left(\frac{\omega^2+1}{2\omega}\right) \pm \frac{3}{2} P^{-1}_{-1/2}\left(\frac{\omega^2+1}{2\omega}\right)\right],$$

$+$ sign for $\omega \geq 1$

- sign for $\omega \leq 1$.

Note: $g(\omega) \rightarrow \sqrt{(\pi)}$ as $\omega \rightarrow 1$ from above or below.

COMPLETELY INTEGRATED MODIFIED PIWINSKI (CIMP)

Finally,

$$\frac{1}{T_p} \approx A2\pi^{3/2}\left\langle\frac{\sigma_H^2}{\sigma_p^2}\left(\frac{\ln(\frac{q^2}{a^2})g(\frac{b}{a})}{a}+\frac{\ln(\frac{q^2}{b^2})g(\frac{a}{b})}{b}\right)\right\rangle,$$

$$\frac{1}{T_h} \approx A2\pi^{3/2}\left\langle -a\ln\left(\frac{q^2}{a^2}\right)g\left(\frac{b}{a}\right) + \frac{\mathcal{H}_h\sigma_H^2}{\varepsilon_h}\left(\frac{\ln(\frac{q^2}{a^2})g(\frac{b}{a})}{a}+\frac{\ln(\frac{q^2}{b^2})g(\frac{a}{b})}{b}\right)\right\rangle,$$

$$\frac{1}{T_v} \approx A2\pi^{3/2}\left\langle -b\ln\left(\frac{q^2}{b^2}\right)g\left(\frac{a}{b}\right) + \frac{\mathcal{H}_v\sigma_H^2}{\varepsilon_v}\left(\frac{\ln(\frac{q^2}{a^2})g(\frac{b}{a})}{a}+\frac{\ln(\frac{q^2}{b^2})g(\frac{a}{b})}{b}\right)\right\rangle,$$

$$a = \frac{\sigma_H}{\gamma}\sqrt{\beta_h/\varepsilon_h} \qquad b = \frac{\sigma_H}{\gamma}\sqrt{\beta_v/\varepsilon_v} \qquad q = \sigma_H\beta\sqrt{2d/r_0}$$

COMMENT

With dispersion, second term in transverse eqs. dominate giving old result

$$\frac{1}{T_{h,v}} \approx \frac{\sigma_p^2 \langle \mathcal{H}_{h,v} \rangle}{\varepsilon_{h,v}} \frac{1}{T_p}$$

For zero dispersion, cannot neglect first term in transverse eqs.

In all formalisms, set

$$\text{Coulomb log} = \ln\left(\frac{q^2}{a^2}\right) \approx \ln\left[\frac{\gamma^2 \sigma_v \varepsilon_h}{r_0 \beta_h}\right].$$

FINALLY

$$\frac{1}{T_p} \approx 2\pi^{3/2} A (\log) \left\langle \frac{\sigma_H^2}{\sigma_p^2} \left(\frac{g(b/a)}{a} + \frac{g(a/b)}{b} \right) \right\rangle$$

$$\frac{1}{T_h} \approx 2\pi^{3/2} A (\log) \left\langle -a g\left(\frac{b}{a}\right) + \frac{\mathcal{H}_h \sigma_H^2}{\varepsilon_h} \left(\frac{g(b/a)}{a} + \frac{g(a/b)}{b} \right) \right\rangle$$

$$\frac{1}{T_v} \approx 2\pi^{3/2} A (\log) \left\langle -b g\left(\frac{a}{b}\right) + \frac{\mathcal{H}_v \sigma_H^2}{\varepsilon_v} \left(\frac{g(b/a)}{a} + \frac{g(a/b)}{b} \right) \right\rangle$$

$$(\log) \equiv \ln\left(\frac{q^2}{a^2}\right) \approx \ln\left[\frac{\gamma^2 \sigma_v \varepsilon_h}{r_0 \beta_h}\right]$$

PRINCIPAL CONTRIBUTIONS TO EMITTANCE GROWTH

- Betatron Coupling
  - If the vertical dispersion is negligible, but there is some H-V coupling, we expect the horizontal IBS emittance growth to feed directly into the vertical plane.
  - In the case that the vertical emittance growth is given entirely by betatron coupling, then

$$\frac{\varepsilon_v}{\varepsilon_{v0}} = \frac{\varepsilon_h}{\varepsilon_{h0}}$$

PRINCIPAL CONTRIBUTIONS TO EMITTANCE GROWTH

- Dispersion
  - Arises from the change in longitudinal momentum in a collision at a location of nonzero dispersion.
  - Leads to an effective change in the transverse coordinates of the colliding particles with respect to off-momentum orbits.
  - Generally larger than the direct contribution
  - The larger the dispersion, the faster the IBS emittance growth from this effect.

PRINCIPAL CONTRIBUTIONS TO EMITTANCE GROWTH

- “Direct” contribution
  - Comes from the fact that two particles moving with zero transverse momentum, after collision, have some nonzero H & V momentum.
  - There is emittance growth even where H & V dispersion are both zero.
  - Dominates vertical emittance growth only for ultra small emittances.
  - Bane approximation ignores this contribution

OTHER CONTRIBUTIONS TO EMITTANCE GROWTH

- Collective effects other than IBS
- Quantum excitations
  - Vertical opening angle of the radiation gives theoretical limit of vertical emittance
  - For high energy beams, contribution from opening angle is small compared to previous contributions.

Comparisons for ATF @ KEK
(no H-V Coupling, Zero Vertical dispersion)

Beam energy	1.28 GeV
Electrons per bunch	10^{10}
Horizontal emittance	1.15 nm
Vertical emittance	4.03 pm
Relative energy spread	5.47 $\times 10^{-4}$
Bunch length (rms)	5.59 mm

Approximation	$\frac{1}{T_p}$	$\frac{1}{T_h}$	$\frac{1}{T_v}$
Ref. [4]	390	267	9.10
Bane	435	291	-
CIMP	449	298	7.47

MATHEMATICA Code for Emittance vs. Bunch Charge

- First, ignore "direct" contribution
- In limit of zero bunch charge

$$\varepsilon_{v0} = \varepsilon_{v0,\eta} + \varepsilon_{v0,\kappa}$$

$$\varepsilon_{v0,\eta} \sim J_\epsilon < \mathcal{H}_v > \sigma_p^2 \quad \text{(Raubenheimer, SLAC-R-387, Thesis)}$$

$$r_\epsilon = \varepsilon_{v0,\kappa}/\varepsilon_{v0}$$

  - Finally, in computer code, we use

$$\varepsilon_v = \left[(1 - r_\varepsilon)\frac{T_v}{T_v - \tau_v} + r_\varepsilon \frac{T_h}{T_h - \tau_h}\right]\varepsilon_{v0}$$

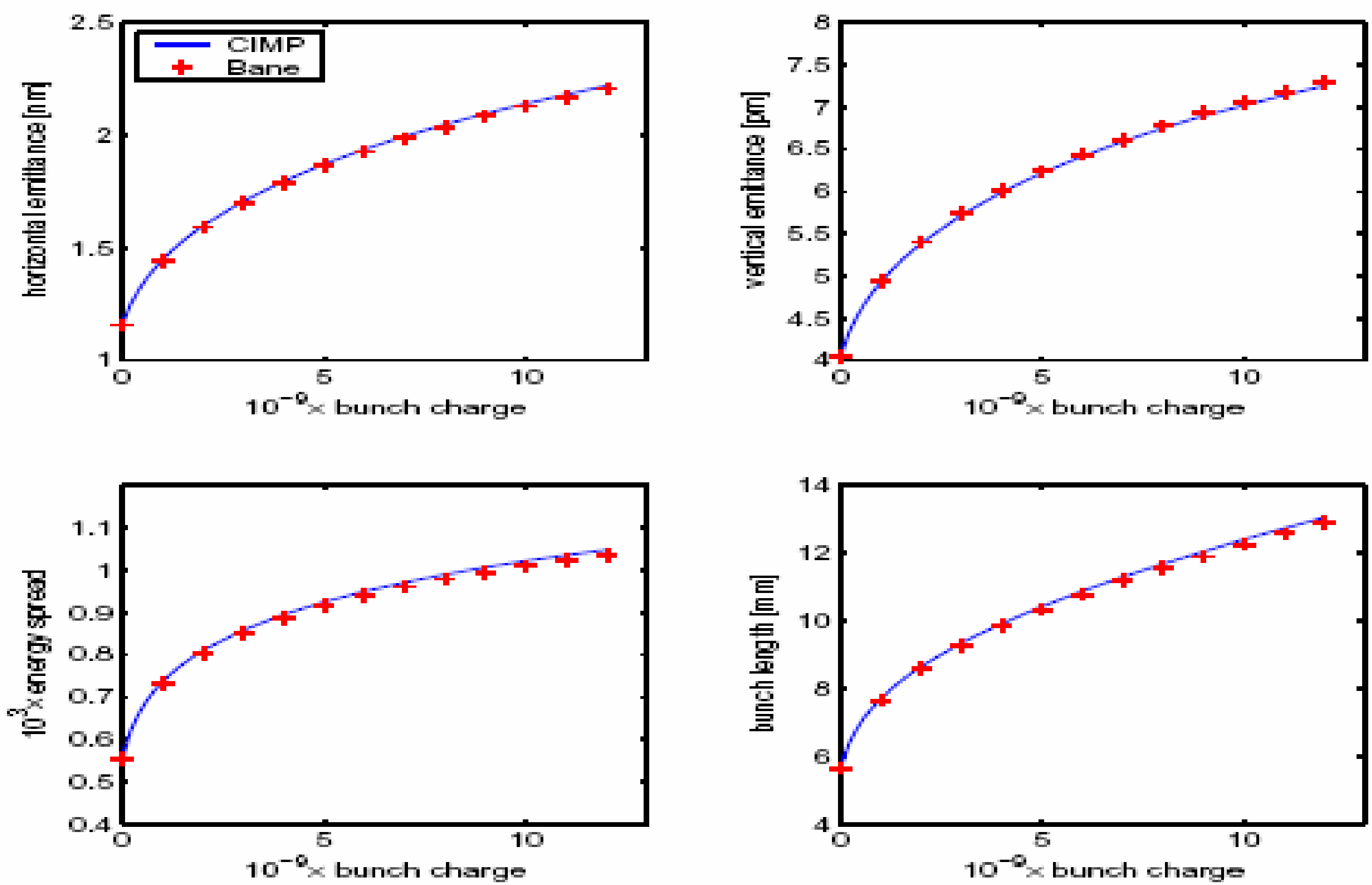
where $\tau_{h,v}$ are the transverse damping times.

Comparison of IBS Growth Rates for ATF, ILC_{Small}, $ILC_{Dogbone}$

(with H-V Coupling)

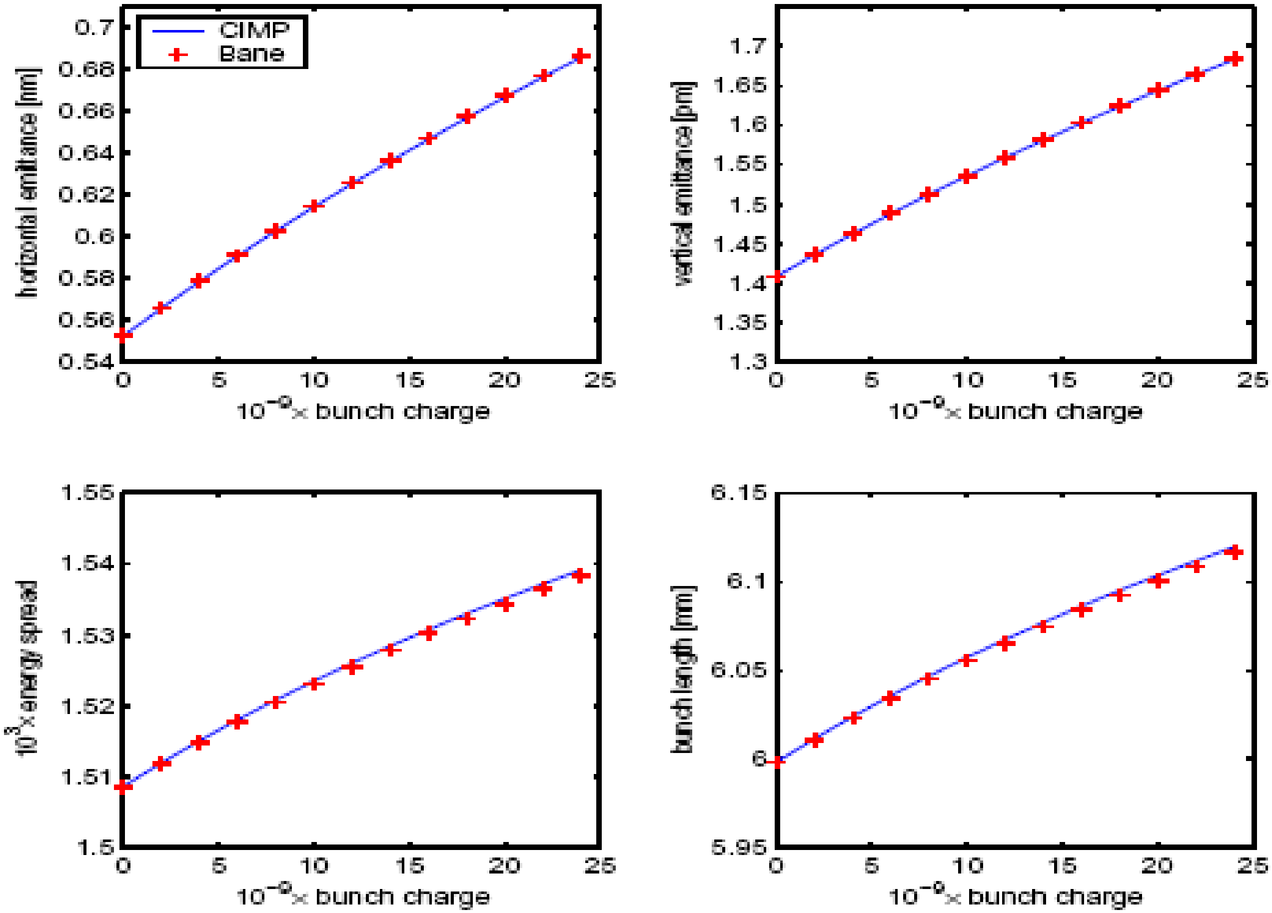
Damping Ring	ATF	ILC_{Small}	$ILC_{Dogbone}$
Beam energy (GeV)	1.28	5.066	5.00
Relative energy spread ($\times 10^{-4}$)	5.47	15.1	12.9
rms Bunch length (mm)	5.59	5.99	6.04
Horizontal emittance (nm)	1.15	0.55	0.51
Vertical emittance (pm)	4.03	1.41	1.29
Horizontal damping time (ms)	17.86	26.75	27.86
Vertical damping time (ms)	28.55	26.74	27.86
Longitudinal damping time (ms)	20.38	13.37	13.93

ATF

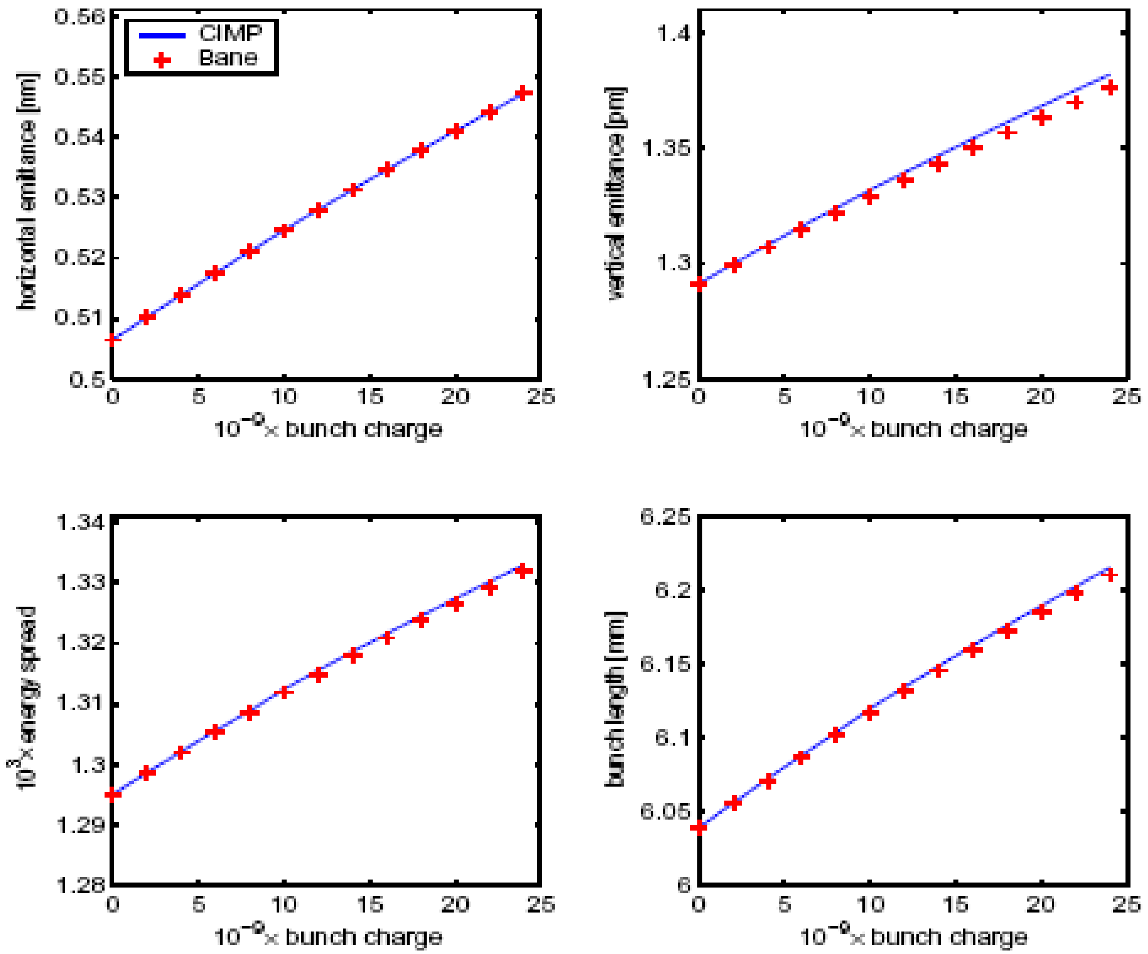


CIMP
Bane
horizontal emittance [nm]
vertical emittance [pm]
$10^3\times$ energy spread
bunch length [mm]
$10^{-9}\times$ bunch charge
$10^{-9}\times$ bunch charge
$10^{-9}\times$ bunch charge
$10^{-9}\times$ bunch charge

ILC$_{Small}$


CIMP
Bane
horizontal emittance [nm]
vertical emittance [pm]
10^3 x energy spread
bunch length [mm]
10^{-9} x bunch charge
0.7
0.68
0.66
0.64
0.62
0.6
0.58
0.56
0.54
1.7
1.65
1.6
1.55
1.5
1.45
1.4
1.35
1.3
1.55
1.54
1.53
1.52
1.51
1.5
6.15
6.1
6.05
6
5.95
0
5
10
15
20
25

ILC$_{Dogbone}$

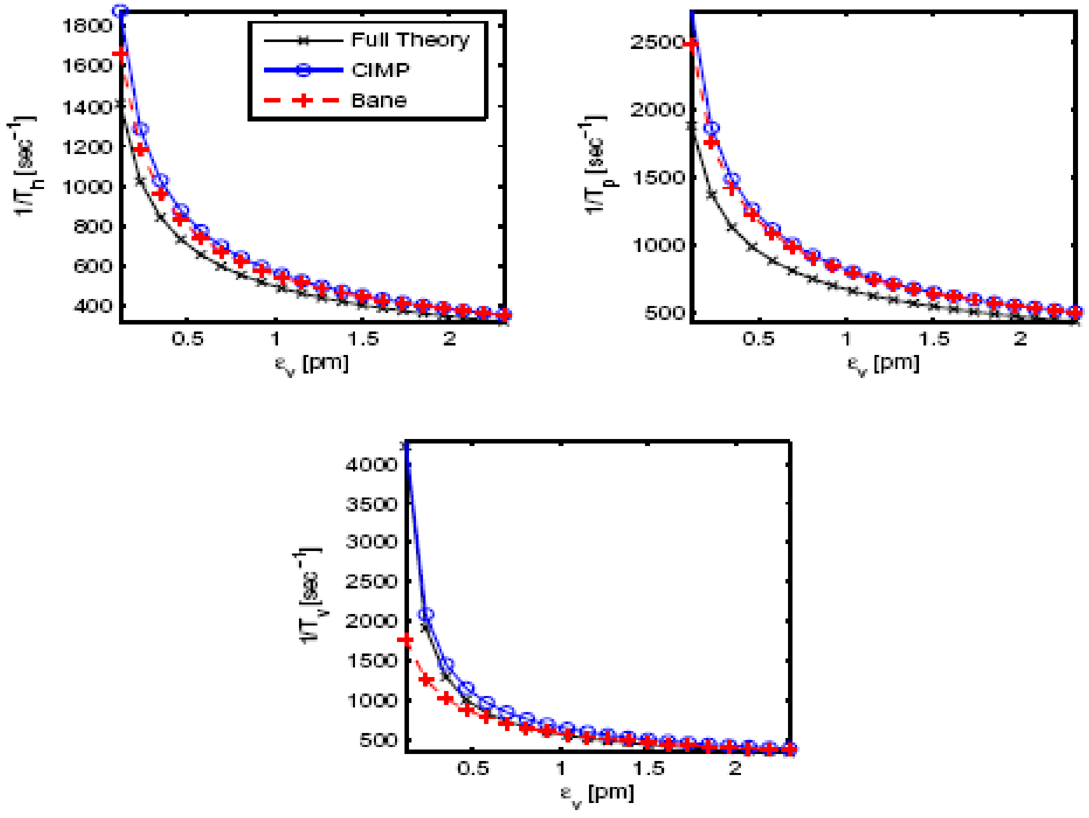

CIMP
Bane
horizontal emittance [nm]
vertical emittance [pm]
10^3 × energy spread
bunch length [mm]
10^-9 × bunch charge

For ATF, Turn off H-V Coupling
(Introduce Small Vertical Dispersion)

Approximation	$\frac{1}{T_p}$	$\frac{1}{T_h}$	$\frac{1}{T_v}$
Ref. [4]	325	238	239
Bane	369	257	263
CIMP	368	257	268

TABLE IV. Comparison of the Bane and CIMP high energy approximations to the more complete theory of Ref. [4] for the ATF prototype damping ring without betatron coupling and with an rms vertical dispersion of 6.6 mm. Quantities are in $\sec^{-1}$.

Consider ATF without H-V coupling for Ultra Small Vertical Emittances



Recent Work by Nagaitsev

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS **8**, 064403 (2005)

Intrabeam scattering formulas for fast numerical evaluation

Sergei Nagaitsev

FNAL, P.O. Box 500, Batavia, Illinois 60510, USA

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Small-angle multiple intrabeam scattering (IBS) emittance growth rates are normally expressed through integrals, which require a numeric evaluation at various locations of the accelerator lattice. In this paper, I demonstrate that the IBS growth rates can be presented in closed-form expressions with the help of the so-called *symmetric elliptic integral*. This integral can be evaluated numerically by a very efficient recursive method by employing the duplication theorem. Several examples of IBS rates for a smooth-lattice approximation, equal transverse temperatures and plasma temperature relaxation are given.

Define Symmetric Elliptic Integral of the Second Kind

$$R_D(x, y, z) = \frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}}.$$

The following are some useful properties of this integ

$$R_D(x, x, x) = x^{-3/2},$$

$$R_D(x, y, z) + R_D(y, z, x) + R_D(z, x, y) = \frac{3}{\sqrt{xyz}},$$

and

$$R_D(hx, hy, hz) = h^{-3/2} R_D(x, y, z) \qquad (\text{for } h > 0).$$

Also, define

$$\Psi(x, y, z) = -2xR_D(y, z, x) + yR_D(z, x, y) + zR_D(x, y, z)$$

Fast recursive methods exist to compute these symmetric elliptic integrals.

See

B.C. Carlson, J. Res. Natl. Inst. Stand. Technol. **107**, 413 (2002), and references therein..
B.C. Carlson, Numer. Math. **33**, 1 (1979).

Definitions

$$a_x = \frac{\beta_x}{\varepsilon_x} \equiv \frac{1}{{\theta_x}^2}, \qquad a_y = \frac{\beta_y}{\varepsilon_y} \equiv \frac{1}{{\theta_y}^2},$$

$$\sigma_x = \sqrt{{D_x}^2{\sigma_p}^2 + \varepsilon_x\beta_x}, \qquad \sigma_y = \sqrt{\varepsilon_y\beta_y},$$

$$a_s = a_x\left(\frac{{D_x}^2}{{\beta_x}^2} + \Phi^2\right) + \frac{1}{{\sigma_p}^2},$$

and

$$a_1 = \frac{1}{2}(a_x + \gamma^2 a_s), \qquad a_2 = \frac{1}{2}(a_x - \gamma^2 a_s).$$

$$\lambda_1 = a_y,$$

$$\lambda_2 = a_1 + \sqrt{{a_2}^2 + \gamma^2 {a_x}^2 \Phi^2},$$

$$\lambda_3 = a_1 - \sqrt{{a_2}^2 + \gamma^2 {a_x}^2 \Phi^2}.$$

$$\Phi = D_x' - \frac{\beta_x' D_x}{2\beta_x}$$

More Definitions

$$R_1 = \frac{1}{\lambda_1} R_D\left(\frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_1}\right),$$

$$R_2 = \frac{1}{\lambda_2} R_D\left(\frac{1}{\lambda_3}, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right),$$

$$R_3 = \frac{1}{\lambda_3} R_D\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\right).$$

$$\Gamma = 8\pi^3 \beta^3 \gamma^3 M^3 c^3 \varepsilon_x \varepsilon_y \sigma_s \sigma_p$$

Emittance Growth Rates

$$\frac{d\sigma_p^2}{dt} = \frac{N r_p^2 c L_C}{12\pi\beta^3\gamma^5\sigma_s} \int_0^L \frac{ds}{L\sigma_x\sigma_y} S_p,$$

$$\frac{d\varepsilon_y}{dt} = \frac{N r_p^2 c L_C}{12\pi\beta^3\gamma^5\sigma_s} \int_0^L \frac{\beta_y ds}{L\sigma_x\sigma_y} \Psi\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}\right),$$

$$\frac{d\varepsilon_x}{dt} = \frac{N r_p^2 c L_C}{12\pi\beta^3\gamma^5\sigma_s} \times \int_0^L \frac{\beta_x ds}{L\sigma_x\sigma_y}\left[S_x + \left(\frac{D_x^2}{\beta_x^2} + \Phi^2\right)S_p + S_{xp}\right],$$

Definition of S Integrals

$$S_p = \frac{\gamma^2}{2}\left[2R_1 - R_2\left(1 - \frac{3a_2}{\sqrt{a_2{}^2 + \gamma^2 a_x{}^2 \Phi^2}}\right) - R_3\left(1 + \frac{3a_2}{\sqrt{a_2{}^2 + \gamma^2 a_x{}^2 \Phi^2}}\right)\right],$$

$$S_x = \frac{1}{2}\left[2R_1 - R_2\left(1 + \frac{3a_2}{\sqrt{a_2{}^2 + \gamma^2 a_x{}^2 \Phi^2}}\right) - R_3\left(1 - \frac{3a_2}{\sqrt{a_2{}^2 + \gamma^2 a_x{}^2 \Phi^2}}\right)\right],$$

$$S_{xp} = \frac{3\gamma^2 \Phi^2 a_x}{\sqrt{a_2{}^2 + \gamma^2 a_x{}^2 \Phi^2}}[R_3 - R_2].$$

Thus, by computing the three integrals R_1, R_2, and R_3 fully defines the IBS rates at a given lattice location.

Recent Work by G. Parzen
Consider Non-Gaussian Beams

$$\delta < (p_i p_j) > \; = \; N \int d^3x \frac{d^3p_1}{\gamma_1} \frac{d^3p_2}{\gamma_2} f(x, p_1) f(x, p_2) F(p_1, p_2) C_{ij} dt$$

$$C_{ij} \; = \; \pi \int_0^\pi d\theta \sigma(\theta) \sin^3\theta \, \Delta^2 [\delta_{ij} - 3 \frac{\Delta_i \Delta_j}{\Delta^2} + \frac{W_i W_j}{W^2}] \;\; i, j = 1, 3$$

$$\Delta_i \; = \; \frac{1}{2}(p_{1i} - p_{2i})$$

$$W_i \; = \; p_{1i} + p_{2i}$$

Lorentz Invariants

$$
\begin{aligned}
F(p_1, p_2) &= c\frac{[(p_1 p_2)^2 - m_1^2 m_2^2 c^4]^{1/2}}{m_1 m_2 c^2} \\
F(p_1, p_2) &= \gamma_1 \gamma_2 c[(\vec{\beta}_1 - \vec{\beta}_2)^2 - (\vec{\beta}_1 \times \vec{\beta}_2)^2]^{1/2} \\
\Delta^2 &= \vec{\Delta}^2 - \Delta_0^2, \quad \Delta_0 = (E_1 - E_2)/(2c) \\
W^2 &= \vec{W}^2 - W_0^2, \quad W_0 = (E_1 + E_2)/c
\end{aligned}
$$

Example: Coulomb Cross Section (Coordinate System Moving with Bunch)

$$
\begin{aligned}
\frac{1}{p_0^2} < \delta(p_{1i} p_{1j}) > &= N \int \; d^3x d^3p_1 d^3p_2 f(x,p_1) f(x,p_2) 2\bar{\beta}c \, C_{ij} \, dt \\
\Delta_i &= \frac{1}{2}(p_{1i} - p_{2i}) \\
\bar{\beta}c &= |\vec{\Delta}|/m \\
C_{ij} &= \frac{2\pi}{p_0^2}(r_0/2\bar{\beta}^2)^2 \ln(1 + (2\bar{\beta}^2 b_{max}/r_0)^2) \\
&\quad [|\vec{\Delta}|^2 \delta_{ij} - 3\Delta_i \Delta_j] \;\; i,j = 1,3 \\
r_0 &= Z^2 e^2/mc^2 \\
\sigma(\theta) &= [\frac{r_0}{2\bar{\beta}^2}]^2 \frac{1}{(1-\cos\theta)^2} \\
\cot(\theta_{min}/2) &= 2\bar{\beta}^2 b_{max}/r_0
\end{aligned}
$$

Consider Bi-Gaussian Distribution (Frame Moving with Bunch)

$$
\begin{aligned}
\frac{1}{p_0^2} < \delta(p_i p_j) > &= \int d^3x d^3p_1 d^3p_2 \left[\frac{N_a}{N}\frac{1}{\Gamma_a} exp[-S_a(x,p_1)] + \frac{N_b}{N}\frac{1}{\Gamma_b} exp[-S_b(x,p_1)]\right] \\
& \left[\frac{N_a}{N}\frac{1}{\Gamma_a} exp[-S_a(x,p_2)] + \frac{N_b}{N}\frac{1}{\Gamma_b} exp[-S_b(x,p_2)]\right] \\
& 2\bar{\beta}c\, C_{ij}\, dt \\
\vec{\Delta} &= \frac{1}{2}(\vec{p_1} - \vec{p_2}) \\
\bar{\beta}c &= |\vec{\Delta}|/m \\
C_{ij} &= \frac{2\pi}{p_0^2}(r_0/2\bar{\beta}^2)^2 \ln(1 + (2\bar{\beta}^2 b_{max}/r_0)^2) \;\; [|\vec{\Delta}|^2\delta_{ij} - 3\Delta_i\Delta_j] \;\; i,j = 1,3
\end{aligned}
$$

$$
f(x,p) = \frac{N_a}{N}\frac{1}{\Gamma_a} exp[-S_a(x,p)] + \frac{N_b}{N}\frac{1}{\Gamma_b} exp[-S_b(x,p)]
$$

$$
r_0 = Z^2 e^2 / mc^2
$$

Finally, Emittance Growth Rates in Lab System in Terms of <p_ip_j> in Frame Moving with Bunch ("~" Denotes Bunch Frame)

$$\frac{d}{dt} < \epsilon_x > = \frac{\beta_x}{\gamma}\frac{d}{d\tilde{t}} < p_x^2/p_0^2 > + \frac{D^2 + \tilde{D}^2}{\beta_x}\gamma\frac{d}{d\tilde{t}} < p_s^2/p_0^2 > -2\tilde{D}\frac{d}{d\tilde{t}} < p_x p_s/p_0^2 >$$

$$\frac{d}{dt} < \epsilon_y > = \frac{\beta_y}{\gamma}\frac{d}{d\tilde{t}} < p_y^2/p_0^2 >$$

$$\frac{d}{dt} < \epsilon_s > = \beta_s\gamma\frac{d}{d\tilde{t}} < p_s^2/p_0^2 >$$

$$\tilde{D} = \beta_x D' + \alpha_x D$$

SUMMARY

- We have used Bane's Ansatz to derive Completely Integrated Modified Piwinski (CIMP) formulae for high energy beams that agree well with the more complete theory for direct, dispersion, and H-V coupling contributions to emittance growth rates.
- Agrees well with Bane's approximation where applicable (eg. nonzero vertical dispersion), except for ultra small vertical emittances, where Bane's approximation breaks down by not including the direct contribution to the vertical emittance growth rate.
- Before using any approximation, always check a few growth rates against the more complete theory to gain confidence.
- Work is being used to compare ILC damping rings.
- Nagaitsev has introduced a method for the fast evaluation of the more complete theory.
- Parzen has developed formalism for non-Gaussian beams and any cross section.