# Cyclotrons: Classic to FFAG

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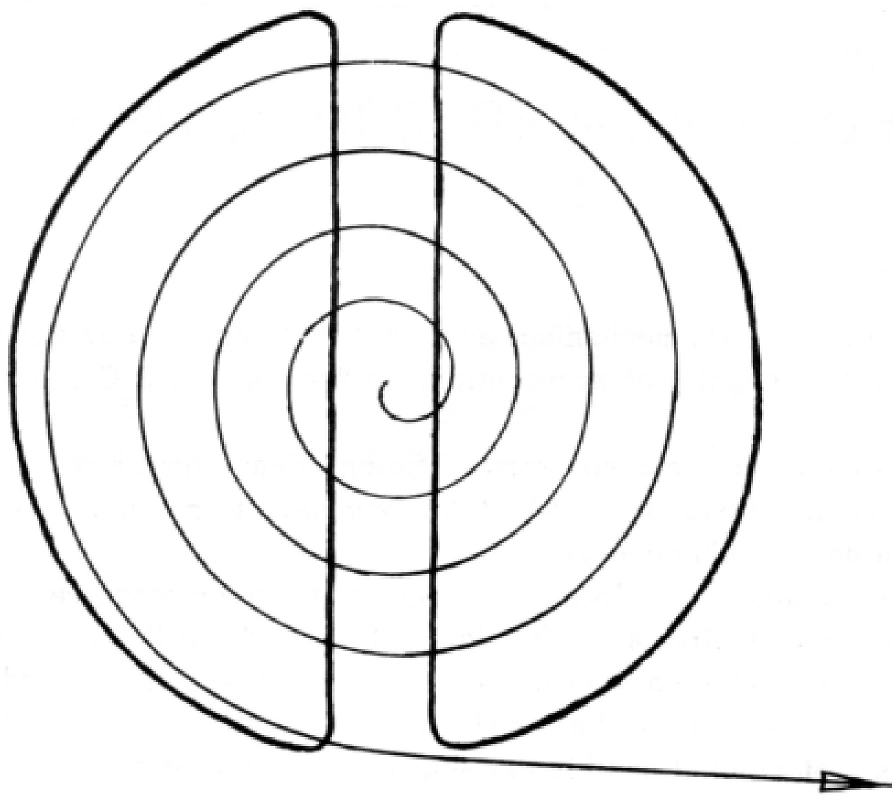
**Abstract**

The cyclotron is much more than a magnet with charged particles spiralling out as they accelerate from the centre of the pole gap. I trace the history and development up to present-day FFAGs, and hopefully convey something regarding their special beam dynamics characteristics.

# Invention (Lawrence, 1930)

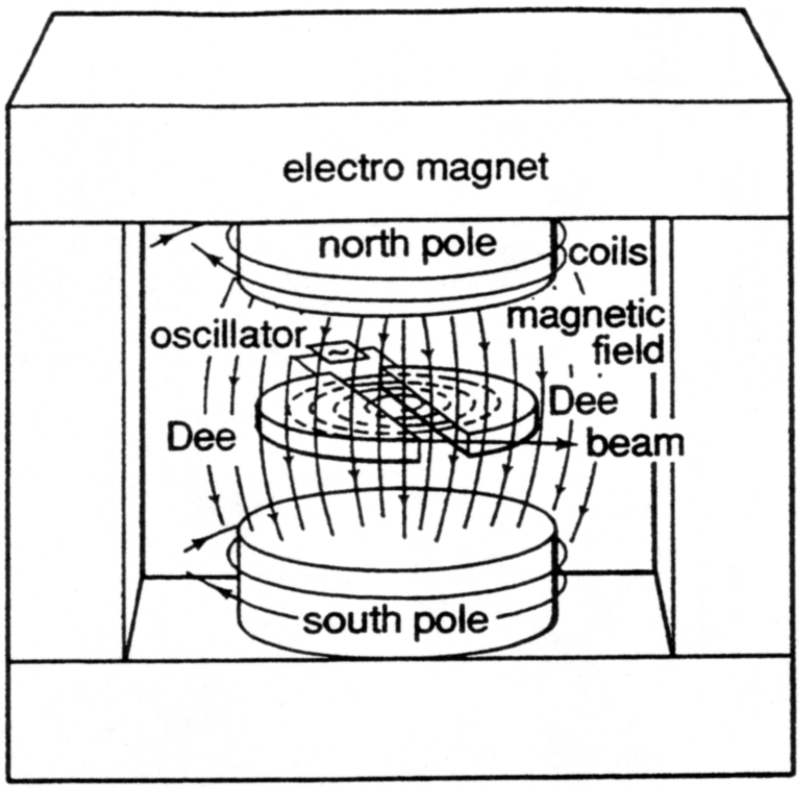
$mv^2/r = qvB$, so $m\omega_0 = qB$, with $r = v/\omega_0$



With $B$ constant in time and uniform in space, as particles gain energy from the rf system, they stay in synchronism, but spiral outward in $r$.

# Development to 1938

Small machines were built, and they worked. It was discovered empirically that the natural decline of $B$ with $r$ actually helped. 1938: (R.R. Wilson) orbit theories developed, the effect is understood.

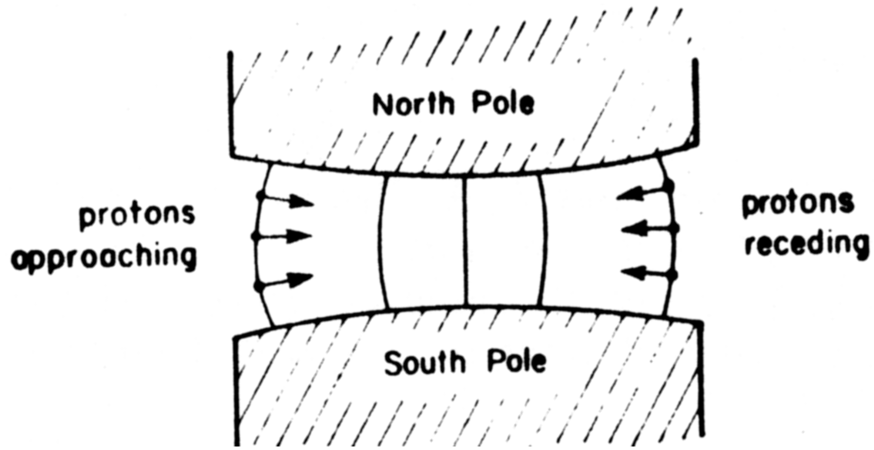



A flat field has no preferred $z$; i.e. $\nu_z = 0$. It also has no preferred centre for the orbit of radius $r$; i.e. $\nu_r = 1$. But a field which falls as $r$ increases provides a restoring force toward the median plane.

No one thought of $B = B(\theta)$; only $B = B(r)$. Why?

# Simple Cyclotron Orbits

Vertical forces result from radial $B$:




$$F_z = qv\, B_r$$

$$\text{Taylor expand: } F_z \approx qv \frac{\partial B_r}{\partial z} z$$

$$\text{since } \nabla \times \vec{B} = \vec{0}\text{: } m\ddot{z} = qv \frac{\partial B_z}{\partial r} z$$

This results in SHM of frequency $\omega_z$:

$$\omega_z^2 = -\frac{qv}{m}\frac{\partial B_z}{\partial r}$$

and tune $\nu_z = \omega_z/\omega_0$:

$$\nu_z^2 = -\frac{qv}{m\omega_0^2}\frac{\partial B_z}{\partial r} = -\frac{r}{B_z}\frac{\partial B_z}{\partial r} \equiv -k$$

($k$ is "field index"). Similarly, $\nu_r^2 = 1 + k$. This sets the requirement $-1 < k < 0$

# Relativity

It turns out that for higher energies, the cyclotron resonance condition remains simple

$$m\gamma\omega_0 = qB, \text{ with } r = \beta c/\omega_0$$

That means

$$k = \frac{\beta}{\gamma}\frac{d\gamma}{d\beta} = \beta^2\gamma^2$$

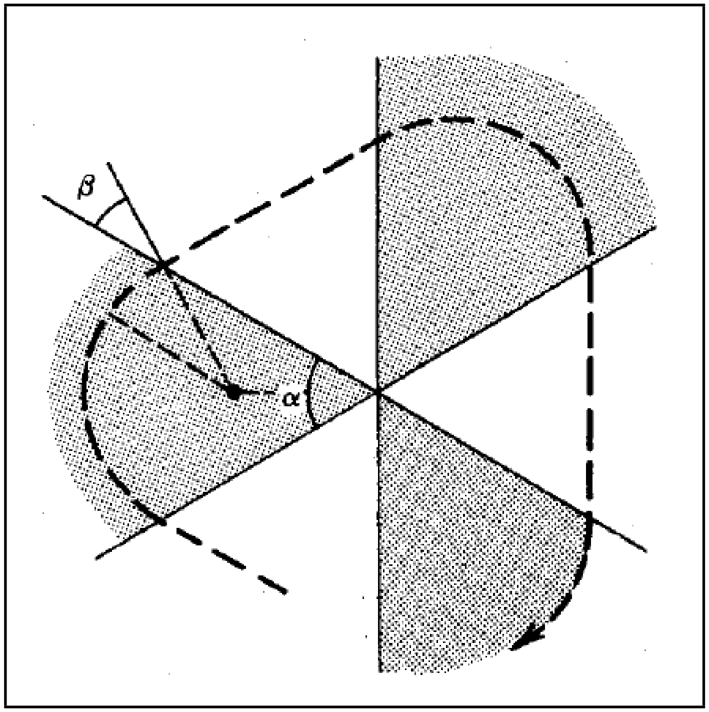
In other words, we cannot satisfy $-1 < k < 0$.

Hans Bethe (1937): *... it seems useless to build cyclotrons of larger proportions than the existing ones... an accelerating chamber of 37 cm radius will suffice to produce deuterons of 11 MeV energy which is the highest possible...*

Such was Bethe's influence, that when a paper appeared in 1938, which appeared to resolve the problem, it was ignored for at least a decade. That paper was *The Paths of Ions in the Cyclotron* by L.H. Thomas.

Frank Cole: *If you went to graduate school in the 1940s, this inequality [$-1 < k < 0$] was the end of the discussion of accelerator theory.*

# AVF, or Thomas focusing

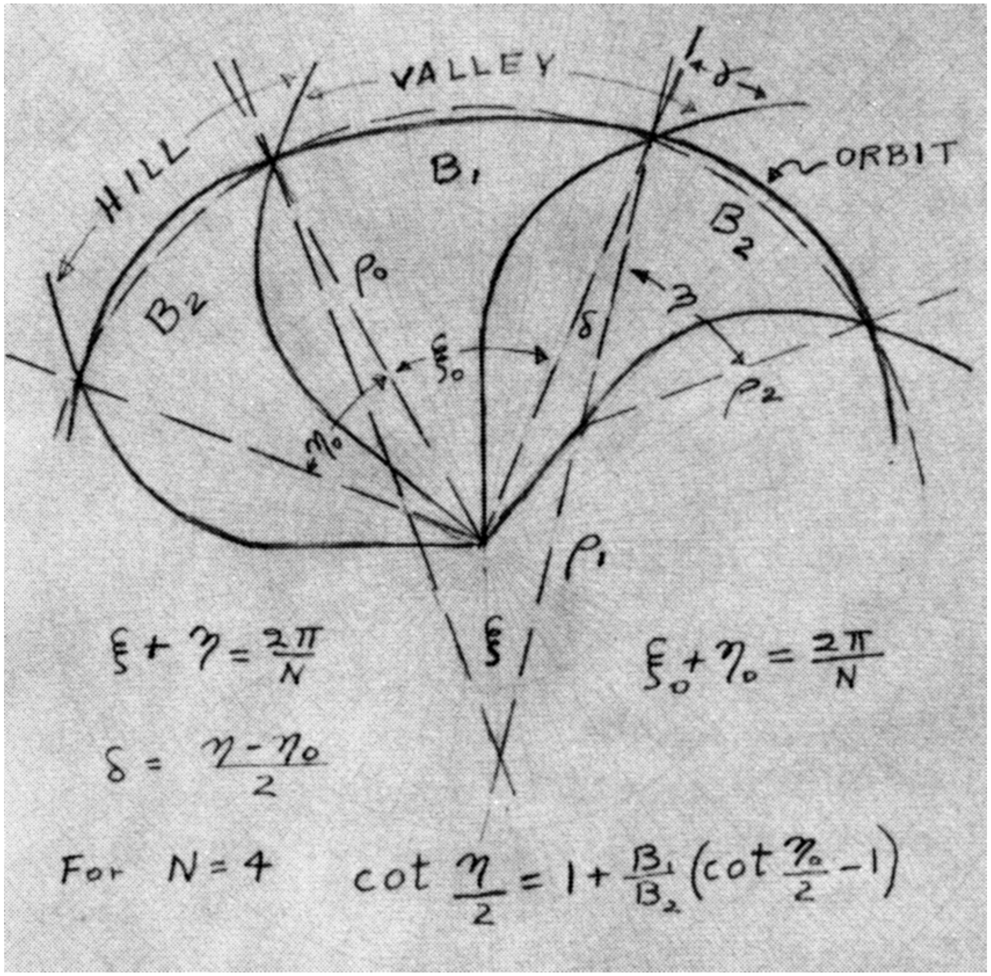



The paper was hard to understand, but knowing today's accelerator theory, it is easy for us to see how it works. Separate the magnet into sector fields and drifts and you can see immediately that you cannot help but have edge focusing at every sector edge.

So to build a relativistic cyclotron, you would allow the field to grow $\propto \gamma$, giving vertical defocusing, and compensate with focusing edges. This is an early form of "strong focusing". If the focusing was still insufficient, you could actually have reverse bends. Thus was invented the Fixed-Field Alternating Gradient machine (FFAG) by Symon and independently by Ohkawa in Japan (1953).

Ernest Courant (1952): *A significant side benefit of inventing strong focusing was that it finally enabled me to understand what Thomas' paper was about.*

# Spiral focusing




In 1954, Kerst realized that the sectors need not be symmetric. By tilting the edges, the one edge became more focusing and the other edge less. But by the strong focusing principle (larger betatron amplitudes in focusing, smaller in defocusing), one could gain nevertheless. This had the important advantage that reverse bends would not be needed (reverse bends made the machine excessively large). (Figure is from J.R. Richardson notes.)

The resulting machines no longer had *alternating gradients*, but Kerst and Symon called them FFAGs anyway. The misnomer is still with us.

# Isochronism

Orbit length $L$ is given by speed and orbit period $T$:

$$L = \oint ds = \oint \rho d\theta = \beta c T.$$

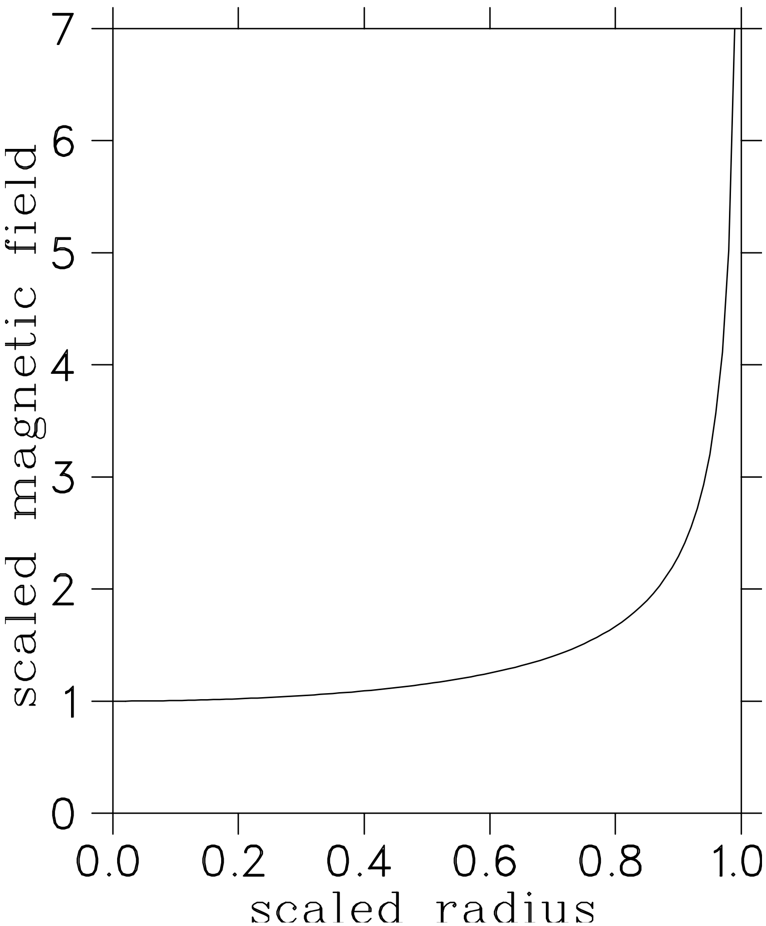
The local curvature $\rho = \rho(s)$ can vary and for reversed-field bends (Ohkawa, 1953) even changes sign. (Along an orbit, $ds = \rho d\theta > 0$ so $d\theta$ is also negative in reversed-field bends.) Of course on one orbit, we always have $\oint d\theta = 2\pi$. What is the magnetic field averaged over the orbit?

$$\overline{B} = \frac{\oint B ds}{\oint ds} = \frac{\oint B \rho \, d\theta}{\beta c T}.$$

But $B\rho$ is constant and in fact is $\beta\gamma mc/q$. Therefore

$$\overline{B} = \frac{2\pi}{T}\frac{m}{q}\gamma \quad \equiv B_{\mathrm{c}}\,\gamma = \frac{B_{\mathrm{c}}}{\sqrt{1-\beta^2}}.$$

# Isochronism, cont'd



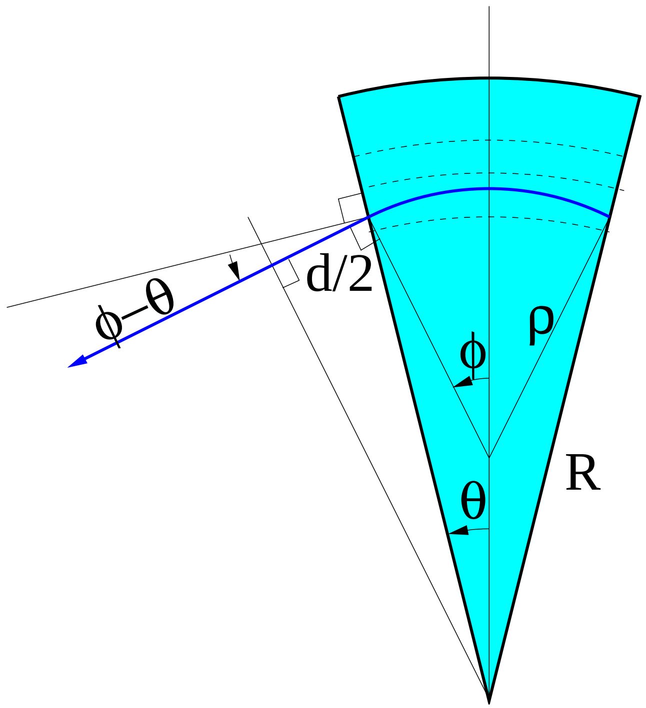

Remember, $\beta$ is related to the orbit length: $\beta = L/(cT) = 2\pi R/(cT) \equiv R/R_\infty$. So

$$\overline{B} = \frac{B_c}{\sqrt{1-(R/R_\infty)^2}}.$$

Of course, this means the field index is $k = \frac{R}{B}\frac{dB}{dR} = \frac{\beta}{\gamma}\frac{d\gamma}{d\beta} = \beta^2\gamma^2 \neq$ constant.

Muon FFAGs contact at only one point...

# Tunes in an FFAG




$$\frac{\sin(\theta)}{\rho} = \frac{\sin(\phi)}{R}$$

$$d/2 = R\sin(\phi-\theta)$$

To make it transparent, let us consider all identical dipoles and drifts; no reverse bends. We have drifts $d$, dipoles with index $k$, radius $\rho$, bend angle $\phi$, and edge angles $\phi - \theta$:

In addition, imagine that the edges are inclined by an extra angle $\xi$. This is called the “spiral angle” (hard to draw).

In this hard-edged case, the “flutter” $F^2 \equiv \langle (B - \overline{B})^2 \rangle / \overline{B}^2 = R/\rho - 1$.

Aside: Notice that the particle trajectory (blue curve) does not coincide with a contour of constant $B$ (dashed curves). This has large implications for using existing transport codes to describe FFAGs.

# resort to Mathematica...

```
In[1]:= s = ρ*ϕ;
θ = ArcSin[(ρ / R) Sin[ϕ]];
f1 = ρ / Tan[ϕ - θ + ξ];
f2 = ρ / Tan[ϕ - θ - ξ];
d = 2 R*Sin[ϕ - θ];
R = ρ (1 + F^2);
k = κ*ρ / R;
kx = Sqrt[1 + k] / ρ;
ky = Sqrt[k] / ρ;
Mz := (({{Cosh[ky s], Sinh[ky s] / ky}, {ky Sinh[ky s], Cosh[ky s]}}).{{1, 0}, {-1/f2, 1}}.{{1, d}, {0, 1}}.{{1, 0}, {-1/f1, 1}}.
   {{Cosh[ky s], Sinh[ky s] / ky}, {ky Sinh[ky s], Cosh[ky s]}})
νz^2 = FullSimplify[(ArcCos[Series[(Mz[[1, 1]] + Mz[[2, 2]]) / 2, {ϕ, 0, 3}]]) / (2 ϕ)]^2

Out[11]= (-κ + F^2 (1 + 2 Tan[ξ]^2)) + O[ϕ]^2

In[12]:= Mr := (({{Cos[kx s], Sin[kx s] / kx}, {-kx Sin[kx s], Cos[kx s]}}).{{1, 0}, {1/f2, 1}}.{{1, d}, {0, 1}}.{{1, 0}, {1/f1, 1}}.
   {{Cos[kx s], Sin[kx s] / kx}, {-kx Sin[kx s], Cos[kx s]}})

νr^2 = FullSimplify[ArcCos[Series[(Mr[[1, 1]] + Mr[[2, 2]]) / 2, {ϕ, 0, 3}]] / (2 ϕ)]^2

Out[13]= (1 + κ) + O[ϕ]^2
```

$\nu_r^2 = 1 + \kappa$, and $\nu_z^2 = -\kappa + F^2(1 + 2\tan^2\xi)$

# Tunes

$$\nu_r^2 = 1 + \kappa, \text{ and } \nu_z^2 = -\kappa + F^2(1 + 2\tan^2\xi)$$

These expressions were originally derived by Symon, Kerst, Jones, Laslett, Terwilliger in the original 1956 Phys. Rev. paper about FFAGs.

Note: Since there is now a distinction between local curvature ($\rho$) and global ($R$), the definition of field index is ambiguous. The local index, used in the dipole transfer matrix, is $k = \frac{\rho}{B}\frac{dB}{d\rho}$, while the Symon formula uses $\kappa = \frac{R}{B}\frac{dB}{dR} \approx k\,\frac{R}{\rho}$. It is in fact this latter quantity which must be equal to $\beta^2\gamma^2$ for isochronism.

For isochronous machines, we therefore have

$$\nu_r = \gamma, \text{ and } \nu_z^2 = -\beta^2\gamma^2 + F^2\left(1 + 2\tan^2\xi\right)$$

# Two kinds of FFAGs...

So this kind of focusing can be used for either of 2 purposes:

1. Make $\nu_z$ real for isochronous machines (cyclotrons). But then horizontal resonances must be crossed.

2. Fix both tunes. But then the machine must be a synchro-cyclotron and so must be pulsed and therefore much lower intensity.

1. *FFAG Cyclotrons* of this kind were built at TRIUMF and PSI. They provide the most economical way of achieving beam power in the 1MW range. Resonance crossing is possible because in this kind of machine traversal is fast: rf frequency is fixed so can use high-$Q$ cavities to achieve large voltage per turn.

2. *FFAG Synchro-cyclotrons* were rapidly overtaken in energy by synchrotrons and so this application was never fully brought to fruition.

# Example: TRIUMF cyclotron

| Energy | $\overline{B}$ | $R$ | $\beta\gamma$ | $\xi$ | $1 + 2\tan^2\xi$ | $F^2$ |
|---|---|---|---|---|---|---|
| 100 MeV | 0.335T | 175 in. | 0.47 | 0° | 1.0 | 0.30 |
| 250 MeV | 0.383T | 251 in. | 0.78 | 47° | 3.3 | 0.20 |
| 505 MeV | 0.466T | 311 in. | 1.17 | 72° | 20.0 | 0.07 |

# TRIUMF Details:

Magnet: 4,000 tons

RF volts per turn = 0.4 MV.

Number of turns to 500 MeV = 1250.

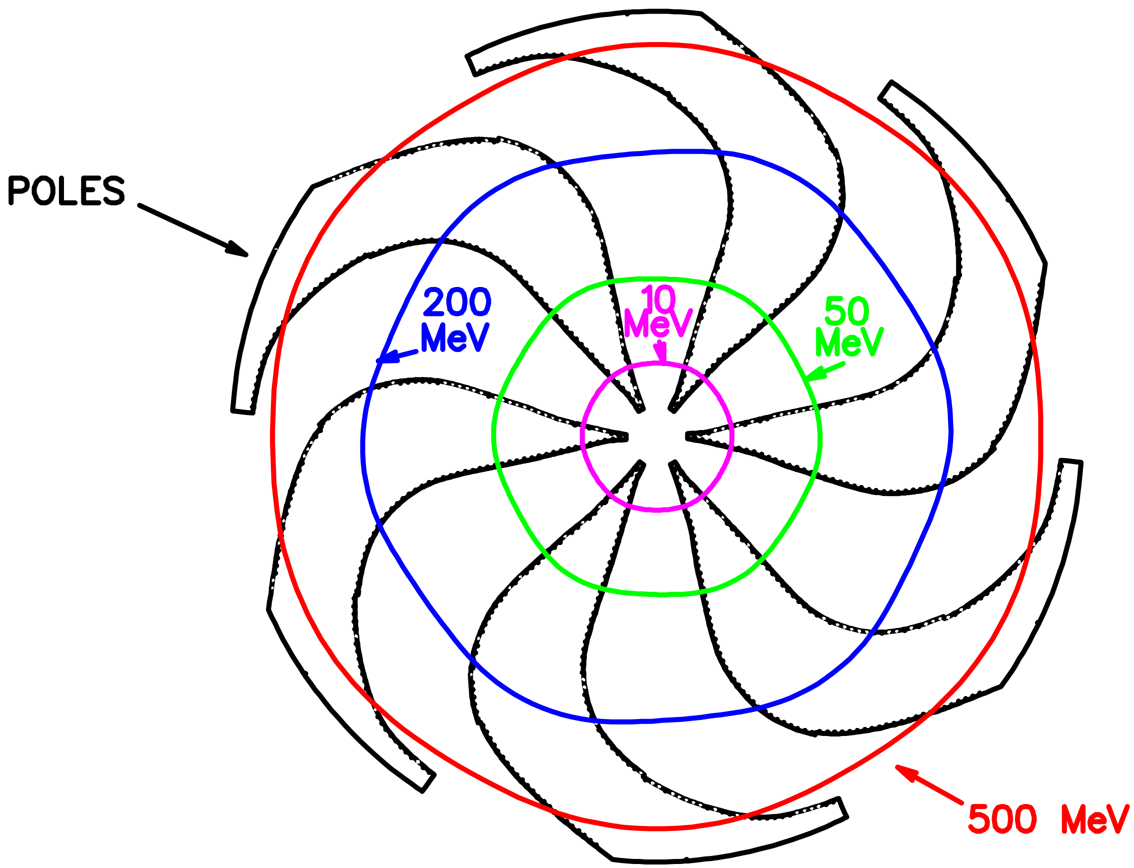
RF harmonic = 5: A magnetic field error of 1:12,500 results in a phase slip of 180°. This means magnetic field tolerance is a few ppm.

Injection energy is 0.3 MeV. That's a momentum range of a factor of 40.

Peak Intensity achieved: 400 $\mu$A. This would be 0.2 MW at full duty cycle.

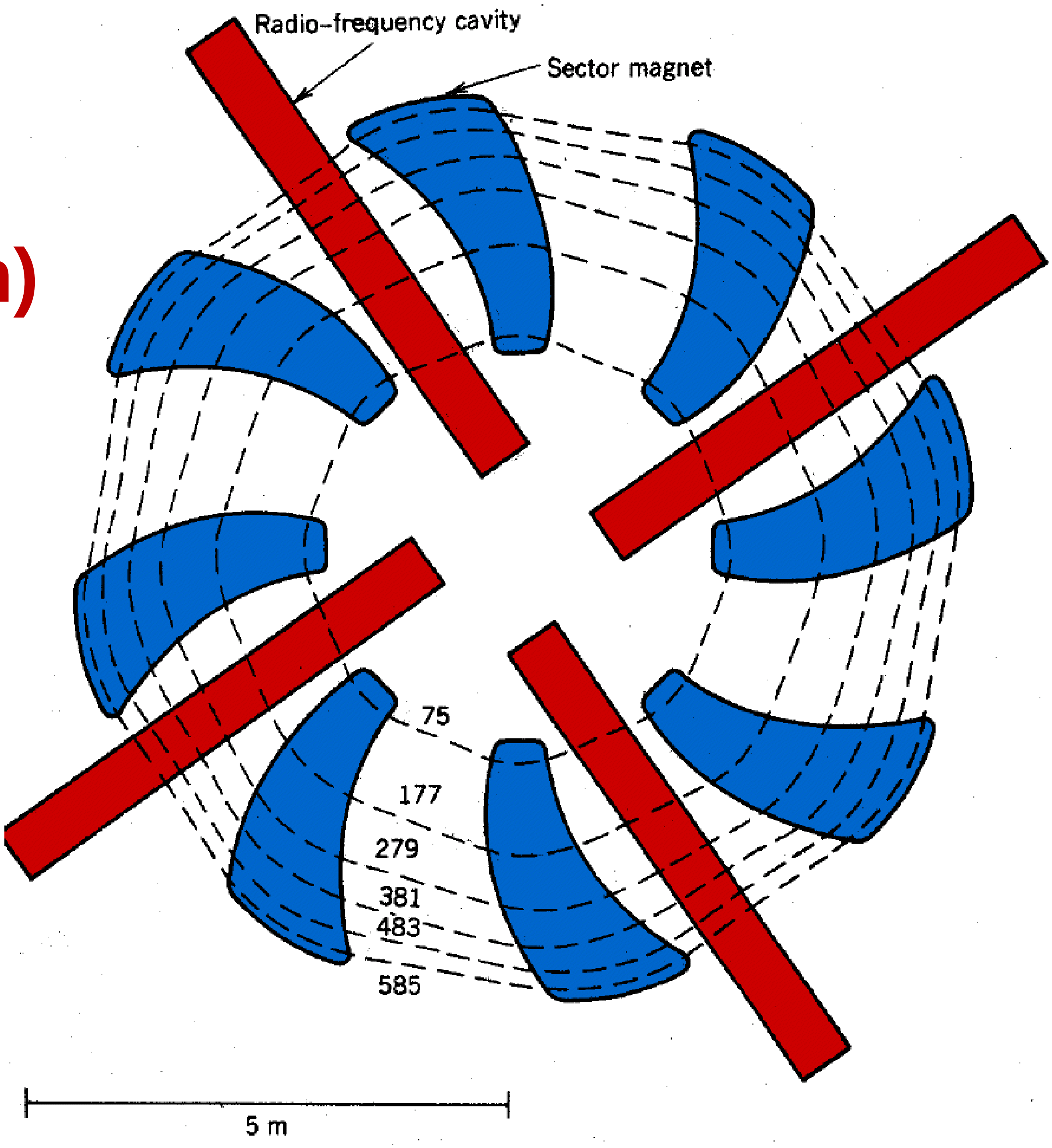
PSI cyclotron has reached 2 mA at 590 MeV, 1.2 MW. The reason is that they have higher injection energy, stronger vertical focusing at injection.

# TRIUMF'S POLES AND EQUILIBRIUM ORBITS



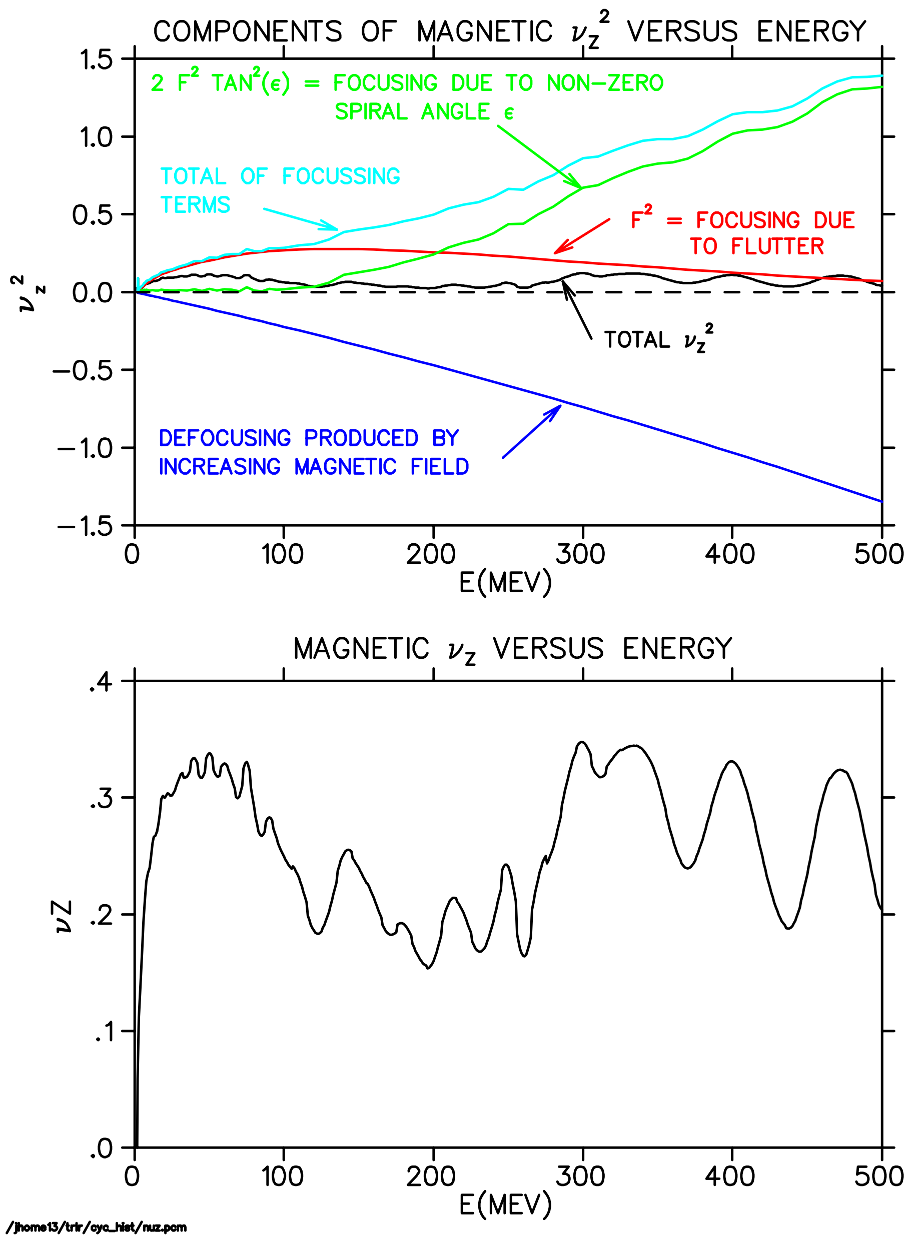
# PSI cyclotron (for comparison)

Outer orbit is 4.5m compared with TRIUMF's 7.6m.

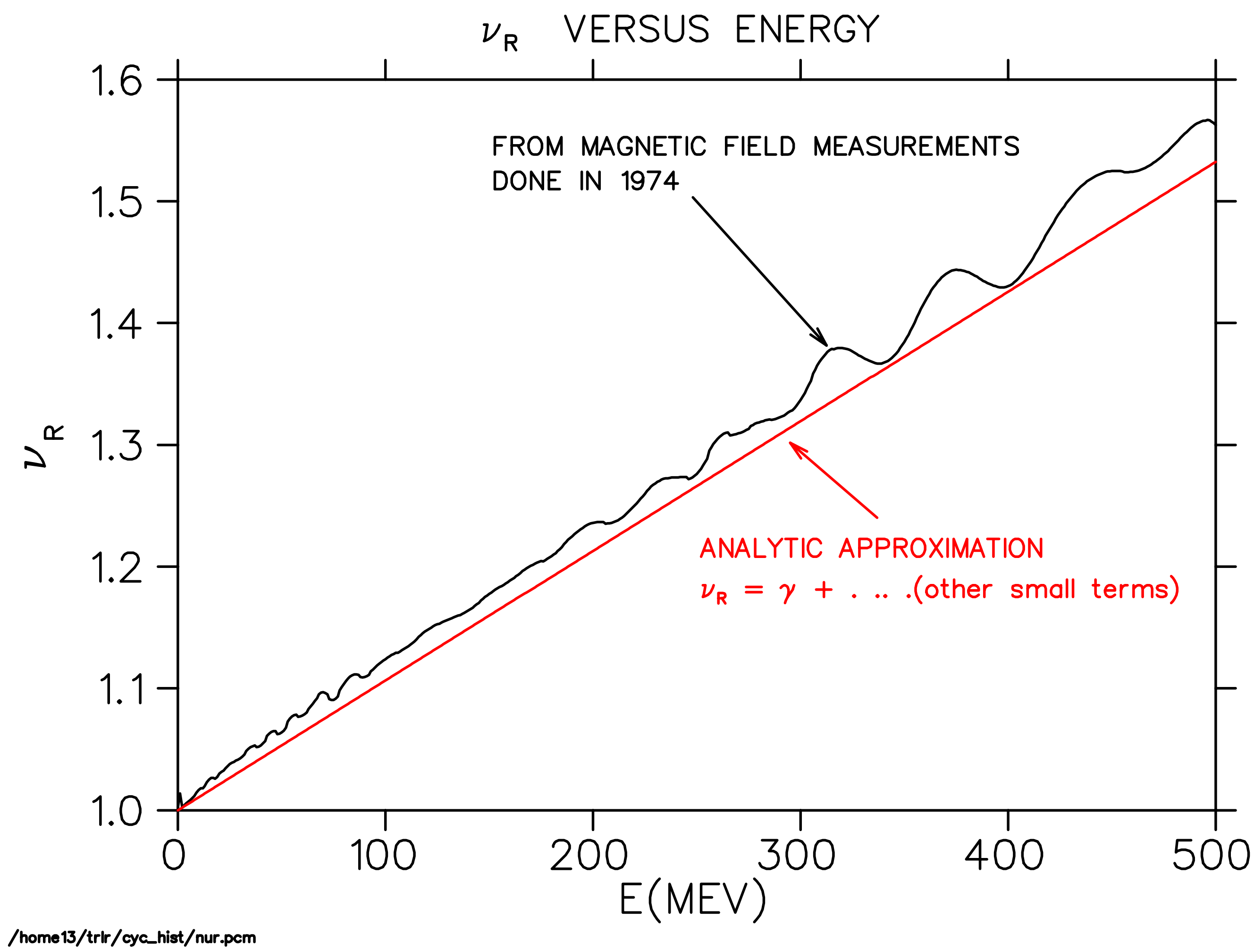


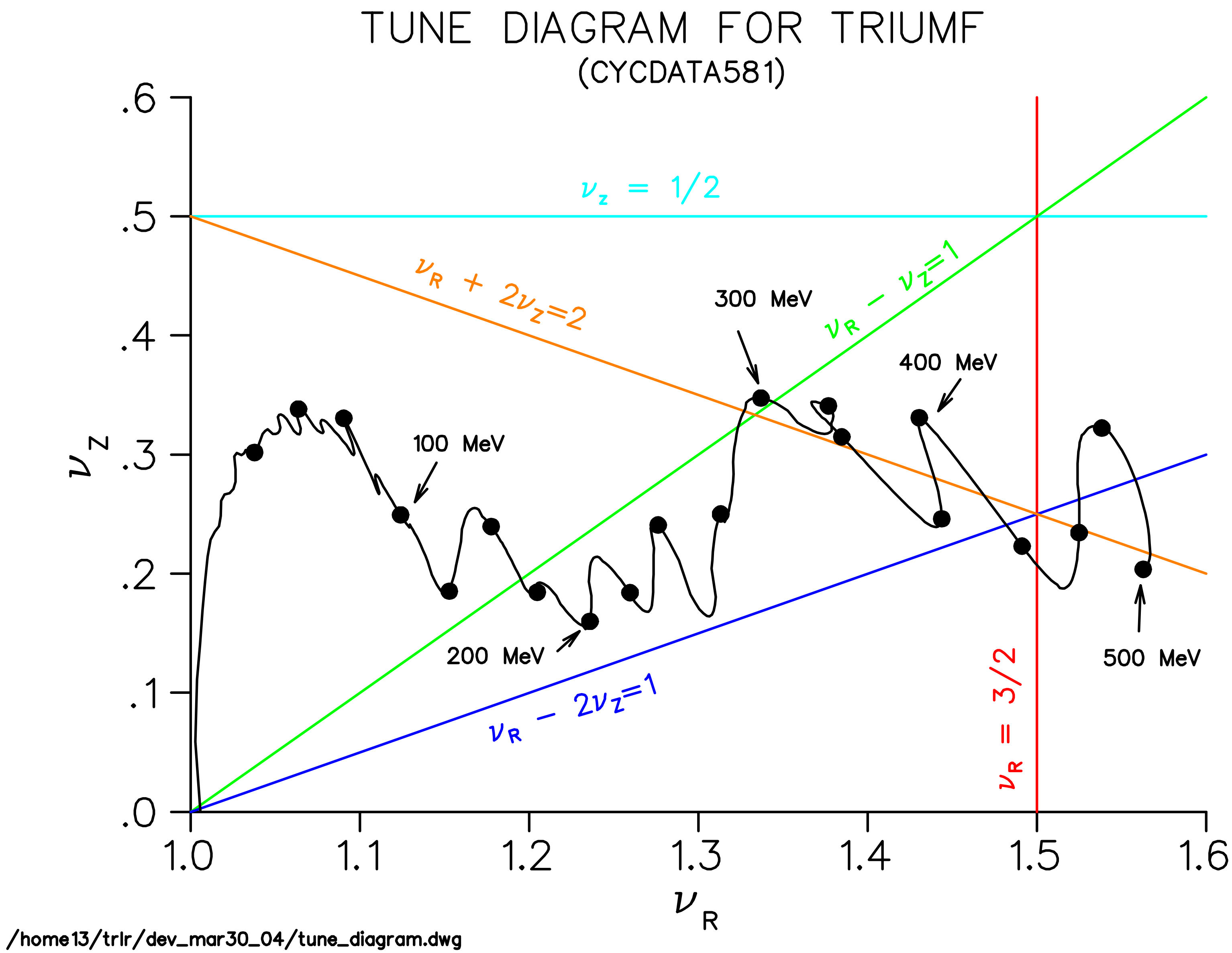
# Vertical focusing in TRIUMF

BTW: $B$ is low because TRIUMF accelerates $H^-$. This prohibits Peak field at 500 MeV from exceeding 0.5 T. This is what makes $F^2$ low at high energy. Compare with PSI (protons), where peak field is 1.65 T.



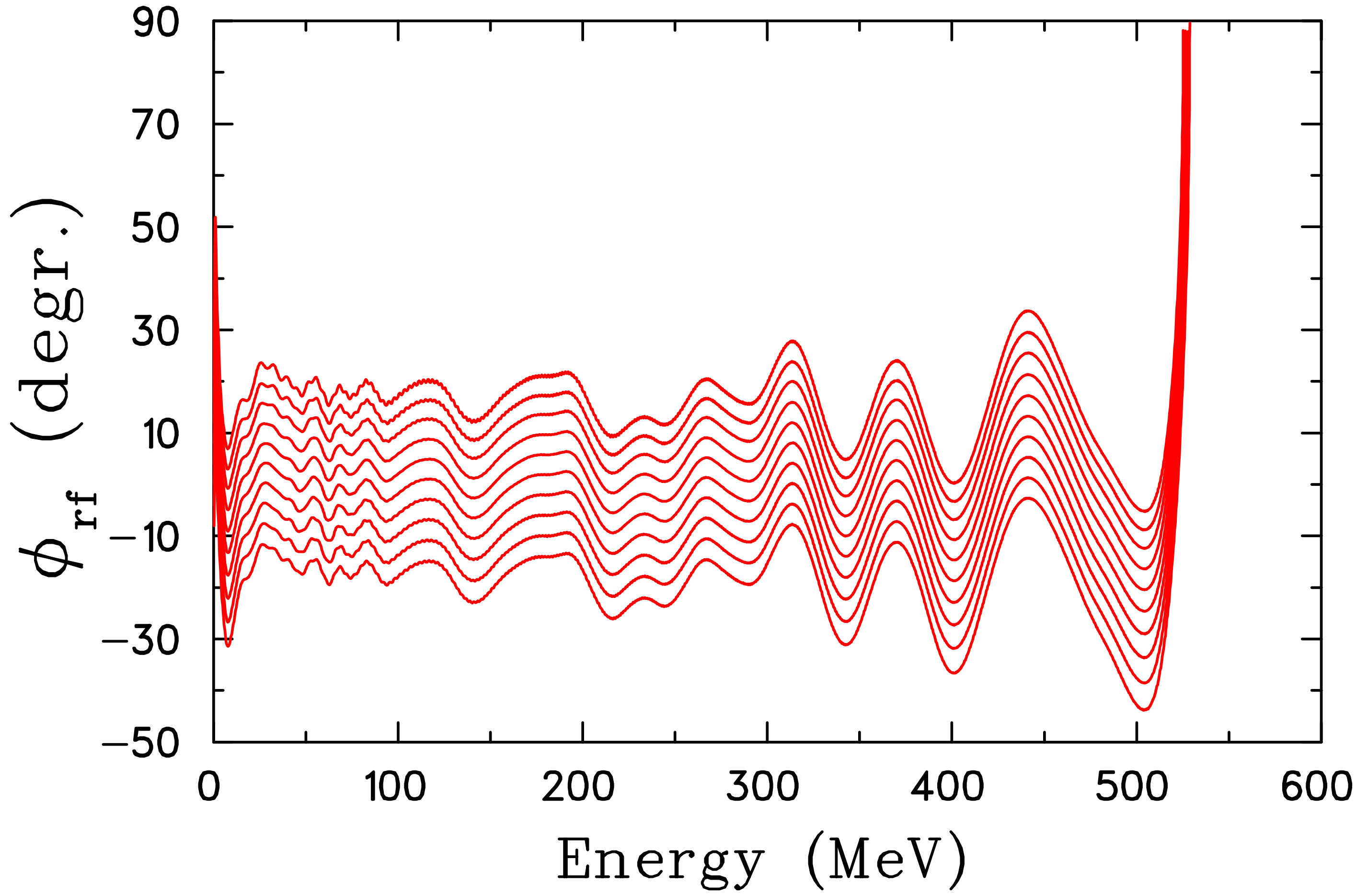
# Radial focusing in TRIUMF





TUNE DIAGRAM FOR TRIUMF
(CYCDATA581)
.6
.5
.4
.3
.2
.1
.0
1.0
1.1
1.2
1.3
1.4
1.5
1.6
$\nu_Z$
$\nu_R$
$\nu_z = 1/2$
$\nu_R + 2\nu_Z=2$
$\nu_R - \nu_Z=1$
$\nu_R - 2\nu_Z=1$
$\nu_R = 3/2$
100 MeV
200 MeV
300 MeV
400 MeV
500 MeV
/home13/trlr/dev_mar30_04/tune_diagram.dwg

# Isochronism (Longitudinal phase space)



# Isochronism (measured)

Take the previous graph, imagine that there is a mirror image at $\phi \to \phi + \pi$, and rotate it 90°.

Here is a longitudinal trajectory as measured by time-of-flight (Craddock et al, 1977 PAC).

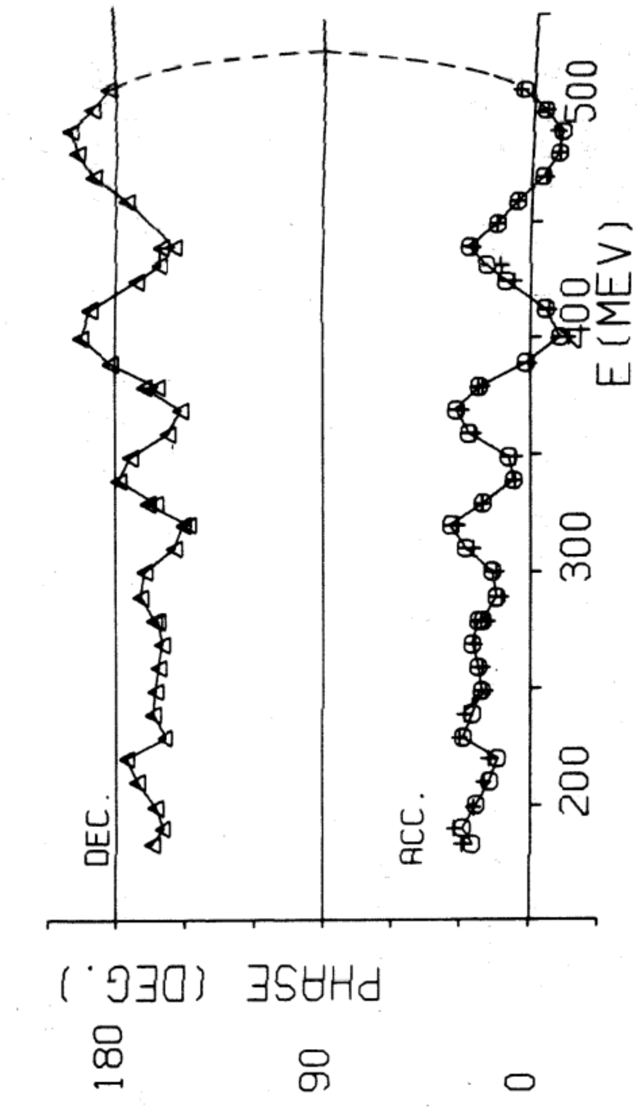
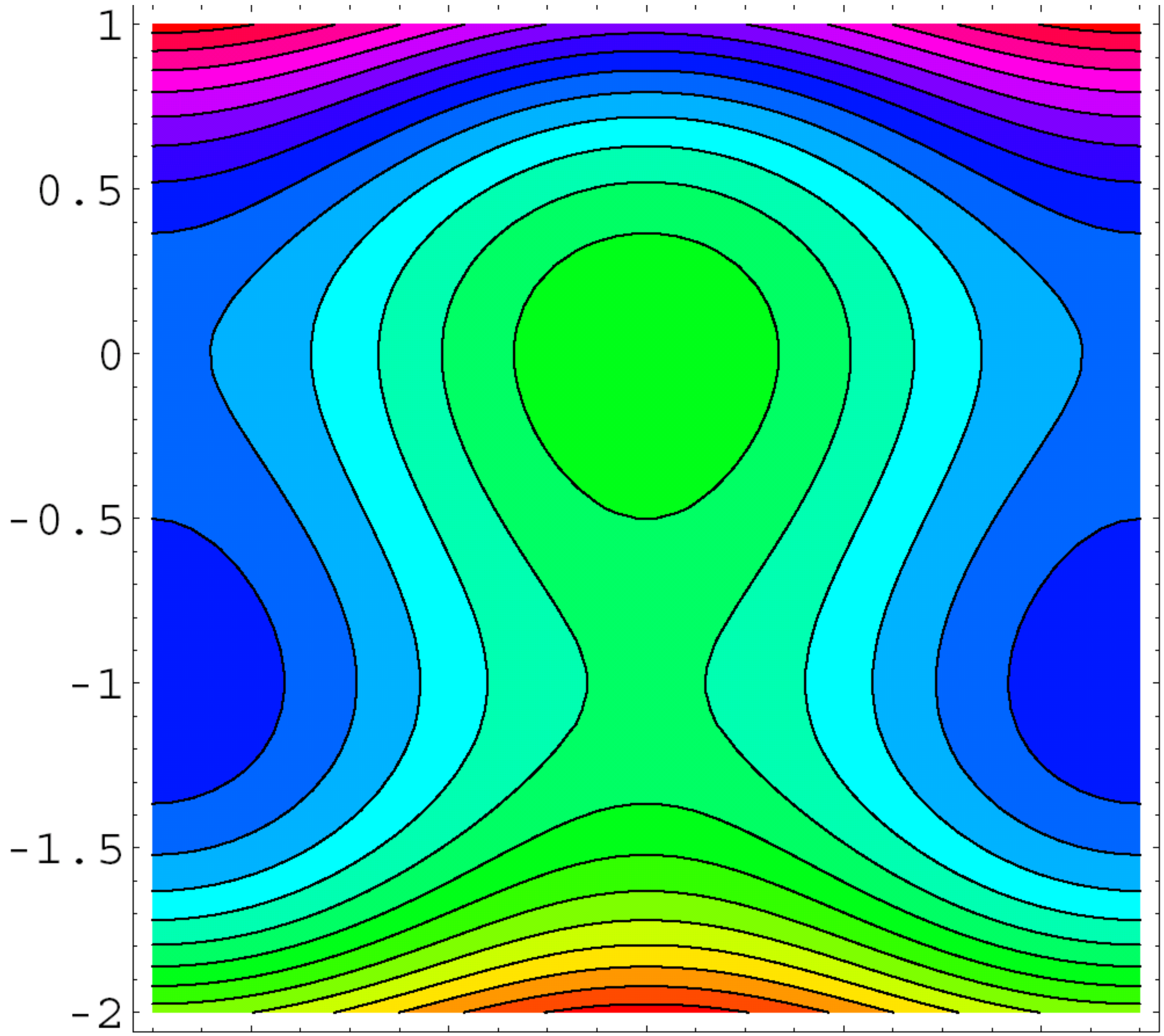



*Fig. 2. Phase histories of accelerating and decelerating beams, obtained by timing an external beam.*

This is what happens when isochronism error has only one “jiggle” i.e. parabolic (from Keil).



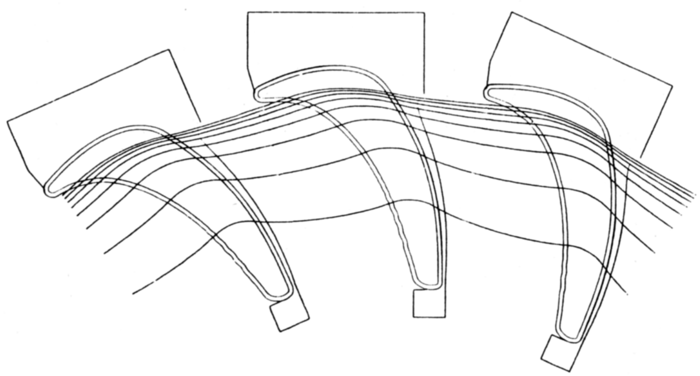
# What about those "FFAG"s proposed for accelerating muons?

*They* have fixed field and fixed frequency and so *are in fact cyclotrons*.

Reminder from Keil's talk: $E$ = 6 to 20 GeV, $\sim$300 cells, rf volts per turn $\sim$ $1.5$ GV(!). They are cyclotrons whose poor isochronism is overcome by brute rf force.

It's also now easier to understand why they cannot be made more nearly isochronous. At 20 GeV, $\gamma = 190$. Isochronism requires $\nu_r = \gamma$, so need 760 cells (sectors) for a final phase advance per cell of around 90$^\circ$. Instead, they are only made "contact isochronous": Isochronous only at one momentum, with a parabolic dependence of orbit time on momentum.

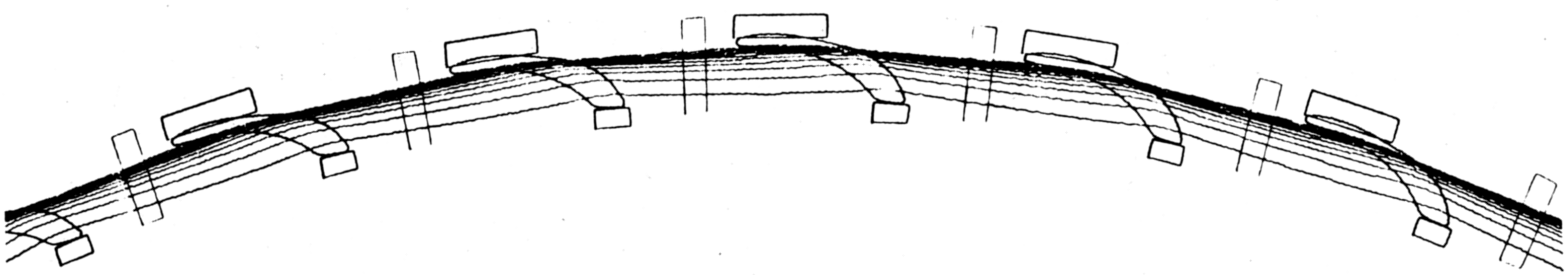
# Other designs (1983, not built)



Sector design and orbits (at 0.5 GeV intervals) for the 3.5 GeV cyclotron using single gullies.

For kaon factories, we designed high energy cyclotrons, but they were never built. Here are two: 3.5 GeV and 12 GeV (protons).

Nowadays (now that superconducting rf has advanced), a 20GeV proton FFAG cyclotron would be much easier to build than a Muon FFAG. Multi-MW would be possible!



Five sectors from the 40 m radius 42 sector cyclotron showing magnets, RF cavities and proton orbits.

# How to design a cyclotron?

Putting dipoles and drifts into a transport code is not going to work. For any momentum, do not know *a priori* where the orbit is, so do not know edge angles, field index in that region. Even bigger problem: the standard dipole model built into these codes is not compatible with FFAG dipoles (see p. 10). Can only do it with a field map and the Equations of motion. The most convenient independent variable is azimuth $\theta$:

$$
\begin{aligned}
p_r' &= Q - rB_z + \frac{r}{Q}\,p_z B_\theta \\
r' &= \frac{r}{Q}\,p_r \\
p_z' &= rB_r - \frac{r}{Q}\,p_r B_\theta \\
z' &= \frac{r}{Q}\,p_z \\
t' &= \frac{r}{Q}\,E
\end{aligned}
$$

where $Q \equiv \sqrt{p^2 - p_r^2 - p_z^2}$. These are in cyclotron units: $B$ in units of central field $m\omega_0/q$, $t$ in units of $\omega_0^{-1}$, lengths in units of $c/\omega_0$, $E$ in units of $mc^2$, $p$ in units of $mc$.

Runge-Kutta integrating is no sweat; the real sweat is in devising an interpolation scheme for $B$ consistent with Maxwell's equations.

We integrate over one sector of an $N$-sector machine. But of course the orbit will not close on itself. To find the closed orbit, we track the first order differentials of motion as well. Let $r \rightarrow r + x$, $z \rightarrow z + y$, etc. and keep only first order. (These are the next element of $r$, $p_r$, etc. if they are considered as DA variables.) Then the previous equations give the following for the "first order equations of motion".

$$\begin{aligned} p_x' &= -\frac{pr}{Q}\, px - \frac{\partial}{\partial r}(rB)\, x \\ x' &= \frac{pr}{Q}\, x + \frac{p^2 r}{Q^3}\, px \\ p_y' &= \left( r\frac{\partial B}{\partial r} - \frac{pr}{Q}\,\frac{\partial B}{\partial \theta} \right) y \\ y' &= \frac{r}{Q}\, py \end{aligned}$$

These are integrated with starting values $(p_x, x) = (1,0), (0,1); (p_y, y) = (1,0), (0,1)$ and so give the transfer matrices. These are used in an iteration to find the equilibrium orbit.

Then the transfer matrices are analyzed to find $\beta$-functions, tunes, etc.

In the design stage, we start with an ideal field ($z_{\text{e.o.}} = p_{z,\text{e.o.}} = 0$). The shapes of the sectors are set up to achieve correct isochronism ($\Delta t = 2\pi/N$) and vertical tune using the approximate formulas. Magnetic fields are calculated with a magnet code, and the equations of motion used to find the e.o., tunes and $\Delta t$. This is repeated at many energies. Often vertical tune is imaginary or traverses dangerous resonances, and $\Delta t \neq 2\pi/N$. And the "shimming" begins.

These techniques were already used in the mid-50s. MURA physicists (e.g. Laslett) discovered many wonderful things: Unstable Fixed-Points, Islands, Resonances, Chaos, Separatrices, etc. They were at the forefront not only of accelerator physics, but also computing. Frank Cole writing about the year 1956:

*IBM was anxious to learn a lot more ... about scientific programming. They even sent two of their advanced programmers for several months, but it was remarkable how far ahead we were. The two IBM programmers laboured for months to produce an orbit-integration program, then had to leave before it was verified. When it was run, it didn't work properly, so Snyder (a MURA physicist) wrote one over the weekend and it worked Monday morning.*