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Canonical Formulations and Cancellation Effect in Electrodynamics of Relativistic Beams on a Curved Trajectory

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Abstract

The cancellation effect has been an important and long-standing controversial issue in the study of coherent synchrotron radiation (CSR) induced electron bunch dynamics in bends. In this paper, equations of motion are derived from the canonical formulation of the dynamics for an ultrarelativistic electron bunch under collective interaction on a curved trajectory. Using *retarded potentials*, we show that the cancellation occurs in the first order of optics between the *local interaction contribution* to the horizontal collective radiative force, which often gives the logarithmic dependence of the horizontal radiative force on the bunch transverse size, and the *local interaction contribution* to the kinetic energy via the potential energy change, which influences the particle's horizontal dynamics through dispersion. With the cancellation taken into account, the particle's horizontal dynamics in a bending system is driven both by the dispersion effect due to the initial kinetic energy deviation from the design energy *together with* the initial potential energy, and by the *effective* collective forces mainly contributed from the *non-local* collective radiative interactions. The local interaction influences the particle dynamics only in the second or higher order terms. Gauge invariance of our results is also discussed.

1. INTRODUCTION

The existence of the centrifugal space charge force F^{CSCF} was first pointed out by Talman [1] when studying a beam of charged particles following a curved path in an external magnetic field. It was understood that this force was due to the nearby (or local) particle interaction via curvature induced radiation fields, giving rise to the non-cancellation of the beam-induced electric and magnetic fields for ultrarelativistic beams. It was shown that when two particles in a bunch on a circular orbit approach zero distance, this transverse radiative force between the two particles exhibits a singular behavior. Even though this singularity can be removed for finite beam size, the resulting logarithmic divergence of the transverse force can cause rapid variation of the horizontal force in the transverse dimensions, causing a shift in horizontal tune and a significant contribution to chromaticity for a coasting beam in electron storage rings. These effects of logarithmic divergence in F^{CSCF} on the horizontal bunch dynamics were later pointed out by Lee [2] to be canceled by the effect of the beam induced electric potential, which enters into the horizontal dynamics through dispersion by changing the kinetic energies of the particles. The residual effect on a *coasting beam* in a storage ring, after taking the cancellation into account, to the first order, is about σ_x/R ($\sigma_x/R \ll 1$) times the F^{CSCF} effect, where σ_x is the horizontal bunch size, and R is the equilibrium radius of the ring.

Recently, due to the possibility of producing high peak current electron beams, transporting these electron bunches while preserving high phase space brightness has become a critical issue. It is then important to understand the effect of the coherent synchrotron radiation (CSR) induced emittance degradation in bends. Even though the cancellation effect was cleared for *coasting beams*, it was in dispute again for the CSR induced horizontal effect for *bunched beams*. In Ref. [3] it was concluded that the effect of F^{CSCF} is no longer canceled by the potential energy for bunched beams, and there exists a longitudinal force named the non-inertial space charge force F^{NISF} [4] in addition to the usual longitudinal space charge and CSR forces. At the same time, it was pointed out [5] that for bunched beams, there is always the cancellation between the effect of F^{CSCF} on the horizontal bunch dynamics and the effect of potential energy. Further analysis [6] shows that the accumulated effect of F^{NISF} contributes to the change of the potential energy; thus its effect on the horizontal bunch dynamics nearly cancels with that of F^{CSCF}. For the steady-state rigid-line bunch case, it was shown [6] that the residual of the cancellation is much smaller in magnitude compared to F^{CSCF}. Most recently, the generality of the cancellation effect was questioned [7] and what exactly the cancellation meant was under discussion again [7]-[9].

In this paper, we study the transverse dynamics of a charged particle distribution in the external and collective electromagnetic (EM) fields. The charged particle distribution is moving on an arbitrary trajectory with ultrarelativistic speed and is observed in an inertial laboratory frame. Usually, the charged particle motion is studied using the Lorentz force in terms of EM fields, where the collective interaction fields for curved trajectories are calculated using Liénard-Wiechert fields. However, in our study, it is found that an interesting relationship, which is between the usual centrifugal force—due to the change of direction of the kinetic momentum—and the radial collective EM force associated with the charged particle distributions on an curved trajectory, cannot be made obvious if one studies particle dynamics using Liénard-Wiechert fields. As an alternative, we analyze the charged particle dynamics via the canonical momentum. We show that the two approaches are equivalent, yet the latter one, with the choice of the retarded potentials in the Lorentz gauge, can provide some interesting insight on the interaction process under study. In particular, we aim to demonstrate the generality of the cancellation effect, which is the cancellation between (1) the effect on the horizontal particle dynamics due to the *local interaction contribution* to the radial collective radiative force and (2) the effect on the horizontal particle dynamics due to the *local interaction contribution* to the kinetic energy change via the potential energy change, which influences the particle's horizontal dynamics through dispersion.

In order to demonstrate the cancellation, in Sec. 2, the equations of motion for particles in an electron bunch, which are moving ultrarelativistically on a curvilinear orbit undergoing collective interaction, are derived from several different points of view. First, the equations of motion are written in a Cartesian frame, where by using both (1) the Lorentz force approach in terms of EM fields and (2) the canonical momentum approach in terms of potentials, we establish the equivalence of the two approaches. Next (view i), the equations of motion in the Cartesian frame based on Lorentz forces are projected to the curvilinear frame for a circular orbit with the EM fields expressed in terms of potentials. Then (view ii), using Lagrangian analysis for charged particle dynamics, the equations of motion are derived for an arbitrary curvilinear Frenet-Serret coordinate system, which yields the equations of motion around a circular orbit (view i) as a special case. Furthermore (view iii), it is shown that the equations of *transverse* motion obtained from view ii can be directly obtained by projecting the equations for the canonical momentum in a Cartesian frame to a Frenet-Serret frame. Lastly (view iv), Hamiltonian analysis is carried out which yields the transverse equation of motion consistent with previous views.

With the equations of motion in terms of the interaction potentials derived in Sec. 2, in Sec. 3, we use retarded potentials to demonstrate the cancellation effect. From view i, the particle's horizontal dynamics is driven by (a) the kinetic energy deviation from design energy and (b) the radial Lorentz force due to collective interactions. We will show that the *local* interaction contributions to the changes of each of the above two driving factors over time almost cancel and the residual effect is negligible compared to the remaining effective terms. However, instead of the aforementioned *cancellation* between effects in (a) and (b) seen from view point i, from view points ii, iii and iv we see that for an ultrarelativistic beam, it is the *canonical* energy, i.e., the kinetic and potential energy *together*, that experiences the curvature induced dispersion effect. This curvature effect for the canonical energy as a whole is depicted by the *generalized centrifugal force*, in contrast to the *usual centrifugal force* related to the curvature effects for the kinetic energy only. Moreover, since CSR changes the canonical energy, it is found out that apart from the dispersion effect related to the initial canonical energy deviation from the designed kinetic energy, only the effective forces (majorly contributed from the *non-local* interactions) are responsible for driving the horizontal dynamics. It is also shown that the cancellation effect is independent of the choice of gauge, as would be expected. In Sec. 4, our understanding of the cancellation effect is summarized, followed by a discussion why the cancellation effect has been a long-standing controversial issue, including discussion of the counter-example raised in Ref. [7]. A conclusion is given in Sec. 5, where the cancellation effect with shielding is highlighted.

2. GENERAL FORMULAS

In this section, based on the general covariant classical field theory reviewed in Appendix A, we first formulate in the nominal way the charged particle dynamics in the inertial laboratory frame with a Cartesian coordinate system. The equations of motion in the Frenet-Serret frame for an curvilinear orbit are then derived by (1) projection of the Lorentz force equation onto the Frenet-Serret bases for an arbitrary reference trajectory with the fields written in terms of potentials, (2) Lagrangian analysis for general curvilinear coordinates, (3) directly projecting the dynamics of canonical momentum in the Cartesian frame onto Frenet-Serret bases, and (4) Hamiltonian analysis. All these methods generate the same set of equations of motion for the particle's transverse dynamics, which set the stage for the discussion of the cancellation effect in the following section.

2.1. Charged Particle Dynamics in a Cartesian Frame

To set the foundation for the analysis of the cancellation effect, here we review some classical EM field theory [10]. Our purpose is to show the equivalence of the first set of dynamics equations in terms of EM *fields*—Eqs. (13), (17), and (19)—with the second set of dynamics equations in terms of *potentials*—Eqs. (8), (11) and (12)—respectively. As will be discussed in Sec. 4, using the first set of equations with the Liénard-Wiechert fields as the collective fields, one may not see the cancellation effect explicitly. On the other hand, the cancellation effect can be demonstrated straightforwardly using the second set of equations in terms of retarded potentials.

For an observer in the inertial laboratory frame using a Cartesian coordinate system, the spacetime is described by the Minkowski metric tensor

$$g_{00} = 1, \quad g_{\mu 0} = g_{0\mu} = 0, \quad g_{ij} = -\delta_{ij} \tag{1}$$

with μ from 1 to 4, and i, j from 1 to 3. Here the 4-spacetime-vector is $x^\mu = (ct, \mathbf{x})$, and Eq. (A5) gives $dt = \gamma d\tau$, with $\gamma = 1/\sqrt{1-\beta^2}$ for $\beta = |\boldsymbol{\beta}|$, $\boldsymbol{\beta} = \mathbf{v}/c$ and $\mathbf{v} = d\mathbf{x}/dt$. The 4-velocity is $V^\mu = dx^\mu/d\tau = \gamma(c, \mathbf{v})$, with $\sqrt{V^\mu V_\mu} = c$, and the 4-kinetic-momentum is $p^\mu = mV^\mu = (E/c, \mathbf{p})$ for $E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$. The Lagrangian in this case can be written as a function of x^μ and V^μ, with τ as the independent variable

$$\tilde{L}(x^\mu, V^\mu) = \tilde{L}_{\text{free}} + \tilde{L}_{\text{int}}, \tag{2}$$

with the free particle Lagrangian

$$\tilde{L}_{\text{free}} = -mc\sqrt{V_\mu V^\mu} = \gamma L_{\text{free}}, \qquad L_{\text{free}} = -mc^2\sqrt{1-\frac{v^2}{c^2}}, \tag{3}$$

and the interaction Lagrangian in terms of the 4-potential $A^\mu = (\Phi, \mathbf{A})$:

$$\tilde{L}_{\text{int}} = -\frac{e}{c}V_\mu A^\mu = \gamma L_{\text{int}}, \tag{4}$$

with

$$L_{\rm int} = -e(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}) = -e(\Phi - \sum_i \beta_i A_i). \tag{5}$$

Here $A_i = \mathbf{A}\cdot\mathbf{e}_i$ and $\beta_i = \boldsymbol{\beta}\cdot\mathbf{e}_i$ with $\mathbf{e}_i$ $(i = 1, 2, 3)$ the Cartesian basis, and $L_{\rm free}$ and $L_{\rm int}$ are Lagrangians with t as the independent variable. The canonical momentum P^μ conjugate to x^μ is

$$P^\mu = -\frac{\partial \tilde{L}}{\partial V_\mu} = p^\mu + \frac{e}{c}A^\mu, \tag{6}$$

with its components

$$P^0 = \frac{E + e\Phi}{c}, \qquad \mathbf{P} = \mathbf{p} + \frac{e}{c}\mathbf{A}. \tag{7}$$

Thus using Eq. (1), the Euler-Lagrangian equation in Eq. (A4) gives the time derivative for the canonical momentum ($ds = cd\tau$)

$$\frac{dP^\mu}{d\tau} = -\frac{\partial \tilde{L}_{\rm int}}{\partial x_\mu} = \frac{e}{c}V_\nu \frac{\partial A^\nu}{\partial x_\mu} \tag{8}$$

where $\partial/\partial x_\mu = (\partial/\partial x_0, -\nabla)$. Equation (8) shows that in the Minkowski spacetime the change of the canonical momentum over time is driven by the gradient of the interaction Lagrangian. For convenience, let us denote $\hat{\nabla}$ and $\hat{\partial}/\partial t$ as operators only acting on potentials Φ and $\mathbf{A}$:

$$\hat{\nabla}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}) \equiv \nabla\Phi - \sum_i \beta_i \nabla A_i, \qquad \frac{\hat{\partial}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A})}{\partial t} \equiv \frac{\partial \Phi}{\partial t} - \sum_i \beta_i \frac{\partial A_i}{\partial t}. \tag{9}$$

The vector components of Eq. (8) can then be written as

$$\frac{d\mathbf{P}}{dt} = \hat{\nabla} L_{\rm int}, \tag{10}$$

or

$$\frac{d\left(\mathbf{p} + e\mathbf{A}/c\right)}{dt} = -e(\nabla\Phi - \sum_i \beta_i \nabla A_i), \tag{11}$$

with $\mathbf{p} = \gamma m \mathbf{v}$, and the zeroth component of Eq. (8) gives the energy relation

$$\frac{d\left(E + e\Phi\right)}{dt} = -\frac{\hat{\partial} L_{\rm int}}{\partial t} \equiv e\left(\frac{\partial \Phi}{\partial t} - \sum_i \beta_i \frac{\partial A_i}{\partial t}\right), \tag{12}$$

with $E = \gamma mc^2$ the kinetic energy, and $E + e\Phi$ the canonical energy.

The equivalence of the above canonical formulations with the nominal electrodynamics in terms of EM fields and Lorentz forces can be traced back to Eq. (8). Using Eqs. (6)-(8), we obtain the time derivative for the kinetic momentum

$$\frac{dp^\mu}{d\tau} = \frac{e}{c}F^{\mu\nu}V_\nu, \tag{13}$$

with the field tensor expressed in terms of the 4-potential A^μ:

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}. \tag{14}$$

The matrix form of the field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{15}$$

where the fields $\mathbf{E}$ and $\mathbf{B}$ are related to the scalar and vector potentials Φ and $\mathbf{A}$ via

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{16}$$

Note that Eq. (13) is the straightforward reduction of Eq. (A6) for Minkowski spacetime, where the connection in Eq. (A8) vanishes. The vector components of Eq. (13) can be written in terms of the Lorentz force $\mathbf{F}$

$$\frac{d\,\mathbf{p}}{dt} = \mathbf{F} \tag{17}$$

with

$$\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{18}$$

and the scalar component of Eq. (13) gives

$$\frac{dE}{dt} = \mathbf{v}\cdot\mathbf{F} = e\mathbf{v}\cdot\mathbf{E}. \tag{19}$$

With notations in Eq. (9), we can write the Lorentz force in Eq. (18) in terms of potentials by combining Eqs. (16) and (18):

$$\mathbf{F} = -\frac{e}{c}\frac{d\mathbf{A}}{dt} - e\hat{\nabla}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}), \quad \text{with} \quad \frac{d\mathbf{A}}{dt} = \frac{\partial A}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}, \tag{20}$$

and

$$\mathbf{v}\cdot\mathbf{E} = -e\frac{d\Phi}{dt} + e\frac{\hat{\partial}}{\partial t}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}). \tag{21}$$

Substituting Eqs. (20) and (21) into Eqs. (17) and (19) respectively, one can obtain the equations for canonical momentum and energy in Eqs. (11) and (12). Note that Eqs. (11) can also be derived from the Hamiltonian (canonical energy)

$$H = E + e\Phi = c\sqrt{(\mathbf{P} - e\mathbf{A}/c)^2 + m^2c^2} + e\,\Phi, \tag{22}$$

with $\mathbf{P}$ defined in Eq. (7), and the Hamiltonian equations

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial P}, \qquad \frac{d\mathbf{P}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}. \tag{23}$$

The energy relation in Eq. (12) is equivalent to

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{24}$$

Since the collective interaction potentials and fields comove with the charge distribution while the external potentials or fields do not, it is useful to separate the potentials, fields and Lorentz force into their collective and external counterparts:

$$\Phi = \Phi^{\text{ext}} + \Phi^{\text{col}}, \qquad \mathbf{A} = \mathbf{A}^{\text{ext}} + \mathbf{A}^{\text{col}}, \tag{25}$$

$$\mathbf{E} = \mathbf{E}^{\text{ext}} + \mathbf{E}^{\text{col}}, \qquad \mathbf{B} = \mathbf{B}^{\text{ext}} + \mathbf{B}^{\text{col}}, \tag{26}$$

with

$$\mathbf{E}^{\text{ext}} = -\nabla\Phi^{\text{ext}} - \frac{1}{c}\frac{\partial \mathbf{A}^{\text{ext}}}{\partial t}, \qquad \mathbf{B}^{\text{ext}} = \nabla \times \mathbf{A}^{\text{ext}}, \tag{27}$$

$$\mathbf{E}^{\text{col}} = -\nabla\Phi^{\text{col}} - \frac{1}{c}\frac{\partial \mathbf{A}^{\text{col}}}{\partial t}, \qquad \mathbf{B}^{\text{col}} = \nabla \times \mathbf{A}^{\text{col}}. \tag{28}$$

The Lorentz force is separated as

$$\mathbf{F} = \mathbf{F}^{\text{ext}} + \mathbf{F}^{\text{col}}, \tag{29}$$

with

$$\mathbf{F}^{\text{ext}} = e(\mathbf{E}^{\text{ext}} + \frac{\mathbf{v}}{c} \times \mathbf{B}^{\text{ext}}), \qquad \mathbf{F}^{\text{col}} = e(\mathbf{E}^{\text{col}} + \frac{\mathbf{v}}{c} \times \mathbf{B}^{\text{col}}), \tag{30}$$

which can also be expressed in terms of potentials using Eq. (20)

$$\mathbf{F}^{\text{ext}} = -\frac{e}{c}\frac{d\mathbf{A}^{\text{ext}}}{dt} + \hat{\nabla} L_{\text{int}}^{\text{ext}}, \qquad \mathbf{F}^{\text{col}} = -\frac{e}{c}\frac{d\mathbf{A}^{\text{col}}}{dt} + \hat{\nabla} L_{\text{int}}^{\text{col}}, \tag{31}$$

where the interaction Lagrangians are expressed in terms of the vector components in the Cartesian frame

$$L_{\text{int}}^{\text{ext}} = -e(\Phi^{\text{ext}} - \boldsymbol{\beta} \cdot \mathbf{A}^{\text{ext}}) = -e(\Phi^{\text{ext}} - \sum_i \beta_i A_i^{\text{ext}}) \tag{32}$$

$$L_{\text{int}}^{\text{col}} = -e(\Phi^{\text{col}} - \boldsymbol{\beta} \cdot \mathbf{A}^{\text{col}}) = -e(\Phi^{\text{col}} - \sum_i \beta_i A_i^{\text{col}}). \tag{33}$$

2.2. View i: Lorentz Force and Centrifugal Force on a Curved Orbit

We now consider a charged particle distribution moving on a curvilinear trajectory, and study the dynamics of the charged particles under external and collective EM fields. We first study the dynamics in terms of the Lorentz force due to external and collective interactions, as well as the centrifugal force due to curvature of the reference trajectory.

Let us denote the reference trajectory predetermined by the external fields to be $\mathbf{r}_0(s)$ in the laboratory frame, where s is the path length along this trajectory measured from a certain fixed initial point

$$s = \int_{s_0}^{s} \sqrt{\frac{d\mathbf{r}_0}{ds} \cdot \frac{d\mathbf{r}_0}{ds}} ds_1. \tag{34}$$

We now consider the Frenet-Serret system [11], which is the curvilinear coordinate system naturally built in along the reference trajectory. A vector around the reference trajectory can be represented as

$$\mathbf{r}(s, x, y) = \mathbf{r}_0(s) + x\,\mathbf{e}_x + y\,\mathbf{e}_y \tag{35}$$

where s, x, and y are the longitudinal, horizontal, and vertical coordinates respectively. The unit vector tangent to the curvilinear trajectory is $\mathbf{e}_s$:

$$\frac{d\,\mathbf{r}_0(s)}{ds} = \mathbf{e}_s. \tag{36}$$

The principal normal $\mathbf{e}_x$ and binormal $\mathbf{e}_y$ of the curve are

$$\mathbf{e}_x = -\frac{d\mathbf{e}_s/ds}{|d\mathbf{e}_s/ds|}, \qquad \mathbf{e}_y = \mathbf{e}_s \times \mathbf{e}_x. \tag{37}$$

The unit vectors $\mathbf{e}_s$, $\mathbf{e}_x$ and $\mathbf{e}_y$ form the bases of the Frenet-Serret system, which are functions of s:

$$\begin{pmatrix} d\,\mathbf{e}_s/ds \\ d\,\mathbf{e}_x/ds \\ d\,\mathbf{e}_y/ds \end{pmatrix} = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_s \\ \mathbf{e}_x \\ \mathbf{e}_y \end{pmatrix}, \tag{38}$$

where $\kappa(s)$ and $\tau(s)$ are respectively the curvature and torsion of the reference curve in the three dimensional space (note here τ is no longer the proper time in Sec. 2.1 and Appendix A). For a curve in the two dimensional plane, the torsion $\tau(s)$ vanishes. From Eq. (35), we get the velocity components for the Frenet-Serret bases

$$\frac{d\mathbf{r}}{dt} = v_s\,\mathbf{e}_s + v_x\,\mathbf{e}_x + v_y\,\mathbf{e}_y \tag{39}$$

for

$$v_s = (1+\kappa x)\dot{s}, \quad v_x = (\dot{x} - \tau y \dot{s}), \quad v_y = (\dot{y} + \tau x \dot{s}), \tag{40}$$

with $\dot{s} = ds/dt, \dot{x} = dx/dt, \dot{y} = dy/dt$.

Typically, for a charge distribution moving on a curvilinear reference trajectory, the charge dynamics given by Eq. (17) is projected to the Frenet-Serret bases. The time derivative of the momentum projected to a Frenet-Serret basis $\mathbf{e}_\lambda$ ($\mathbf{e}_\lambda$ is any one of $\mathbf{e}_s, \mathbf{e}_x, \mathbf{e}_y$) is

$$\frac{d\left(\mathbf{p}\cdot\mathbf{e}_\lambda\right)}{dt} = \frac{d\,\mathbf{p}}{dt}\cdot\mathbf{e}_\lambda + \dot{s}\,\mathbf{p}\cdot\frac{d\,\mathbf{e}_\lambda}{ds}. \tag{41}$$

Using Eq. (38) for $d\mathbf{e}_\lambda/ds$, the equation of motion in Eq. (17) becomes

$$\frac{dp_s}{dt} = -\dot{s}\kappa p_x + \mathbf{e}_s\cdot\mathbf{F}^{\rm ext} + \mathbf{e}_s\cdot\mathbf{F}^{\rm col} \tag{42}$$

$$\frac{dp_x}{dt} = \dot{s}\left(\kappa p_s + \tau p_y\right) + \mathbf{e}_x\cdot\mathbf{F}^{\rm ext} + \mathbf{e}_x\cdot\mathbf{F}^{\rm col} \tag{43}$$

$$\frac{dp_y}{dt} = -\dot{s}\tau p_x + \mathbf{e}_y\cdot\mathbf{F}^{\rm ext} + \mathbf{e}_y\cdot\mathbf{F}^{\rm col}, \tag{44}$$

where Eq. (30) is used for the Lorentz forces. Note that in Eq. (43), the term $\dot{s}\kappa p_s$ is the usual centrifugal force due to the curvature effect. The conventional approach to Eqs. (42)-(44) at this point is to use the Liénard-Wiechert fields in Eqs. (D3), (D4), and (D6) to calculate $\mathbf{F}^{\rm col}$ in Eq. (30) based on the history of the charge distribution along the curved orbit. As will be discussed in Sec. 4, although this Liénard-Wiechert fields approach is equivalent to the analysis based on potentials, the latter can be used to *explicitly* demonstrate the cancellation effect, while the former can only take care of the cancellation implicitly if *both* the longitudinal and transverse collective force (especially the radiative force) are taken into account. For this reason, we express the fields in terms of potentials as in Ref. [5].

Let us look at the case of a bunch moving on a circular orbit of radius R, which has $\kappa = 1/R$ and $\tau = 0$, and write Eqs. (42)-(44) in terms of potentials. The reference orbit is now

$$\mathbf{r}_0(s) = R\,\mathbf{e}_x(s) \tag{45}$$

with the Frenet-Serret basis $\mathbf{e}_\lambda$ ($\lambda = s, x, y$) related to the Cartesian basis $\mathbf{e}_i$ ($i = 1, 2, 3$) by

$$\mathbf{e}_s = -\sin\frac{s}{R}\,\mathbf{e}_1 + \cos\frac{s}{R}\,\mathbf{e}_2, \quad \mathbf{e}_x = \cos\frac{s}{R}\,\mathbf{e}_1 + \sin\frac{s}{R}\,\mathbf{e}_2, \quad \mathbf{e}_y = -\mathbf{e}_3. \tag{46}$$

The momentum components in this cylindrical coordinate system are obtained from Eq. (40)

$$p_\lambda = \gamma m v_\lambda, \quad \text{with} \quad \gamma = \sqrt{1 - v^2/c^2} \tag{47}$$

for

$$v_s = r\dot{\theta}, \quad v_x = \dot{x}, \quad v_y = \dot{y}, \tag{48}$$

where $r \equiv R + x$ is the distance from the test particle to the center of the circular reference orbit, and $\theta = s/R$. Here instead of using the Liénard-Wiechert fields, we write out the collective fields in terms of potentials using Eq. (28) with cylindrical coordinates:

$$\mathbf{E}^{\rm col} = \left(-\frac{\partial \Phi^{\rm col}}{r\partial\theta} - \frac{\partial A_s^{\rm col}}{c\partial t}\right)\mathbf{e}_s + \left(-\frac{\partial \Phi^{\rm col}}{\partial x} - \frac{\partial A_x^{\rm col}}{c\partial t}\right)\mathbf{e}_x + \left(-\frac{\partial \Phi^{\rm col}}{\partial y} - \frac{\partial A_y^{\rm col}}{c\partial t}\right)\mathbf{e}_y, \tag{49}$$

and

$$\mathbf{B}^{\rm col} = \left(\frac{\partial A_y^{\rm col}}{\partial x} - \frac{\partial A_x^{\rm col}}{\partial y}\right)\mathbf{e}_s + \left(\frac{\partial A_s^{\rm col}}{\partial y} - \frac{\partial A_y^{\rm col}}{r\partial\theta}\right)\mathbf{e}_x + \frac{1}{r}\left(\frac{\partial A_x^{\rm col}}{\partial\theta} - \frac{\partial (rA_s^{\rm col})}{\partial x}\right)\mathbf{e}_y. \tag{50}$$

For the interaction Lagrangian in terms of the vector components in the Frenet-Serret frame with subscript $\lambda = (s, x, y)$

$$\mathcal{L}_{\rm int} = -e(\Phi - \sum_\lambda \beta_\lambda A_\lambda), \tag{51}$$

as opposed to Eq. (5) for a Cartesian frame, we define $\hat{\nabla}$ and $\hat{\partial}/\partial t$ as

$$\hat{\nabla}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}) \equiv \nabla\Phi - \sum_i \beta_\lambda \nabla A_\lambda, \qquad \frac{\hat{\partial}(\Phi - \boldsymbol{\beta}\cdot\mathbf{A})}{\partial t} \equiv \frac{\partial\Phi}{\partial t} - \sum_i \beta_\lambda \frac{\partial A_\lambda}{\partial t}. \tag{52}$$

The Lorentz force in Eq. (30) due to the collective fields in Eqs. (49) and (50) can then be written as

$$F_s^{\rm col} = \frac{\hat{\partial}\mathcal{L}_{\rm int}^{\rm col}}{r\partial\theta} - \frac{e}{c}\frac{dA_s^{\rm col}}{dt} - e\beta_x\frac{A_s^{\rm col}}{r} \tag{53}$$

$$F_x^{\rm col} = \frac{\hat{\partial}\mathcal{L}_{\rm int}^{\rm col}}{\partial x} - \frac{e}{c}\frac{dA_x^{\rm col}}{dt} + e\beta_s\frac{A_s^{\rm col}}{r} \tag{54}$$

$$F_y^{\rm col} = \frac{\hat{\partial}\mathcal{L}_{\rm int}^{\rm col}}{\partial y} - \frac{e}{c}\frac{dA_y^{\rm col}}{dt} \tag{55}$$

where $\beta_s = r\dot{\theta}/c$, $\beta_x = \dot{x}/c$, and for subscript $\lambda, \eta = (s, x, y)$, one has

$$\frac{dA_\lambda^{\rm col}}{dt} = \frac{\partial A_\lambda^{\rm col}}{\partial t} + (\sum_\eta v_\eta \partial_\eta)A_\lambda. \tag{56}$$

The equations of motion in Eqs. (42)-(44) thus become (using $\kappa\dot{s} = v_s/r$ and $\dot{s}\kappa p_x = v_x p_s/r$)

$$\frac{d(p_s + eA_s^{\rm col}/c)}{dt} = F_s^{\rm ext} + \frac{\hat{\partial}\mathcal{L}_{\rm int}^{\rm col}}{r\partial\theta} - v_x\frac{p_s + eA_s^{\rm col}/c}{r} \tag{57}$$

$$\frac{d(p_x + eA_x^{\rm col}/c)}{dt} = F_x^{\rm ext} + \frac{\hat{\partial}\mathcal{L}_{\rm int}^{\rm col}}{\partial x} + v_s\frac{p_s + eA_s^{\rm col}/c}{r} \tag{58}$$

$$\frac{d(p_y + eA_y^{\rm col}/c)}{dt} = F_y^{\rm ext} + \frac{\hat{\partial}\mathcal{L}_{\rm int}^{\rm col}}{\partial y}. \tag{59}$$

The energy relation is not affected by the curvature of the trajectory, and is obtained from Eqs. (19), (21) and (26)

$$\frac{d(E + e\Phi^{\rm col})}{dt} = e\mathbf{v} \cdot \mathbf{E}^{\rm ext} - \frac{\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}}{\partial t}. \tag{60}$$

The difference between $\hat{\nabla}$ (or $\hat{\partial}_t$) acting on $L_{\rm int}$ in Eq. (5) and $\mathcal{L}_{\rm int}$ in Eq. (51) is discussed in Appendix B.

For the horizontal dynamics, the last term on the right-hand side of Eq. (58) (which is $\kappa \dot{s} P_s$ in Eq. (71) of Sec. 2.3) shows that $e\beta_s A_s^{\rm col}/r$, which is a part of the radial collective force $\mathbf{F}_x^{\rm col}$ (see Eq. (54)) and is centrifugal in direction, turns on (becoming nonzero) simultaneously with the usual centrifugal force $v_s p_s/r$ (which is $\dot{s}\kappa p_s$ in Eq. (43)) at the instant when the bunch enters a circular orbit ($\kappa \neq 0$) from a straight path ($\kappa = 0$), and it turns off simultaneously with the centrifugal force as well at the instant when the bunch exits to a straight path again. For this reason, we denote this part of the collective radial force as the centrifugal force due to space charge interaction, or, the "*centrifugal space charge force*" $F^{\rm CSCF}$:

$$F^{\rm CSCF} = \frac{e\beta_s A_s^{\rm col}}{r}, \tag{61}$$

which arises from $\mathbf{B}^{\rm col}$ in Eq. (50) for a circular orbit. Eq. (58) shows clearly that $v_s p_s/r$ always works together with $e\beta_s A_s^{\rm col}/r$, meaning the usual centrifugal force—due to the radial acceleration—always works together with the centrifugal space charge force caused by collective radiative force—also due to the radial acceleration. On the other hand, the fact that the two seemingly unrelated terms in Eq. (43), one being the usual centrifugal force $\dot{s}\kappa p_s$ and the other being the collective radial force $\mathbf{e}_x \cdot \mathbf{F}^{\rm col}$, actually are strongly correlated—in the way of cancellation of the local interaction contributions to the two terms (see Sec. 3)—has caused many puzzles. This point will be further discussed in Sec. 4.

2.3. View ii: Lagrangian Analysis of Charged Particle Dynamics in Curvilinear Coordinate Systems

We have by now derived the equations of motion around a circular orbit, with fields written in terms of potentials, and show the terms representing the curvature effects. In this subsection, we will consider the more general case of a charge distribution moving on an arbitrary curvilinear reference trajectory, and derive the equations of motion using

Lagrangian analysis with Frenet-Serret coordinates. For the particular case of a circular orbit, the general equations of motion from the Lagrangian agree (as expected) with results given in Sec. 2.2 obtained by projecting the equations of motion in terms of Lorentz forces onto the Frenet-Serret bases.

With the Frenet-Serret coordinates defined in Eq. (35), and the velocity components in Eq. (40), the Lagrangian using Eqs. (3) and (5) with t as the independent variable is now

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \\ &= -mc^2 \left[1 - \frac{[(1+\kappa x)\dot{s}]^2 + (\dot{x} - \tau y \dot{s})^2 + (\dot{y} + \tau x \dot{s})^2}{c^2}\right]^{1/2} \\ &\quad -e\left[\Phi - \frac{(1+\kappa x)\dot{s} A_s + (\dot{x} - \tau y \dot{s}) A_x + (\dot{y} + \tau x \dot{s}) A_y}{c}\right]. \end{aligned} \tag{62}$$

The momentums conjugate to the Frenet-Serret coordinates are

$$\mathcal{P}_s = \frac{\partial \mathcal{L}}{\partial \dot{s}} = (1+\kappa x) P_s - \tau (y P_x - x P_y), \tag{63}$$

$$\mathcal{P}_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = P_x, \quad \mathcal{P}_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = P_y, \tag{64}$$

where P_s, P_x, and P_y are the direct projections of the canonical momentum in the Cartesian frame in Eq. (7) to the Frenet-Serret bases

$$P_s = \gamma m v_s + e A_s / c, \quad P_x = \gamma m v_x + e A_x / c, \quad P_y = \gamma m v_s + e A_y / c, \tag{65}$$

with v_s, v_x and v_y given in Eq. (40).

The Euler-Lagrangian equations for the Frenet-Serret coordinates s, x and y

$$d\mathcal{P}_s/dt = \partial \mathcal{L}/\partial s, \quad d\mathcal{P}_x/dt = \partial \mathcal{L}/\partial x, \quad d\mathcal{P}_y/dt = \partial \mathcal{L}/\partial y \tag{66}$$

now become

$$\frac{d[(1+\kappa x)P_s - \tau(yP_x - xP_y)]}{dt} = \dot{s}\,[\kappa' x P_s - \tau'(yP_x - xP_y)] + \frac{\hat{\partial} \mathcal{L}_{\text{int}}}{\partial s} \tag{67}$$

with $\kappa' = d\kappa/ds$ and $\tau' = d\tau/ds$, and

$$\frac{dP_x}{dt} = \dot{s}\,(\kappa P_s + \tau P_y) + \frac{\hat{\partial} \mathcal{L}_{\text{int}}}{\partial x}, \tag{68}$$

$$\frac{dP_y}{dt} = \dot{s}\,(-\tau P_x) + \frac{\hat{\partial} \mathcal{L}_{\text{int}}}{\partial y}. \tag{69}$$

For a reference orbit in the two dimensional plane ($\tau = 0$), with arbitrary curvature $\kappa(s)$, the above equations become

$$\frac{dP_s}{dt} = \frac{-\kappa \dot{x}}{1+\kappa x} P_s + \frac{1}{1+\kappa x} \frac{\hat{\partial} \mathcal{L}_{\rm int}}{\partial s}, \tag{70}$$

$$\frac{dP_x}{dt} = \kappa \dot{s} P_s + \frac{\hat{\partial} \mathcal{L}_{\rm int}}{\partial x}, \tag{71}$$

$$\frac{dP_y}{dt} = \frac{\hat{\partial} \mathcal{L}_{\rm int}}{\partial y}. \tag{72}$$

For a circular reference orbit with radius R, $s = R\theta$, $\kappa(s) = 1/R$, and $r = R + x$, Eqs. (70)-(72) reduce to Eqs. (57)-(59) after separating the contributions of the external fields and the collective interaction fields. The energy relation in Eq. (12) or Eq. (60) stays untouched for the Lagrangian in Eq. (62). Note that in Eqs. (70)-(72), we have

$$\frac{\hat{\partial} \mathcal{L}_{\rm int}}{\partial x} = \mathbf{e}_x \cdot \hat{\nabla} \mathcal{L}_{\rm int}, \qquad \frac{\hat{\partial} \mathcal{L}_{\rm int}}{r \partial \theta} = \mathbf{e}_s \cdot \hat{\nabla} \mathcal{L}_{\rm int}, \qquad \frac{\hat{\partial} \mathcal{L}_{\rm int}}{\partial y} = \mathbf{e}_y \cdot \hat{\nabla} \mathcal{L}_{\rm int}, \tag{73}$$

with $\mathcal{L}_{\rm int}$ and $\hat{\nabla}$ given in Eqs. (51) and (52).

2.4. View iii: Projection of $\mathbf{p} + e\mathbf{A}/c$ in a Cartesian frame onto Frenet-Serret Bases

We now look at the relation between Eqs. (68) and (69) for Frenet-Serret coordinates and Eq. (10) for Cartesian coordinates by simply projecting the equation for the canonical momentum in the Cartesian frame onto the bases of the Frenet-Serret frame.

For $\mathbf{e}_\lambda$ being $\mathbf{e}_x$ or $\mathbf{e}_y$, and for P_λ given in Eq. (65), we have

$$\frac{d\,(\mathbf{P} \cdot \mathbf{e}_\lambda)}{dt} = \frac{d\,\mathbf{P}}{dt} \cdot \mathbf{e}_\lambda + \mathbf{P} \cdot \frac{d\,\mathbf{e}_\lambda}{dt}, \tag{74}$$

with $d\mathbf{e}_\lambda/dt = \dot{s}\, d\mathbf{e}_\lambda/ds$ as used earlier in Eq. (41). Applying Eqs. (10), (38) and (B10) to Eq. (74), we find for the transverse components

$$\frac{dP_x}{dt} = \dot{s}(\kappa \mathbf{e}_s + \tau \mathbf{e}_y) \cdot \mathbf{P} + \mathbf{e}_x \cdot \hat{\nabla} \mathcal{L}_{\rm int} \tag{75}$$

$$\frac{dP_y}{dt} = \dot{s}(-\tau \mathbf{e}_x) \cdot \mathbf{P} + \mathbf{e}_y \cdot \hat{\nabla} \mathcal{L}_{\rm int} \tag{76}$$

which agrees with Eqs. (68) and (69). After separating the external and collective counterparts of the Lorentz interaction, as we did in Eq. (58), we obtain Eqs. (58)-(59) from

Eqs. (75)-(76) for general curvature $\kappa(s) = 1/R(s)$, $\tau = 0$, and $r = R(s) + x$:

$$\frac{d(p_x + eA_x^{\text{col}}/c)}{dt} = F_x^{\text{ext}} + \frac{\hat{\partial}\mathcal{L}_{\text{int}}^{\text{col}}}{\partial x} + \dot{s}\frac{p_s + eA_s^{\text{col}}/c}{R}. \tag{77}$$

$$\frac{d(p_y + eA_y^{\text{col}}/c)}{dt} = F_x^{\text{ext}} + \frac{\hat{\partial}\mathcal{L}_{\text{int}}^{\text{col}}}{\partial y}. \tag{78}$$

The longitudinal dynamics equation in Eq. (57) is also reproduced for constant curvature $\kappa = 1/R$ and $\tau = 0$ by using Eqs. (74) and (B11)

$$\frac{d(p_s + eA_s^{\text{col}}/c)}{dt} = F_s^{\text{ext}} + \frac{\hat{\partial}\mathcal{L}_{\text{int}}^{\text{col}}}{r\partial\theta} - \dot{x}\frac{p_s + eA_s^{\text{col}}/c}{r}. \tag{79}$$

2.5. Discussions

Comparing the equations of motion in Eqs. (57)-(59) with Eq. (11) for a Cartesian coordinate system, which is rewritten as

$$\frac{d\left(\mathbf{p} + e\mathbf{A}^{\text{col}}/c\right)}{dt} = \mathbf{F}^{\text{ext}} + \hat{\nabla}L_{\text{int}}^{\text{col}}, \tag{80}$$

we can see that the last terms on the right-hand sides of Eqs. (57) and (58) depict the geometrical effect induced by the curvature of the reference orbit. Let us first consider $-v_x p_s/r$ in the last term of Eq. (57) and $v_s p_s/r$ in the last term of Eq. (58). For example, when an electron moves in a potential $\Phi^{\text{ext}}(\mathbf{r}, t)$ with $\mathbf{F}^{\text{ext}} = -e\nabla\Phi^{\text{ext}}$, there is no collective interaction, and Eq. (58) reduces to

$$\frac{dp_x}{dt} = v_s\frac{p_s}{r} - e\frac{\partial\Phi^{\text{ext}}}{\partial x} \tag{81}$$

where $v_s p_s/r = \gamma m r\dot{\theta}^2$ is the usual centrifugal force due to the rotation of the Frenet-Serret bases, and Eq. (57) reduces to

$$\frac{dp_s}{dt} = -v_x\frac{p_s}{r} - e\frac{\partial\Phi^{\text{ext}}}{r\partial\theta}. \tag{82}$$

The term $-v_x p_s/r$ in Eq. (82), which gives the curvature effect, can be combined with the term proportional to κ in dp_s/dt (using p_s in Eq. (47) with v_s in Eq. (40)), and then yields

$$r\frac{d\gamma m\dot{\theta}}{dt} = -2\gamma m\dot{r}\dot{\theta} - e\frac{\partial\Phi^{\text{ext}}}{r\partial\theta}, \tag{83}$$

where $-2\gamma m\dot{r}\dot{\theta}$ is the usual Coriolis force related to the rotation frame. These usual centrifugal and Coriolis forces are considered to be *fictitious* because they arise when the bases of the Frenet-Serret frame rotates, while the particle tends to maintain its inertial motion.

We now focus on the transverse dynamics for a relativistic electron bunch on a curved orbit (for $\tau = 0$ and $\Phi^{\rm ext} = 0$). As discussed in Sec. 2.2, the usual centrifugal force, $v_s p_s/r$, always works together with the centrifugal space charge force. This can be clearly seen from $\mathbf{P} \cdot d\mathbf{e}_x/dt$ in Eq. (74), which yield the generalized centrifugal force $F^{\rm GCF}$ related to the collective interaction potentials in Eq. (77) [8]:

$$F^{\rm GCF} \equiv \dot{s}\frac{p_s + eA_s^{\rm col}/c}{R} = v_s\frac{p_s + eA_s^{\rm col}/c}{r}. \tag{84}$$

For a bunch moving on a straight path with all the particles having the same speed $\boldsymbol{\beta} = \beta_s \mathbf{e}_s$, the retarded potentials in Eqs. (D10) and (D11) satisfy $A_s^{\rm col} = \beta_s \Phi^{\rm col}$, and thus the following equality holds exactly:

$$p_s + \frac{e}{c}A_s^{\rm col} = \beta_s\frac{E + e\Phi^{\rm col}}{c} = \beta_s\frac{H}{c}, \tag{85}$$

with the canonical energy defined as

$$H = E + e\Phi^{\rm col}. \tag{86}$$

As will be shown in Sec. 3, for an ultrarelativistic bunch moving on a curvilinear orbit, Eq. (85) is still a good approximation (assuming $\gamma_s^{-1} = \sqrt{1-\beta_s^{-2}} \ll 1$)

$$p_s + \frac{e}{c}A_s^{\rm col} \approx \beta_s\frac{H}{c}, \tag{87}$$

indicating that the retarded potentials due to the collective interaction, which are dominated by the local interaction contributions, travel together with the charged particle distribution (the external potentials are separated out because they do not have this property). The canonical energy H in Eq. (87) can be obtained by integrating Eq. (12) or Eq. (60) with $\mathbf{E}^{\rm ext} = 0$,

$$H(t) = H(t_0) - \int_{t_0}^{t} \frac{\hat{\partial}\mathcal{L}_{\rm ext}^{\rm col}}{\partial t} dt'. \tag{88}$$

Combining Eqs. (84), (87) and (88), one gets for $\beta_s \simeq 1$

$$F^{\rm GCF} \simeq \frac{H}{r} = \frac{H(t_0)}{r} - \frac{1}{r}\int_{t_0}^{t} \frac{\hat{\partial}\mathcal{L}_{\rm ext}^{\rm col}}{\partial t} dt', \tag{89}$$

and Eq. (77) becomes

$$\frac{d(p_x + eA_x^{\rm col}/c)}{dt} \simeq F_x^{\rm ext} + \frac{H(t_0)}{r} - \frac{1}{r}\int_{t_0}^{t} \frac{\hat{\partial}\mathcal{L}_{\rm ext}^{\rm int}}{\partial t} dt' + \frac{\hat{\partial}L_{\rm int}^{\rm col}}{\partial x}, \tag{90}$$

which shows that the change of the transverse canonical momentum is driven by the interaction Lagrangian induced effective forces, and by the dispersion effect related to the *initial canonical energy* (as opposed to the *initial kinetic energy* in usual treatments when the radial collective radiation force is not included).

2.6. View iv: Hamiltonian Analysis

In many applications, Hamiltonian (instead of Lagrangian) analysis is used to study charged particle dynamics. Starting from the canonical momentums in Eqs. (63) and (64), the Hamiltonian with time t as independent variable is obtained as

$$\begin{aligned} H &= \left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\dot{s} + \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{x} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\dot{y}\right) - \mathcal{L} \\ &= c\sqrt{\left(\frac{\mathcal{P}_s + \tau y \mathcal{P}_x - \tau x \mathcal{P}_y}{1+\kappa x} - \frac{e}{c}A_s\right)^2 + \left(\mathcal{P}_x - \frac{e}{c}A_x\right)^2 + \left(\mathcal{P}_y - \frac{e}{c}A_y\right)^2 + m^2c^2} + e\Phi. \end{aligned} \tag{91}$$

Converting the independent variable from t to the path length s, and for $A_\lambda = \mathbf{A}\cdot\mathbf{e}_\lambda$ with $\lambda = (s, x, y)$, the Hamiltonian conjugate to s can be obtained from Eq. (91) (see Ref. [11], [13] and Appendix C):

$$\begin{aligned} \mathcal{H} &= -\mathcal{P}_s = \tau(y\mathcal{P}_x - x\mathcal{P}_y) \\ &\quad -(1+\kappa x)\left[\frac{e}{c}A_s + \sqrt{\left(\frac{H - e\Phi}{c}\right)^2 - \left(\mathcal{P}_x - \frac{e}{c}A_x\right)^2 - \left(\mathcal{P}_y - \frac{e}{c}A_y\right)^2 - m^2c^2}\right] \end{aligned} \tag{92}$$

which contains the longitudinal kinetic momentum for the ultrarelativistic particle ($\beta_s \simeq 1$)

$$\begin{aligned} p_s &= \sqrt{\left(\frac{H - e\Phi}{c}\right)^2 - \left(\mathcal{P}_x - \frac{e}{c}A_x\right)^2 - \left(\mathcal{P}_y - \frac{e}{c}A_y\right)^2 - m^2c^2} \\ &= p - \frac{(\mathcal{P}_x - eA_x/c)^2}{2p} - \frac{(\mathcal{P}_y - eA_y/c)^2}{2p} + \cdots \end{aligned} \tag{93}$$

with the total kinetic momentum

$$p \equiv \frac{1}{c}\sqrt{E^2 - m^2c^4} \simeq \frac{E}{c}, \qquad E = H - e\Phi \gg mc^2. \tag{94}$$

Consider the case when the external potential is only present in $A_s = A_s^{\rm ext} + A_s^{\rm col}$. The Hamiltonian in Eq. (92) can then be expanded

$$\begin{aligned} \mathcal{H} &= \tau(y\mathcal{P}_x - x\mathcal{P}_y) - \frac{e}{c}\mathcal{A}_s^{\rm ext} \\ &\quad -(1+\kappa x)\left[\frac{e(A_s^{\rm col} - \Phi^{\rm col}) + H}{c} - \frac{(\mathcal{P}_x - eA_x^{\rm col}/c)^2}{2p} - \frac{(\mathcal{P}_y - eA_y^{\rm col}/c)^2}{2p}\right] + \cdots \end{aligned} \tag{95}$$

where the canonical external potential $\mathcal{A}_s^{\rm ext}$ is defined by

$$\mathcal{A}_s^{\rm ext} = (1+\kappa x)A_s^{\rm ext}. \tag{96}$$

For external magnetic fields

$$B_y = B_0 + B_1(s)x + \cdots, \quad \text{and} \quad B_x = B_1(s)y + \cdots, \tag{97}$$

with

$$B_0 = \frac{p_0 c}{eR}, \quad K_1(s) = \frac{eB_1}{p_0 c}, \tag{98}$$

for $p_0 = \gamma_0 \beta_0 mc$ with $\beta_0 = \sqrt{1-\gamma_0^{-2}}$, $\mathcal{A}_s^{\rm ext}$ is given by [14]

$$\mathcal{A}_s^{\rm ext} = -\frac{p_0 c}{e}\left[\frac{x}{R} + \left(\frac{1}{R^2} - K_1(s)\right)\frac{x^2}{2} + \frac{K_1 y^2}{2}\right] + \cdots. \tag{99}$$

The transverse equation of motion is thus

$$\frac{dx}{ds} = \frac{\partial \mathcal{H}}{\partial \mathcal{P}_x} = \tau y + \frac{P_x - eA_x^{\rm col}/c}{p} + \cdots \tag{100}$$

and

$$\begin{aligned} \frac{d\mathcal{P}_x}{ds} &= -\frac{\partial \mathcal{H}}{\partial x} \\ &= -\tau P_y + \frac{e}{c}\frac{\partial \mathcal{A}_s^{\rm ext}}{\partial x} + \mathcal{K}^{\rm GCF} + (1+\kappa x)\left[\frac{-e}{c}\frac{\hat{\partial}(\Phi^{\rm col} - \boldsymbol{\beta}\cdot\mathbf{A}^{\rm col})}{\partial x}\right] + \cdots \end{aligned} \tag{101}$$

with $\mathcal{K}^{\rm GCF}$ related to the generalized centrifugal force in Eq. (84)

$$\mathcal{K}^{\rm GCF} = \kappa\left(p_s + \frac{e}{c}A_s^{\rm col}\right) \simeq \frac{[e(A_s^{\rm col} - \Phi^{\rm col}) + H]/c}{R}, \tag{102}$$

where Eqs. (93) and (94) are used. As will be shown in Sec. 3, for ultrarelativistic beams, the retarded potentials $A_s^{\rm col}$ and $\Phi^{\rm col}$ are nearly canceled, and the residual of their cancellation is negligible compared to the other effective terms; thus $\mathcal{K}^{\rm GCF}$ only depends on the canonical energy

$$\mathcal{K}^{\rm GCF} \simeq \frac{H}{cR}. \tag{103}$$

The canonical energy H in the above equation satisfies

$$\frac{dH}{ds} = \frac{\partial \mathcal{H}}{\partial t} \simeq (1+\frac{x}{R})\frac{e}{c}\frac{\partial(\Phi^{\rm col} - \boldsymbol{\beta}\cdot\mathbf{A}^{\rm col})}{\partial t} \tag{104}$$

which, after integration, becomes

$$H(s) = H(s_0) + \int_{s_0}^{s}\left(1+\frac{x}{R}\right)F_v^{\rm eff} ds', \tag{105}$$

which is equivalent to Eq. (88) with $F_v^{\rm eff}$ defined in Eq. (119). Combination of Eqs. (99)-(101), (103) and (105) yields the perturbative expansion of the equation for horizontal motion

$$\frac{d^2x}{ds^2}+\frac{1}{1+\delta}\left[\frac{1}{R^2}-K_1(s)\right]x$$
$$\simeq\frac{1}{1+\delta}\left[\frac{H(s_0)-E_0}{E_0R}+\frac{1}{E_0R}\int_{s_0}^{s}\left(1+\frac{x}{R}\right)F_v^{\rm eff}\,ds'+\left(1+\frac{x}{R}\right)\frac{F_x^{\rm eff}}{E_0}\right]+\cdots,\quad(106)$$

with $F_x^{\rm eff}$ defined in Eq. (117), and

$$\frac{1}{1+\delta}=1-\delta+\delta^2-\delta^3+\cdots\qquad\text{for}\qquad\delta=\frac{E-E_0}{E_0}.\quad(107)$$

Note that due to the cancellation of the local interaction contributions in $A_s^{\rm col}$ and $\Phi^{\rm col}$, and the fact that the effective forces are dominated by non-local interaction (See Sec. 3 and Appendix E), apart from the initial canonical energy spread $H(s_0)-E_0$, the local charge interaction has negligible ($\propto\gamma^{-2}$ or $\beta_\perp^2$) contributions to the first order optics. However, one should note that the local charge interaction will still contribute to the second or higher order optics, because in Eq. (107), $E=H-\Phi^{\rm col}$ where $\Phi^{\rm col}(s)$ is the retarded potential dominated by the local charge interaction depending on the present charge distribution.

3. CANCELLATION EFFECTS ON A PLANE ORBIT

Based on the equations of motion in the previous section, here we demonstrate explicitly the cancellation of the local interaction effects on the bunch horizontal dynamics. Gauge invariance of the cancellation effect is also discussed.

3.1. Equation for the Horizontal Motion

The perturbative equation of motion was derived from the Hamiltonian in Sec. 2.6 with s as the independent variable. Here a similar equation of motion can be obtained following the Lagrangian analysis with t as the independent variable. For a bunch with design energy $E_0=\gamma_0mc^2$ circulating on an orbit with design radius $R(s)$, one has

$$\mathbf{E}^{\rm ext}=0,\qquad\mathbf{B}^{\rm ext}=B_0\mathbf{e}_y\quad(108)$$

where B_0 is given in Eq. (98). The external force is then

$$\mathbf{F}^{\rm ext}=e(\frac{\mathbf{v}}{c}\times\mathbf{B}^{\rm ext})=\frac{eB_0}{c}\,(-r\dot\theta\mathbf{e}_x+\dot x\mathbf{e}_s).\quad(109)$$

For particles with energy $E = \gamma mc^2$ and radius $r = R + x$, where x is the radial offset from the design orbit, we get from Eq. (43) ($\tau = 0$)

$$\frac{d(\gamma m\dot{x})}{dt} = r\dot{\theta}\left(\frac{\gamma m\beta_s c}{r} - \frac{\gamma_0 m\beta_0 c}{R}\right) + F_x^{\rm col}, \tag{110}$$

where $\beta_s = r\dot{\theta}/c$, and $F_x^{\rm col}$ is the horizontal component of the Lorentz force due to bunch collective interaction. Eq. (110) can be further written as

$$\frac{1}{\gamma_0}\frac{d(\gamma\dot{x})}{c^2dt} - \beta_s\left(\frac{\beta_s}{r} - \frac{\beta_0}{R}\right) = \frac{G}{E_0} \tag{111}$$

with the driving term

$$G = \beta_s^2\frac{\Delta E(t)}{r} + F_x^{\rm col}, \tag{112}$$

where ΔE is the kinetic energy deviation from the design energy:

$$E(t) = \gamma(t)mc^2, \quad E_0 = \gamma_0 mc^2, \quad \Delta E(t) = [\gamma(t) - \gamma_0]mc^2. \tag{113}$$

Assuming terms $\propto \gamma_0^{-2}$ or $\beta_\perp^2$ are negligible, Eq. (111) becomes

$$\frac{d(1+\delta)\dot{x}}{c^2dt} - \frac{1}{R}\left[(1+\frac{x}{R})^{-1} - 1\right] \simeq \frac{G}{E_0} \tag{114}$$

with δ in Eq. (107). The equation of horizontal motion, approximated to the first order of δ and x/R, can be obtained from Eq. (114):

$$\frac{d^2x}{c^2dt^2} + \frac{x}{R^2} \simeq \frac{G}{E_0} \tag{115}$$

which shows that the horizontal oscillation around the equilibrium orbit is driven by the combined effect of the kinetic energy deviation $\Delta E(t)$ and the horizontal collective force $F_x^{\rm col}$ in Eq. (112).

We now take a closer look at the two driving terms in Eq. (112). Let us rewrite Eq. (54) as

$$F_x^{\rm col} = F^{\rm CSCF} + F_x^{\rm eff}, \tag{116}$$

where $F^{\rm CSCF}$ is given in Eq. (61), and the effective radial force is defined as

$$F_x^{\rm eff} \equiv \frac{\partial\mathcal{L}_{\rm int}^{\rm col}}{\partial x} - e\frac{dA_x^{\rm col}}{cdt} = -e\left(\frac{\partial\Phi^{\rm col}}{\partial x} - \boldsymbol{\beta}\cdot\frac{\partial\mathbf{A}^{\rm col}}{\partial x}\right) - e\frac{dA_x^{\rm col}}{cdt}. \tag{117}$$

We next study the kinetic energy deviation $\Delta E(t)$ in Eq. (112). From Eq. (60), we have for $\mathbf{E}^{\rm ext} = 0$

$$\frac{dE}{cdt} = \boldsymbol{\beta}\cdot\mathbf{F}^{\rm col} = -e\frac{d\Phi^{\rm col}}{cdt} + F_v^{\rm eff} \tag{118}$$

with

$$F_v^{\rm eff} \equiv -\frac{\hat{\partial}\mathcal{L}_{\rm int}^{\rm col}}{c\partial t} = e\left(\frac{\partial\Phi^{\rm col}}{c\partial t} - \boldsymbol{\beta}\cdot\frac{\partial\mathbf{A}^{\rm col}}{c\partial t}\right). \tag{119}$$

Integrating Eq. (118) over time yields

$$[E + e\Phi^{\rm col}]_t = [E + e\Phi^{\rm col}]_{t_0} + \int_{t_0}^{t} F_v^{\rm eff}(t')cdt' \tag{120}$$

which is equivalent to Eq. (105), or

$$\Delta E = \Delta E^{\rm tot}(t_0) + \int_{t_0}^{t} F_v^{\rm eff}(t')cdt' - e\Phi^{\rm col}(t) \tag{121}$$

with ΔE in Eq. (113), and $\Delta E^{\rm tot}(t_0)$ the initial total (kinetic + potential) energy deviation from the design energy

$$\Delta E^{\rm tot}(t_0) = H(t_0) - E_0 = [\gamma(t_0)mc^2 + e\Phi^{\rm col}(t_0)] - \gamma_0 mc^2. \tag{122}$$

Substituting Eqs. (116) and (121) into Eq. (112) yields

$$G = G_0 + G_{\rm res} + G_v + F_x^{\rm eff} \tag{123}$$

with

$$G_0 = \beta_s^2\frac{\Delta E^{\rm tot}(t_0)}{r}, \tag{124}$$

$$G_{\rm res} = F^{\rm CFSC} - e\frac{\Phi^{\rm col}}{r} = e\beta_s\frac{A_s^{\rm col} - \beta_s\Phi^{\rm col}}{r} \tag{125}$$

$$G_v = \frac{\beta_s^2}{r}\int_{t_0}^{t} F_v^{\rm eff}(t')cdt' = \frac{\beta_s^2}{r}\int_{t_0}^{t}\frac{\hat{\partial}\mathcal{L}_{\rm int}^{\rm col}}{\partial t}dt'. \tag{126}$$

As in Sec. 2.6, one can expand Eq. (114) perturbatively and show that $E = H - e\Phi^{\rm col}$ (with $e\Phi^{\rm col}$ dominated by the local interaction contributions) will affect the horizontal optics via chromaticity or other higher order terms.

3.2. Retarded Potentials and Cancellation

In classical electrodynamics, only EM fields are observable quantities; thus one is free to choose potentials in any gauge (as reviewed in Eqs. (A10)-(A11)). Here we use the retarded potentials in the Lorentz gauge to analyze terms in Eqs. (123)-(126). It turns out that with the retarded potentials, $F_v^{\rm eff}$ and $F_x^{\rm eff}$ give the effective longitudinal and transverse force studied earlier [5, 6]. At the same time, the retarded potentials efficiently (1) gather the effect of the near-neighbor generated fields on a particle's kinetic energy change into $-e\beta_s^2\Phi^{\rm col}/r$ in Eq. (125), and (2) gather the contributions of the near-neighbor interaction fields to the radial collective radiative force on a particle into $e\beta_s A_s^{\rm col}/r$ in Eq. (125). We will show that these two local interaction effects on the particle transverse dynamics are nearly canceled; as a result, in Eq. (123), $G_{\rm res}$ is basically negligible; therefore in addition to G_0 related to the initial canonical energy, the transverse dynamics is driven by G_v and $F_x^{\rm eff}$ dominated by the non-local interaction due to the effective forces. The use of other gauges will be discussed in Sec. 3.3.

3.2.1. Single Particle Liénard-Wiechert Potentials

In the Lorentz gauge, the Lorentz condition requires

$$\partial_\alpha \partial^\alpha A^\mu = \frac{4\pi}{c} J^\mu \quad \text{and} \quad \partial_\mu A^\mu = 0. \tag{127}$$

The Liénard-Wiechert potentials generated from a source particle at $(\mathbf{x}', t')$ on a test particle at $(\mathbf{x}, t)$ are given in Eq. (D1). The source particle velocity $\boldsymbol{\beta}$ at retarded time in Eq. (D1) is projected on the Frenet-Serret bases at the retarded position

$$\boldsymbol{\beta}'(t') = \beta'_{s'}\mathbf{e}_{s'} + \beta'_{x'}\mathbf{e}_{x'} + \beta'_{y'}\mathbf{e}_{y'} \tag{128}$$

and the test particle velocity at the present time is projected on the Frenet-Serret bases at the present position

$$\boldsymbol{\beta}(t) = \beta_s\mathbf{e}_s + \beta_x\mathbf{e}_x + \beta_y\mathbf{e}_y. \tag{129}$$

The ultrarelativity implies that the source particle velocity at the retarded (or present) time is ultrarelativistic in the retarded (or present) longitudinal direction:

$$\beta'_{s'} = \boldsymbol{\beta}' \cdot \mathbf{e}_{s'} \approx 1 - \frac{1}{2\gamma'^2} - \frac{\beta'^2_{\perp'}}{2} \simeq 1, \quad \gamma'^{-2} \ll 1, \quad \beta'^2_{\perp'} \ll 1, \tag{130}$$

with $\beta'^2_{\perp'} = \beta'^2_{x'} + \beta'^2_{y'}$, and

$$\beta_s = \boldsymbol{\beta} \cdot \mathbf{e}_s \approx 1 - \frac{1}{2\gamma^2} - \frac{\beta_\perp^2}{2} \simeq 1, \quad \gamma^{-2} \ll 1, \quad \beta_\perp^2 \ll 1, \tag{131}$$

with $\beta_\perp^2 \equiv \beta_x^2 + \beta_y^2$. Let $\Delta\theta$ be the angle between $\mathbf{e}_{s'}$ at retarded position and $\mathbf{e}_s$ for the test particle at present time. Then

$$\mathbf{e}_{s'} = \cos\Delta\theta\,\mathbf{e}_s + \sin\Delta\theta\,\mathbf{e}_x, \quad \mathbf{e}_{x'} = -\sin\Delta\theta\,\mathbf{e}_s + \cos\Delta\theta\,\mathbf{e}_x. \tag{132}$$

Consider a uniform motion for both the source and the test particle: $\boldsymbol{\beta} = \boldsymbol{\beta}' = \beta_s \mathbf{e}_s$ and $\Delta\theta = 0$. The Liénard-Wiechert potentials in Eq. (D1) satisfy

$$A_{0s} = \beta_s \Phi_0 \tag{133}$$

and

$$\mathcal{L}_{\text{int}}^{\text{col}} = -e(\Phi_0 - \boldsymbol{\beta} \cdot \mathbf{A}_0) = -e \left[\frac{1}{\gamma^2 (1 - \mathbf{n} \cdot \boldsymbol{\beta}) R} \right]_{\text{ret}} . \tag{134}$$

The relativistic cancellation of Φ_0 and $\boldsymbol{\beta} \cdot \mathbf{A}_0$ in Eq. (134) corresponds to the γ^{-2} behavior for the transverse collective force on the test particle due to the relativistic cancellation between $\mathbf{E}^{\text{col}}$ and $\mathbf{B}^{\text{col}}$, and the γ^{-2} behavior of the longitudinal collective force due to the relativistic dilation of the longitudinal distance between particles in the test particle's instantaneous rest frame.

For a relativistic beam on a curved trajectory, Eqs. (133) and (134) (for a straight path) can only be satisfied approximately for near-neighbor ($|\Delta\theta| \ll 1$) interaction, since locally the test and the source particles are approximately in parallel motion. From Eqs. (D1), (128), (130) and (132), the Liénard-Wiechert potentials on the test particle obeys

$$A_{0s} = \mathbf{A}_0 \cdot \mathbf{e}_s = \left[\frac{\beta'_{s'} \cos \Delta\theta - \beta'_{x'} \sin \Delta\theta}{(1 - \boldsymbol{\beta} \cdot \mathbf{n}) R} \right]_{\text{ret}} \simeq \Phi_0. \tag{135}$$

Combining Eq. (135) with $p_s = \beta_s E/c$ and Eq. (131) for a relativistic test particle, one has

$$p_s + \frac{e}{c} A_{0s} \simeq \frac{E + e\Phi_0}{c}. \tag{136}$$

This implies that for local interactions described by Eq. (D1), eA_{0s}/c is approximately the momentum of $e\Phi_0$ in the same manner as p_s relates to E, and both the local interaction Liénard-Wiechert potential energy and the test particle kinetic energy move at relativistic speed. The total canonical energy tries to keep its inertial motion while the Frenet-Serret frame rotates; thus, instead of the particle experiencing the *usual centrifugal force* due to the inertia of its *kinetic energy*, here the particle experiences the *generalized centrifugal force* due to its *total canonical energy*.

3.2.2. Centrifugal Space Charge Force and Cancellation

With the Liénard-Wiechert potentials satisfying Eqs. (135) and (136) for the interaction between two nearby particles in an ultrarelativistically moving charged distribution, we now show that the retarded potentials for the interaction of each particle with the relativistic bunch satisfies a similar relationship, since as shown in Appendix D, the retarded potentials are closely related to the single particle Liénard-Wiechert potentials. We can then use these relations for retarded potentials to estimate the centrifugal space charge force and demonstrate the cancellation effect.

Consider the retarded potentials in Eqs. (D10) and (D11). For convenience we define the velocity field

$$\mathbf{u}(\mathbf{x},t) = \frac{\mathbf{J}(\mathbf{x},t)}{\rho(\mathbf{x},t)}. \tag{137}$$

For each retarded position vector $\mathbf{x}'$, we find its Frenet-Serret coordinates s', x' and y' according to Eq. (35):

$$\mathbf{x}' = r_0(s') + x'\mathbf{e}_{x'}(s') + y'\mathbf{e}_{y'}(s'), \tag{138}$$

with $\mathbf{e}_{s'}$, $\mathbf{e}_{x'}$, and $\mathbf{e}_{y'}$ the Frenet-Serret bases at the retarded longitudinal position s' . For $\mathbf{u}(\mathbf{x},t)$ in Eq. (137), let us define $\boldsymbol{\beta}'_u$ at a retarded position, which is decomposed to the Frenet-Serret components at s',

$$\boldsymbol{\beta}'_u(\mathbf{x}',t') \equiv \frac{\mathbf{u}(\mathbf{x}',t')}{c} = \beta'_{us'}\mathbf{e}_{s'} + \beta'_{ux'}\mathbf{e}_{x'} + \beta'_{uy'}\mathbf{e}_{y'}. \tag{139}$$

The vector potential in Eq. (D11) is then written as

$$\mathbf{A}^{\mathrm{col}}(\mathbf{x},t) = \int \frac{\rho(\mathbf{x}',t')\boldsymbol{\beta}'_u(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'. \tag{140}$$

The ultrarelativistic condition for $\boldsymbol{\beta}'_u(\mathbf{x}',t')$ is

$$\beta'_{us'} = \boldsymbol{\beta}'_u \cdot \mathbf{e}_{s'} \approx 1 - \frac{1}{2\gamma_u'^2} - \frac{\beta'^2_{u\perp'}}{2} \simeq 1, \quad \gamma_u'^{-2} \ll 1, \quad \beta'^2_{u\perp'} \ll 1 \tag{141}$$

with $\beta'^2_{u\perp'} = \beta'^2_{ux'} + \beta'^2_{uy'}$. Using Eqs. (132), (139) and (140), we get

$$A_s^{\mathrm{col}} = \mathbf{A}^{\mathrm{col}}(\mathbf{x},t)\cdot\mathbf{e}_s = \int \frac{\rho(\mathbf{x}',t')\left[\beta'_{us'}\cos(\Delta\theta) - \beta'_{ux'}\sin(\Delta\theta)\right]}{|\mathbf{x}'-\mathbf{x}|}d\mathbf{x}', \tag{142}$$

which can be further written as

$$A_s^{\mathrm{col}} \simeq \Phi^{\mathrm{col}} + \Delta A_s^{\mathrm{col}} = \int \frac{\rho(\mathbf{x}',t')(1+\Delta_s)}{|\mathbf{x}'-\mathbf{x}|}d\mathbf{x}' \tag{143}$$

with Φ^{col} given in Eq. (D10), and $\Delta A_s^{\mathrm{col}}$ defined as

$$\Delta A_s^{\mathrm{col}} = \int d\mathbf{x}'\Delta_s \frac{\rho(\mathbf{x}',t')}{|\mathbf{x}'-\mathbf{x}|} \tag{144}$$

for
$$\Delta_s \simeq -2\sin^2\left(\frac{\Delta\theta}{2}\right) - \beta'_{ux'}\sin\Delta\theta - \frac{1}{2}\left(\frac{1}{\gamma_u'^2} + \beta'^2_{u\perp'}\right)\cos\Delta\theta, \tag{145}$$
where we used Eq. (141) and
$$1 - \cos\Delta\theta = 2\sin^2(\Delta\theta/2). \tag{146}$$

For near-neighbor ($|\Delta\theta| \ll 1$) interaction, Eqs. (145) yields
$$|\Delta_s| \ll 1. \tag{147}$$

In general, $\Phi^{\rm col}$ is dominated by the short range interaction when $1/|\mathbf{x} - \mathbf{x}'| \gg 1$. However, for $\Delta A_s^{\rm col}$ in Eq. (144), when $|\Delta\theta| \ll 1$, this short range singularity due to $1/|\mathbf{x} - \mathbf{x}'|$ is suppressed by $|\Delta_s| \ll 1$. Even though the long-range ($|\Delta\theta| \sim 1$) interaction contributions to $\Delta A_s^{\rm col}$ and $\Phi^{\rm col}$ in Eq. (143) are of the same magnitude, they are both negligible compared to the short range interaction contributions. As a result, we always have
$$|\Delta A_s^{\rm col}| \ll \Phi^{\rm col} \quad \text{or} \quad A_s^{\rm col} \simeq \Phi^{\rm col}. \tag{148}$$

This equation, together with $p_s = \beta_s E/c$ for the test particle, proves Eq. (87). In the above discussions, the retarded position of the source particle can be either ahead or behind the test particle.

Due to the singular integrand of $A_s^{\rm col}$ in Eq. (143), after integration over a finite charge distribution, $F^{\rm CSCF}$ in Eq. (61) often has logarithmic dependence over particles' transverse offset. The possible consequences of such divergent behavior on the tune spread and chromaticity in a storage ring were raised by Talman [1], and later was pointed out by Lee [2] to be largely canceled by the potential energy effect for a coasting beam. Here we show the general validity of this cancellation effect.

After combining the effect of $F_x^{\rm col}$ with the effect of $\Delta E(t)$ in Eq. (112), the horizontal dynamics is found to be driven by the terms G_0, $G_{\rm res}$, G_v and $F_x^{\rm eff}$ in Eq. (123). In particular, the term $G_{\rm res}$ represents the combined effects of the centrifugal space charge force $F^{\rm CSCF}$ in Eq. (116) with that of the potential energy $e\Phi^{\rm col}(t)$ in $\Delta E(t)$ of Eq. (121). With the potentials in Eqs. (D10) and (143), and with $\beta_s \simeq 1$ in Eq. (131), $G_{\rm res}$ in Eq. (125) becomes
$$G_{\rm res} \simeq e\frac{\Delta A_s}{r}. \tag{149}$$

Comparing $G_{\rm res}$ in Eq. (149) with $F^{\rm CSCF}$ in Eq. (61), and using Eq. (148), one gets
$$|G_{\rm res}| \ll |F^{\rm CSCF}|. \tag{150}$$

This result shows that the joint effects of the potential energy $e\Phi^{\rm col}(t)$ and the centrifugal space charge force $F^{\rm CSCF}$ are almost canceled. The residual of cancellation, $G_{\rm res}$, is negligibly small compared to $F^{\rm CSCF}$.

In Appendix E, it is shown that the effective terms $F_x^{\rm eff}$ in Eq. (117) and $F_v^{\rm eff}$ in Eq. (119) are dominated by non-local interaction contributions, and the residual of the cancellation, $G_{\rm res}$, is negligible compared to the other effective terms in Eq. (123).

3.3. Gauge Transformation

Let $\Phi^{\rm col}$ and $\mathbf{A}^{\rm col}$ be the retarded potentials in Eqs. (D10) and (D11), and consider the gauge transformation

$$\Phi'^{\rm col} = \Phi^{\rm col} - \frac{1}{c}\frac{\partial\Lambda}{\partial t}, \qquad \mathbf{A}'^{\rm col} = \mathbf{A}^{\rm col} + \nabla\Lambda. \tag{151}$$

For example, the gauge transformation from the Lorentz gauge to the Coulomb gauge ($\nabla\cdot\mathbf{A}=0$) is achieved via [15]

$$\Lambda(\mathbf{x},t) = -\int d\mathbf{x}' \int_0^1 d\lambda\, \rho(\mathbf{x}', t-\lambda R/c) + \Lambda_0 \tag{152}$$

with $R = |\mathbf{x}-\mathbf{x}'|$, Λ_0 a constant, and $\rho(\mathbf{x},t)$ the charge distribution function. The simplicity of the scalar potential in the Coulomb gauge (compared to the retarded potential), i.e., the instantaneous Coulomb potential,

$$\Phi_C(\mathbf{x},t) = \int \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}', \tag{153}$$

comes together with the more complicated form of the vector potential compared to the retarded vector potential:

$$\mathbf{A}_C(\mathbf{x},t) = \mathbf{A}_R(\mathbf{x},t) + \Delta\mathbf{A} \tag{154}$$

where $\mathbf{A}_R(\mathbf{x},t)$ is the retarded potential in Eq. (D11), and

$$\Delta\mathbf{A}(\mathbf{x},t) = \int d\mathbf{x}' \left[-\hat{R}\rho(\mathbf{x}',t') + \frac{c\hat{R}}{R^2}\int_0^{R/c} d\tau \rho(\mathbf{x}', t-\tau)\right], \tag{155}$$

with

$$\mathbf{R} = \mathbf{x}-\mathbf{x}', \qquad R = |\mathbf{R}|, \qquad \hat{R} = \frac{\mathbf{R}}{R}, \qquad t' = t - \frac{R}{c}. \tag{156}$$

With the new potentials in Eq. (151), the interaction Lagrangian in Eq. (51) becomes

$$\mathcal{L}'^{\rm col}_{\rm int} \equiv -e(\Phi'^{\rm col} - \boldsymbol{\beta}\cdot\mathbf{A}'^{\rm col}) = \mathcal{L}^{\rm col}_{\rm int} + \frac{e}{c}\frac{d\Lambda}{dt}. \tag{157}$$

Similarly, relations between the effective forces, $F_v^{\rm eff}$ in Eq. (119) and $F_x^{\rm eff}$ in Eq. (117), in the new gauge and old gauge can be found:

$$F_v^{\prime\rm eff} \equiv -\frac{\hat{\partial}\mathcal{L}_{\rm int}^{\prime\rm col}}{c\partial t} = F_v^{\rm eff} - \frac{e}{c}\frac{d}{dt}\left[\frac{\partial\Lambda}{\partial t}\right] \tag{158}$$

and

$$F_x^{\prime\rm eff} \equiv -\mathbf{e}_x\cdot\hat{\nabla}\mathcal{L}_{\rm int}^{\prime\rm col} - e\frac{d\,A_x^{\prime\rm eff}}{cdt} = F_x^{\rm eff} - \frac{e\beta_s\partial_s\Lambda}{r}, \tag{159}$$

where the following equality is used

$$\frac{d}{cdt}[\mathbf{e}_x\cdot\nabla\Lambda] = \frac{d\,\mathbf{e}_x}{cdt}\cdot\nabla\Lambda + \mathbf{e}_x\cdot\nabla\frac{d\Lambda}{cdt}, \tag{160}$$

with $d\mathbf{e}_x/dt = \kappa\dot{s}/R = \beta_s/r$. In the new gauge, using potentials in Eq. (151), the centrifugal space charge force in Eq. (61) becomes

$$F^{\prime\rm CSCF} = F^{\rm CSCF} + \frac{e\beta_s\partial_s\Lambda}{r}. \tag{161}$$

The energy relation in the new gauge is obtained by integrating Eq. (118) and using Eq. (158)

$$[E + e\Phi^{\prime\rm col}]_t = [E + e\Phi^{\prime\rm col}]_{t_0} + \int_{t_0}^{t} F_v^{\prime\rm eff}(t')cdt' - \left[\frac{1}{c}\frac{d}{dt}\frac{\partial\Lambda}{\partial t}\right]_{t_0}^{t}, \tag{162}$$

which is equivalent to Eq. (120) with $\Phi^{\prime\rm col}$ defined in Eq. (151). This shows that the change of $\Phi^{\rm col}$ to $\Phi^{\prime\rm col}$ is actually compensated by the change of $F_v^{\rm eff}$ to $F_v^{\prime\rm eff}$; thus the resulting kinetic energy change $E - E_0$ is gauge invariant. The gauge invariance of the total collective radial force in Eq. (116) can be seen by using Eqs. (159) and (161)

$$F_x^{\prime\rm col} \equiv F^{\prime\rm CSCF} + F_x^{\prime\rm eff} = F_x^{\rm col}, \tag{163}$$

indicating that the change of $F^{\rm CSCF}$ to $F^{\prime\rm CSCF}$ is compensated by the change of $F_x^{\rm eff}$ to $F_x^{\prime\rm eff}$. Due to the arbitrariness of Λ, in the new gauge the cancellation between effects of $F^{\prime\rm CSCF}$ and $e\Phi^{\prime\rm col}/r$ no longer holds, or, Eq. (148) is no longer valid for the new gauge. However, this non-cancellation is compensated by the changes in the effective forces in the new gauge.

Note here that the names of the effective "forces" and the centrifugal space charge "force" may be misleading, because unlike $F_x^{\rm col}$ and ΔE in Eq. (112) which are gauge invariant, these "forces" are only different terms in the actual forces and thus they vary when the gauge is changed. However, with the proper choice of gauge—such as the Lorentz gauge—these effective "forces" can efficiently separate the local and long-range behavior of the collective interaction.

4. WHY THE CANCELLATION EFFECT HAS BEEN A CONTROVERSIAL ISSUE

The canonical formulations with the Lorentz gauge in this paper provide us with the following understanding (the potentials in the canonical momentum or energy here refer to that due to collective interactions):

(1) It is the canonical momentum, rather than the kinetic momentum, which undergoes the curvature effect due to the rotation of the Frenet bases along the curved trajectory.

(2) For ultrarelativistic beams, the longitudinal canonical momentum represents the motion of canonical energy at speed $v_s \simeq c$. Here the potential energy is dominated by the local interaction contributions, and locally the interaction acts as if the bunch moves on a straight path.

(3) Other than the curvature effect, the dynamics of the canonical momentum on a curved trajectory is basically the same as that in a Cartesian frame. Especially, locally the interaction of a test particle with its neighboring distribution is similar to the collective interaction of a relativistic bunch on a straight path, where the transverse collective force scales as γ^{-2} due to the relativistic cancellation between $\mathbf{E}^{\rm col}$ and $\mathbf{B}^{\rm col}$. This relativistic cancellation, however, does not apply to the long-range interaction where the bunch deviates from the local straight path.

(4) Due to (1) and (2), the horizontal dynamics depends on the canonical energy through the curvature effect, where the change of the canonical energy has negligible ($\propto \gamma^{-2}$ or $\beta_\perp^2$) dependence on the local interaction because locally the bunch acts as if moving on a straight path. As in (3), this local interaction property does not apply to the long-range interaction.

(5) As the result of (1)-(4), there are negligible local interaction contributions to the changes of the two driving terms in Eq. (112) over time, or, the local interaction contributions to the change of each of the two driving terms in Eq. (112) over

time are canceled. Eq. (120) gives

$$\frac{\Delta E(t) - \Delta E(t_0)}{r} = -\frac{e[\Phi^{\mathrm{col}}(t) - \Phi^{\mathrm{col}}(t_0)]}{r} + \frac{1}{r}\int_{t_0}^{t} F_v^{\mathrm{eff}}(t')cdt', \tag{164}$$

and from Eqs. (61) and (116), one has

$$F_x^{\mathrm{col}}(t) - F_x^{\mathrm{col}}(t_0) = \frac{e[\beta_s(t)A_s^{\mathrm{col}}(t) - \beta_s(t_0)A_s^{\mathrm{col}}(t_0)]}{r} + [F_x^{\mathrm{eff}}(t) - F_x^{\mathrm{eff}}(t_0)]. \tag{165}$$

Using Eqs. (131) and (148) for both t and t_0, one gets from Eqs. (164) and (165)

$$\left[\frac{\Delta E(t)}{r} + F_x^{\mathrm{col}}(t)\right] - \left[\frac{\Delta E(t_0)}{r} + F_x^{\mathrm{col}}(t_0)\right]$$
$$\simeq \frac{1}{r}\int_{t_0}^{t} F_v^{\mathrm{eff}}(t')cdt' + [F_x^{\mathrm{eff}}(t) - F_x^{\mathrm{eff}}(t_0)]. \tag{166}$$

This shows that the *change* of the driving terms *together* $\Delta E(t)/r + F_x^{\mathrm{col}}(t)$ over time depends only on the effective forces which are majorly contributed from non-local interactions.

Since the centrifugal space charge force was first introduced in 1986 by Talman [1], its effect on the transverse bunch dynamics and the cancellation of this effect with the effect of the potential energy has been a long-standing controversial issue [2]-[9]. This is because the potential energy change causes the change of the kinetic energy E, which further causes the change of the usual centrifugal force $\kappa\dot{s}p_s = \beta_s^2 E/r$ in Eq. (43), while F^{CSCF} is the logarithmically divergent part of the radial collective radiative force in $\mathbf{e}_x \cdot F^{\mathrm{col}}$ of Eq. (43). The two terms are often considered as unrelated physical quantities; therefore in many papers one effect is studied without considering the other. For example, the effect of potential energy change on the transverse dynamics was studied for the case of the transverse offset of particles in bends [4, 6], and for the converging beam (using instantaneous Coulomb potential) before the last bend in a chicane [16]. Here our point is that this potential energy change induced kinetic energy change is strongly coupled with the change of F^{CSCF}, and in fact, the local interaction contributions to the changes of the two physical quantities are canceled, as illustrated in Eqs. (164)-(166). For example, in a bunch compression chicane, the bunch initially carries with it certain kinetic and potential energy spread. There may be no correlation between these two initial energy spreads. As the bunch dynamics evolves in the chicane, in addition to the long range effective forces, the kinetic energy of a particle varies due to its potential energy change, and meanwhile the horizontal collective radiative force

(Talman's force) on the particle also changes due to the change of the charge distribution. Our result shows that the total change of $\Delta E/r + F_x^{\rm col}$ has negligible dependence on the local charge interaction at the present time.

In Ref. [7], counter-examples are raised to dispute the cancellation effect using *instantaneous Coulomb potential.* One should emphasize the consistency of using the same gauge for the calculation of *both* the effective CSR "forces" and potentials. The cancellation effect was previously proved for various systems using *retarded potentials* [2, 5, 6], since the Lorentz gauge turns out to be the most efficient gauge to separate the local interaction effects (in $F^{\rm CSCF}$ and in $e\Phi^{\rm col}/r$) from the non-local interaction effects (in $F_v^{\rm eff}$ and in $F_x^{\rm eff}$) and to display the cancellation of the local interaction effects on transverse dynamics. If one uses potentials in the Coulomb gauge, such as the *instantaneous Coulomb potential*, one should make sure that the effective "forces" are changed accordingly using Coulomb gauge potentials (Sec. 3.3).

Another reason for the controversy may arise due to the use of Liénard-Wiechert fields, which was the foundation for some analyses of the CSR interaction forces [7, 17]. Even though the Liénard-Wiechert field approach is equivalent to the analysis in this paper (see Appendix D), and the cancellation effect is implicitly carried out once the equations of motion are integrated including *both* the radial and longitudinal collective forces, in this approach the energy relation (Eq. (121)) with the scalar potential and the scalar potential relation with $F^{\rm CSCF}$ (Eqs. (61) and (148)) are not obvious. As we notice, unlike in a Cartesian frame where the fields are gradients of potentials, in the Frenet-Serret frame the term $e\beta_s A_s^{\rm col}/r$ shows up in the radial collective force in Eq. (54)—without gradient or derivative acting on the potential $A_s^{\rm col}$. At the same time, due to radial acceleration, the particles experience the centrifugal force $v_s p_s/r \simeq E/r$ in Eq. (58), where E is the kinetic energy in Eq. (121) obtained after the integration of Eq. (118) over time, which yielded $e\Phi^{\rm col}/r$—again without gradient or derivative acting on the potential $\Phi^{\rm col}$. Both $e\beta_s A_s^{\rm col}/r$ and $e\Phi^{\rm col}/r$ terms are results of rotation of the Frenet-Serret bases; thus they are strongly correlated. Consequently, analysis using *potentials* is more advantageous in explicitly revealing the relationship between *potentials*, i.e., the cancellation effect, compared to analysis based on the Liénard-Wiechert *fields*.

5. CONCLUSION

In this paper, we address the controversial issue of the cancellation effect in the dynamics of an ultrarelativistic beam on a curved trajectory. Equations of motion using Frenet-Serret coordinates in frames along the curvilinear orbit are obtained via both the Lagrangian and the Hamiltonian approaches, which are shown to be equivalent to the direct projection of the equations for the canonical momentum in the Cartesian frame to the Frenet-Serret frame. It is illustrated through these analyses that instead of the usual centrifugal force due to the inertia of a particle's kinetic energy, the particle experiences the general centrifugal force due to the inertia of the particle's canonical energy. It turns out that the retarded potentials (or the Liénard-Wiechert potentials) in the Lorentz gauge are the natural choice to describe the cancellation between effects due to the local interaction contributions to the change of the kinetic energy and to the change of the centrifugal space charge forces. After the cancellation, the local interaction influences the particle dynamics only via the second or higher order terms. Note that boundary condition is not included in this paper. In general, fields under arbitrary boundary conditions can be considered as the superposition of (1) fields generated by the bunch in free space and (2) fields satisfying the homogeneous Maxwell equations with boundary conditions representing the reflection of the free space fields by the boundary. The free space case is treated in this paper, and the cancellation of local effects has been shown using retarded potentials. However, the reflection of free space fields by the boundaries, which relates to non-local (via reflection) interactions, needs to be carefully solved using specific boundary conditions for the fields.

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APPENDIX A: REVIEW OF CLASSICAL THEORY ON CHARGED PARTICLE DYNAMICS

Consider a test charged particle moving in an EM field. The covariance form of the action integral is [10]

$$S = S_{\text{free}} + S_{\text{int}} \tag{A1}$$

with the free particle action and Lagrangian

$$S_{\text{free}} = \int_{s_1}^{s_2} \mathcal{L}_{\text{free}}\, ds, \quad \mathcal{L}_{\text{free}}(x, U) = -mc\sqrt{g_{\mu\nu}(x)U^\mu U^\nu}, \quad U^\mu = \frac{dx^\mu}{ds} \tag{A2}$$

and the interaction action and Lagrangian

$$S_{\text{int}} = \int_{s_1}^{s_2} \mathcal{L}_{\text{int}}\, ds \quad \mathcal{L}_{\text{int}}(x, U) = -\frac{e}{c} g_{\mu\nu}(x) A^\mu(x) U^\nu. \tag{A3}$$

In the above equations s is a parameter which is a monotonically increasing function of the proper time τ of the particle, $g_{\mu\nu}(x)$ is the metric tensor of spacetime in which the particle motion is under concern, m is the charged particle's rest mass and e the charge of the charged particle, and the 4-potential $A^\mu = (\Phi, \mathbf{A})$ includes potentials both due to the particle's interaction with external fields and due to the collective interaction in a charged particle distribution. The least action principle requires the particle's classical trajectory $x^\mu(s)$ from s_1 to s_2 to satisfy the Euler-Lagrangian equation:

$$\frac{d}{ds}\frac{\partial \mathcal{L}}{\partial U^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \quad \text{with} \quad \mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}. \tag{A4}$$

After applying Eqs. (A2) and (A3), and using

$$\sqrt{g_{\mu\nu}U^\mu U^\nu} ds = c d\tau \tag{A5}$$

with c the speed of light, Eq. (A4) becomes

$$m\frac{d^2 x^\eta}{d\tau^2} + m\Gamma^\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{e}{c}F^\eta_{\ \nu}\frac{dx^\nu}{d\tau} \tag{A6}$$

with the field tensor

$$F^\eta_{\ \nu} = g^{\eta\mu}F_{\mu\nu}, \quad F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}, \tag{A7}$$

and the Christoffel second symbol (or connection)

$$\Gamma^\eta_{\ \mu\nu} = g^{\eta\lambda}\left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right). \tag{A8}$$

The EM theory is invariant with gauge transformation

$$A'^{\mu} = A^{\mu} + \partial^{\mu}\Lambda \tag{A9}$$

for an arbitrary scalar function $\Lambda(x)$. This can be seen from the interaction action in Eq. (A3). After the gauge transformation in Eq. (A9), the new interaction action becomes

$$\begin{aligned} S'_{\rm int} &= -\frac{e}{c}\int_{s_1}^{s_2} g_{\mu\nu}(x)[A^{\mu}(x) + \partial^{\mu}\Lambda]dx^{\nu} \\ &= -\frac{e}{c}\int_{s_1}^{s_2} g_{\mu\nu}(x)A^{\mu}(x)\,dx^{\nu} - \frac{e}{c}[\Lambda(s_1) - \Lambda(s_2)] \end{aligned} \tag{A10}$$

where $\Lambda(s) = \Lambda[x(s)]$. This shows that the gauge transformation does not change the path dependence of action from s_1 to s_2; therefore the classical trajectory which minimizes the action will remain the same. The gauge invariance can be also seen from the field tensor invariance under the gauge transformation in Eq. (A9)

$$F'_{\mu\nu} = \frac{\partial A'_{\nu}}{\partial x^{\mu}} - \frac{\partial A'_{\mu}}{\partial x^{\nu}} = F_{\mu\nu}, \tag{A11}$$

which leads to the invariance of classical trajectory of the charged particle determined by Eq. (A6).

APPENDIX B: RELATION BETWEEN $\hat{\nabla}$ (OR $\hat{\partial}_t$) ACTING ON $L_{\rm int}$ AND $\mathcal{L}_{\rm int}$

In Sec. 2.1, the dynamics of the charged particle distribution is analyzed in a Cartesian frame, with basis vector $\mathbf{e}_i$ ($i = 1, 2, 3$), and the interaction Lagrangian $L_{\rm int}$ in terms of vector components in the Cartesian frame is (Eq. (5))

$$L_{\rm int} = -e(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}) = -e(\Phi - \sum_i \beta_i A_i). \tag{B1}$$

On the other hand, in Sec. 2.2-2.5, the dynamics is analyzed using coordinates in the Frenet-Serret frame associated with the curvilinear reference orbit, with basis vector $\mathbf{e}_{\lambda}$ ($\lambda = s, x, z$), and the interaction Lagrangian $\mathcal{L}_{\rm int}$ in terms of vector components in the Frenet-Serret frame is (Eq. (51))

$$\mathcal{L}_{\rm int} = -e(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}) = -e(\Phi - \sum_{\lambda} \beta_{\lambda} A_{\lambda}). \tag{B2}$$

The two sets of bases satisfy the orthogonal condition

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}, \quad \mathbf{e}_{\lambda}\cdot\mathbf{e}_{\lambda'} = \delta_{\lambda\lambda'}, \tag{B3}$$

and they are related by

$$\mathbf{e}_\lambda = \sum_i (\mathbf{e}_i \cdot \mathbf{e}_\lambda)\, \mathbf{e}_i. \tag{B4}$$

Combining Eqs. (B3) and (B4), we get

$$\sum_i (\mathbf{e}_i \cdot \mathbf{e}_\lambda)\, (\mathbf{e}_i \cdot \mathbf{e}_{\lambda'}) = \delta_{\lambda\lambda'}. \tag{B5}$$

The components β_i and A_i in Eq. (B1) are related to β_λ and A_λ in Eq. (B2) by

$$\beta_i = \sum_\lambda \beta_\lambda (\mathbf{e}_i \cdot \mathbf{e}_\lambda), \qquad A_i = \sum_{\lambda'} A_{\lambda'} (\mathbf{e}_i \cdot \mathbf{e}_{\lambda'}). \tag{B6}$$

From Eq. (B5), we have for $L_{\rm int}$ in Eq. (B1) and $\mathcal{L}_{\rm int}$ in Eq. (B2)

$$L_{\rm int} = \mathcal{L}_{\rm int}. \tag{B7}$$

Next, using Eqs. (B5) and (B6), we have

$$\begin{aligned} \sum_i \beta_i \nabla A_i &= \sum_i \sum_\lambda \beta_\lambda (\mathbf{e}_i \cdot \mathbf{e}_\lambda) \nabla \sum_{\lambda'} A_{\lambda'} (\mathbf{e}_i \cdot \mathbf{e}_{\lambda'}) \\ &= \sum_\lambda \beta_\lambda \nabla A_\lambda - \sum_{\lambda\lambda'} \beta_\lambda A_{\lambda'} \sum_i (\mathbf{e}_i \cdot \mathbf{e}_\lambda)(\mathbf{e}_i \cdot \nabla \mathbf{e}_{\lambda'}). \end{aligned} \tag{B8}$$

Since the Frenet-Serret bases are only functions of s as in Eq. (38), Eq. (B8) gives for transverse Frenet-Serret bases $\mathbf{e}_\eta$ ($\eta = x, y$)

$$(\mathbf{e}_\eta \cdot \nabla)\Phi - \sum_i \beta_i (\mathbf{e}_\eta \cdot \nabla A_i) = \partial_\eta \Phi - \sum_\lambda \beta_\lambda (\partial_\eta A_\lambda) \tag{B9}$$

or, with the definition of $\hat{\nabla}$ in Eqs. (9) and (52), we have

$$\mathbf{e}_x \cdot \hat{\nabla} L_{\rm int} = \frac{\hat{\partial} \mathcal{L}_{\rm int}}{\partial x}, \qquad \mathbf{e}_y \cdot \hat{\nabla} L_{\rm int} = \frac{\hat{\partial} \mathcal{L}_{\rm int}}{\partial y}. \tag{B10}$$

For a circular orbit, one can use Eq. (B8) to show that the gradient in the longitudinal direction obeys

$$\mathbf{e}_s \cdot \hat{\nabla} L_{\rm int} = \frac{\hat{\partial} \mathcal{L}_{\rm int}}{r \partial \theta} - e \frac{\beta_x A_s - \beta_s A_x}{r}. \tag{B11}$$

Due to $\partial_t \, \mathbf{e}_\lambda = 0$, with $\hat{\partial}_t$ defined in Eqs. (9) and (52), one can also show

$$\frac{\hat{\partial} L_{\rm int}}{\partial t} = \frac{\hat{\partial} \mathcal{L}_{\rm int}}{\partial t}. \tag{B12}$$

APPENDIX C: HAMILTONIAN DERIVED FROM THE KLEIN-GORDON EQUATION

In Sec. 2.6, the Hamiltonian $\mathcal{H}$ conjugate to s is derived from the Lagrangian. Here we show that it can also be derived from the Klein-Gordon equation in relativistic quantum mechanics. The relativistic energy-momentum relation for a free particle of mass m and 4-momentum P^μ is

$$P^\mu P_\mu = m^2c^2. \tag{C1}$$

Using the minimal coupling principle for the interaction of the particle with EM fields

$$P^\mu \to P^\mu - \frac{e}{c}A^\mu \tag{C2}$$

and the operator representation of the canonical momentum (for $g_{\mu\nu}$ in Eq. (1))

$$P^\mu \to i\hbar\frac{\partial}{\partial x_\mu}, \tag{C3}$$

one gets the Klein-Gordon equation for the wave function $\Psi(s,x,y,t)$ for the Frenet-Serret coordinates with curvature κ (torsion $\tau = 0$):

$$\left[\left(\frac{H-e\Phi}{c}\right)^2 - \left(P_x - \frac{e}{c}A_x\right)^2 - \left(P_y - \frac{e}{c}A_y\right)^2 - \left(P_s - \frac{e}{c}A_s\right)^2\right]\Psi - (mc)^2\Psi = 0 \tag{C4}$$

where $A_\lambda = \mathbf{A}\cdot\mathbf{e}_\lambda$ for $\lambda = (s,x,z)$, or

$$\left[\left(i\hbar\frac{\partial}{c\partial t} - \frac{e}{c}\Phi\right)^2 - \left(-i\hbar\frac{\partial}{\partial x} - \frac{e}{c}A_x\right)^2 - \left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}A_y\right)^2 - \left(-i\hbar\frac{\partial}{(1+\kappa x)\partial s} - \frac{e}{c}A_s\right)^2\right]\Psi$$
$$-(mc)^2\Psi = 0. \tag{C5}$$

Rearranging the operators gives

$$\left[-i\hbar\frac{\partial}{\partial s} - (1+\kappa x)\frac{e}{c}A_s\right]^2\Psi$$
$$-(1+\kappa x)^2\left[\left(i\hbar\frac{\partial}{c\partial t}\frac{e}{c}\Phi\right)^2 - \left(-i\hbar\frac{\partial}{\partial x} - \frac{e}{c}A_x\right)^2 - \left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}A_y\right)^2 - m^2c^2\right]\Psi = 0. \tag{C6}$$

Defining $\mathcal{H} \equiv i\hbar\partial/\partial s$, we get from Eq. (C6)

$$\mathcal{H} = -(1+\kappa x)\frac{eA_s}{c} - (1+\kappa x)\left[\left(\frac{H-e\Phi}{c}\right)^2 - \left(P_x - \frac{e}{c}A_x\right)^2 - \left(P_y - \frac{e}{c}A_y\right)^2 - (mc)^2\right]^{1/2}, \tag{C7}$$

which agrees with Eq. (92) for $\tau = 0$.

APPENDIX D: LIÉNARD-WIECHERT POTENTIALS AND FIELDS

Let us consider the collective interaction of a charged particle distribution around a curvilinear reference orbit. The Liénard-Wiechert potentials generated from a source particle at $(\mathbf{x}', t')$ on a test particle at $(\mathbf{x}, t)$ is

$$\Phi_0(\mathbf{x}, t) = \left[\frac{e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{\rm ret}, \quad \mathbf{A}_0(\mathbf{x}, t) = \left[\frac{e\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{\rm ret}, \tag{D1}$$

where the subscript "ret" refers to the retarded time t', and

$$\mathbf{R} = \mathbf{x} - \mathbf{x}', \quad R = |R|, \quad \mathbf{n} = \frac{\mathbf{R}}{R}, \quad t' = t - \frac{R}{c}. \tag{D2}$$

Using Eq. (16), the Liénard-Wiechert fields on the test particle are obtained from the potentials in Eq. (D1)

$$\mathbf{E}_0(\mathbf{x}, t; \mathbf{s}) = e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{\rm ret} + \frac{e}{c}\left[\frac{\mathbf{n}\times(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{\rm ret} \tag{D3}$$

$$\mathbf{B}_0(\mathbf{x}, t; \mathbf{s}) = [\mathbf{n}\times\mathbf{E}_0]_{\rm ret}. \tag{D4}$$

Using the single particle Liénard-Wiechert potential $(\Phi_0, \mathbf{A}_0)$ in Eq. (D1) and the Liénard-Wiechert fields $\mathbf{E}_0$ and $\mathbf{B}_0$ in Eq. (D4) as the Green's functions, we can find the potentials and fields on a particle due to its interaction with the whole charge distribution. Let a particle be labeled by its initial offset $\mathbf{s}$ from the bunch centroid, and let $n(\mathbf{s})$ be the bunch initial distribution. The trajectory of a particle labeled $\mathbf{s}$ is $\mathbf{x}_0(\mathbf{s}, t)$, and the velocity is $\mathbf{v}_0(\mathbf{s}, t)$. The potentials on the test particle at $(\mathbf{x}, t)$ due to its interaction with the whole charge distribution is then

$$\Phi^{\rm col} = \int \Phi_0(\mathbf{x}, t; \mathbf{s}')n(\mathbf{s}')d\mathbf{s}', \quad \mathbf{A}^{\rm col} = \int \mathbf{A}_0(\mathbf{x}, t; \mathbf{s}')n(\mathbf{s}')d\mathbf{s}', \tag{D5}$$

and the total EM fields on the particle is

$$\mathbf{E}^{\rm col} = \int \mathbf{E}_0(\mathbf{x}, t; \mathbf{s}')n(\mathbf{s}')d\mathbf{s}', \quad \mathbf{B}^{\rm col} = \int \mathbf{B}_0(\mathbf{x}, t; \mathbf{s}')n(\mathbf{s}')d\mathbf{s}', \tag{D6}$$

where $(\Phi_0, \mathbf{A}_0)$ in Eq. (D5) and $\mathbf{E}_0$ and $\mathbf{B}_0$ in Eq. (D4) are evaluated at retarded time $t'(\mathbf{x}, t; \mathbf{s}')$ satisfying

$$t' = t - |\mathbf{x}'_0(\mathbf{s}', t') - \mathbf{x}|/c. \tag{D7}$$

The charge distribution $\rho(\mathbf{x}, t)$ and the current density $\mathbf{J}(\mathbf{x}, t)$ are related to $n(\mathbf{s})$ by

$$\rho(\mathbf{x}, t) = e\int n(\mathbf{s}')\delta(\mathbf{x}-\mathbf{x}_0(\mathbf{s}', t))d\mathbf{s}', \quad \mathbf{J}(\mathbf{x}, t) = e\int \mathbf{v}_0(\mathbf{s}', t)n(\mathbf{s}')\delta(\mathbf{x}-\mathbf{x}_0(\mathbf{s}', t))d\mathbf{s}'. \tag{D8}$$

With the change of variable from $\mathbf{s}'$ to $\mathbf{x}'$

$$\mathbf{x}' = \mathbf{x}_0(\mathbf{s}', t - |\mathbf{x} - \mathbf{x}'|/c), \tag{D9}$$

the potentials in Eq. (D5) becomes the retarded potentials

$$\Phi^{\rm col}(\mathbf{x}, t) = \int \frac{\rho(\mathbf{x}', t')}{|\mathbf{x}' - \mathbf{x}|} d\mathbf{x}', \tag{D10}$$

and

$$\mathbf{A}^{\rm col}(\mathbf{x}, t) = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{x}', t')}{|\mathbf{x}' - \mathbf{x}|} d\mathbf{x}', \tag{D11}$$

where t' is the retarded time

$$t'(\mathbf{x}, t) = t - |\mathbf{x}' - \mathbf{x}|/c. \tag{D12}$$

Therefore the retarded potentials in Eqs. (D10) and (D11) are equivalent to the potentials in Eq. (D5) with the Liénard-Wiechert potentials in Eq. (D1) as the Green's function.

The equivalence of the Liénard-Wiechert field approach and the analyses in this paper lies in Eqs. (54) and (118), or

$$(\mathbf{E}^{\rm col} + \mathbf{v} \times \mathbf{B}^{\rm col}) \cdot \mathbf{e}_x = -e \frac{\hat{\partial}}{\partial x}(\Phi^{\rm col} - \boldsymbol{\beta} \cdot \mathbf{A}^{\rm col}) - \frac{e}{c} \frac{dA_x^{\rm col}}{dt} + \frac{e\beta_s A_s^{\rm col}}{r}, \tag{D13}$$

$$\mathbf{v} \cdot \mathbf{E}^{\rm col} = -e \frac{d\Phi^{\rm col}}{dt} + e \frac{\hat{\partial}}{\partial t}(\Phi^{\rm col} - \boldsymbol{\beta} \cdot \mathbf{A}^{\rm col}). \tag{D14}$$

Substituting the Liénard-Wiechert *fields* in Eqs. (D3) and (D4) into the left-hand sides of Eqs. (D13) and (D14) gives the results in Ref. [7] and [17]. On the other hand, if one substitutes the Liénard-Wiechert *potentials* in Eq. (D1) into the right-hand sides of Eqs. (D13) and (D14), one gets the cancellation effect demonstrated in this paper (the relation of Liénard-Wiechert field and potentials in Eq. (D14) was discussed in Ref. [19]).

APPENDIX E: EFFECTIVE TERMS ON A CIRCULAR ORBIT

Let us start from the interaction Lagrangian for collective interactions

$$\mathcal{L}_{\rm int}^{\rm col} = -e(\Phi^{\rm col} - \boldsymbol{\beta} \cdot \mathbf{A}^{\rm col}) = -e \int \frac{\rho(\mathbf{x}', t') - \boldsymbol{\beta} \cdot \mathbf{J}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}', \tag{E1}$$

with t dependence of the integrand hidden in the retarded time t' given in Eq. (D12). We now write $\mathcal{L}_{\rm int}^{\rm col}$ into two parts,

$$\mathcal{L}_{\rm int}^{\rm col} = \left[\mathcal{L}_{\rm int}^{\rm col}\right]^{(0)} + \left[\mathcal{L}_{\rm int}^{\rm col}\right]^{(1)} \tag{E2}$$

for

$$\left[\mathcal{L}_{\rm int}^{\rm col}\right]^{(0)} = \int \Pi^{(0)}(\mathbf{x},t;\mathbf{x}')d\mathbf{x}', \quad \Pi^{(0)}(\mathbf{x},t;\mathbf{x}') = -e\frac{\mathcal{N}^{(0)}\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}, \tag{E3}$$

$$\left[\mathcal{L}_{\rm int}^{\rm col}\right]^{(1)} = \int \Pi^{(1)}(\mathbf{x},t;\mathbf{x}')d\mathbf{x}', \quad \Pi^{(1)}(\mathbf{x},t;\mathbf{x}') = -e\frac{\mathcal{N}^{(1)}\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}, \tag{E4}$$

with

$$\mathcal{N}^{(0)} \equiv (1-\cos\Delta\theta) = 2\sin^2\frac{\Delta\theta}{2}, \tag{E5}$$

and using Eqs. (129)-(132) and (139),

$$\begin{aligned}\mathcal{N}^{(1)} &\equiv (\cos\Delta\theta - \boldsymbol{\beta}\cdot\boldsymbol{\beta}'_u) \\ &\approx \frac{1}{2}\left(\frac{1}{\gamma^2}+\frac{1}{\gamma_u'^2}+\beta_\perp^2+\beta_{u\perp'}'^2-2\beta_x\beta'_{ux'}\right)\cos\Delta\theta - (\beta_x\beta'_{us'}-\beta_s\beta'_{ux'})\sin\Delta\theta - \beta_y\beta'_{uy'}.\end{aligned} \tag{E6}$$

For the bunch self-interaction on a straight path ($\mathbf{e}_{s'}\cdot\mathbf{e}_s = 0$ or $\Delta\theta = 0$), one has for Eq. (E3)

$$\Pi^{(0)}(\mathbf{x},t;\mathbf{x}') = 0, \tag{E7}$$

and using Eq. (E6)

$$\Pi^{(1)}(\mathbf{x},t;\mathbf{x}') \simeq \frac{1}{2}\left(\frac{1}{\gamma^2}+\frac{1}{\gamma_u'^2}+\beta_\perp^2+\beta_{u\perp'}'^2-2\beta_x\beta'_{ux'}-2\beta_y\beta'_{uy'}\right)\frac{\rho(\mathbf{x}',t')}{|\mathbf{x}'-\mathbf{x}|}, \tag{E8}$$

which gives a negligibly small effect of local space charge interaction for high energy beams. For near-neighbor interaction on a curvilinear orbit, when $x = x'$ and $y = y'$, and when $s' \to s$ or $|\Delta\theta| \to 0$, $\Pi^{(0)}(\mathbf{x},t;\mathbf{x}')$ and $\Pi^{(1)}(\mathbf{x},t;\mathbf{x}')$ in Eqs. (E3) and (E4) becomes

$$\Pi^{(0)}(\mathbf{x},t;\mathbf{x}') \simeq -e\frac{\Delta\theta^2\rho(\mathbf{x}',t')}{2|\mathbf{x}'-\mathbf{x}|} \overset{|\Delta\theta|\to 0}{\rightarrow} 0 \tag{E9}$$

and

$$\Pi^{(1)}(\mathbf{x},t;\mathbf{x}') \overset{|\Delta\theta|\to 0}{\rightarrow} \frac{1}{2}\left(\frac{1}{\gamma^2}+\frac{1}{\gamma_u'^2}+\beta_\perp^2+\beta_{u\perp'}'^2-2\beta_x\beta'_{ux'}-2\beta_y\beta'_{uy'}\right)\frac{\rho(\mathbf{x}',t')}{|\mathbf{x}'-\mathbf{x}|}. \tag{E10}$$

Here Eq. (E9) shows that the near-neighbor interaction does not contribute to $\left[\mathcal{L}_{\rm int}^{\rm col}\right]^{(0)}$ in Eq. (E3), whereas due to the non-perfect relativistic longitudinal flow of the bunch, the near-neighbor interaction at $|\Delta\theta| \to 0$ has a small ($\propto \gamma^{-2}$ and $\beta_\perp^2$) contribution to $\left[\mathcal{L}_{\rm int}^{\rm col}\right]^{(1)}$, similar to the straight path case discussed in Eq. (E8).

Our discussion of the derivatives of $\mathcal{L}_{\rm int}^{\rm col}$ will focus only on the case of a circular orbit. For a test particle in a bunch on a circular orbit (κ = constant), $\mathcal{L}_{\rm int}^{\rm col}$ in Eq. (E1) is contributed from interactions of the test particle with source charge distributions both on the circle ($s_{\min} < s' < s_{\max}$) and outside the circle:

$$\mathcal{L}_{\text{int}}^{\text{col}} = [\mathcal{L}_{\text{int}}^{\text{col}}]_{\text{circle}} + [\mathcal{L}_{\text{int}}^{\text{col}}]_{s'<s_{\min}} + [\mathcal{L}_{\text{int}}^{\text{col}}]_{s'>s_{\max}}. \tag{E11}$$

We will only study the first term in Eq. (E11), which encompasses the contribution of the near-neighbor interaction to $\mathcal{L}_{\text{int}}^{\text{col}}$. For this term, both the observation and the retarded position are around the circular orbit,

$$\mathbf{x} = (R+x)\mathbf{e}_x + y\mathbf{e}_y, \quad \mathbf{x}' = (R+x')\mathbf{e}_{x'} + y'\mathbf{e}_{y'} \tag{E12}$$

where the bases are given by Eq. (46) for $s = R\theta$ and $s' = R\theta'$. One then has

$$[\mathcal{L}_{\text{int}}^{\text{col}}]_{\text{circle}} = [\mathcal{L}_{\text{int}}^{\text{col}}]_{\text{circle}}^{(0)} + [\mathcal{L}_{\text{int}}^{\text{col}}]_{\text{circle}}^{(1)}, \tag{E13}$$

where by denoting $\Delta s' = s' - s$ and

$$\int_{\text{circle}} d\mathbf{x}' = \int_{s_{\min}-s}^{s_{\max}-s} d\Delta s' \int_{-\infty}^{\infty} (1+\frac{x'}{R})dx' \int_{-\infty}^{\infty} dz', \tag{E14}$$

we have

$$[\mathcal{L}_{\text{int}}^{\text{col}}]_{\text{circle}}^{(0)}(\mathbf{x},t) = \int_{\text{circle}} d\mathbf{x}'\, \Pi^{(0)}(\mathbf{x},t;\Delta s',x',y'), \tag{E15}$$

$$[\mathcal{L}_{\text{int}}^{\text{col}}]_{\text{circle}}^{(1)}(\mathbf{x},t) = \int_{\text{circle}} d\mathbf{x}'\, \Pi^{(1)}(\mathbf{x},t;\Delta s',x',y'). \tag{E16}$$

Here $\Pi^{(0)}$ and $\Pi^{(1)}$ are given by Eqs. (E3) and (E4), in which $\rho(\mathbf{x}',t')$ automatically goes to zero when x' and y' get outside of the distribution, and for $\Delta\theta = -\Delta s'/R$

$$|\mathbf{x}'-\mathbf{x}| = \sqrt{\left(1+\frac{x}{R}\right)\left(1+\frac{x'}{R}\right)\left(2R\sin\frac{\Delta\theta}{2}\right)^2 + (x'-x)^2 + (y'-y)^2}. \tag{E17}$$

Let us rewrite the function ρ and the velocity field β_u' in Eqs. (E3) and (E4) as

$$\rho(\mathbf{x}',t') = \varrho[s'-s_c(t'),x',y',t'] \tag{E18}$$

$$\beta_{u\lambda'}'(\mathbf{x}',t') = \tilde{\beta}_{u\lambda'}'[s'-s_c(t'),x',y',t'] \tag{E19}$$

with the subscript $\lambda' = (s',x',y')$, and $s_c(t')$ the longitudinal position of the bunch centroid at retarded time t'. The derivatives of ρ and β_u' will be useful for the following discussions:

$$\frac{\partial\rho}{c\partial t} = -\beta_{sc}\frac{\partial\varrho}{\partial s'} + \frac{\partial\varrho}{c\partial t'}, \tag{E20}$$

$$\frac{\partial\rho}{\partial x} = \left(-\beta_{sc}\frac{\partial\varrho}{\partial s'} + \frac{\partial\varrho}{c\partial t'}\right)\frac{-\partial|\mathbf{x}'-\mathbf{x}|}{\partial x}, \tag{E21}$$

$$-\frac{\partial\mathcal{N}^{(1)}}{c\partial t} = \beta\cdot\frac{\partial\beta'_u}{c\partial t}$$
$$= (\beta_s\beta'_{us',t} + \beta_x\beta'_{ux',t})\cos\Delta\theta + (-\beta_s\beta'_{ux',t} + \beta_x\beta'_{us',t})\sin\Delta\theta + \beta_y\beta'_{uy',t}, \tag{E22}$$

$$\beta'_{u\lambda',t} \equiv \frac{\partial\beta'_{u\lambda'}}{c\partial t} = -\beta_{sc}\frac{\partial\tilde{\beta}'_{u\lambda'}}{\partial s'} + \frac{\partial\tilde{\beta}'_{u\lambda'}}{c\partial t'}, \tag{E23}$$

$$-\frac{\partial\mathcal{N}^{(1)}}{\partial x} = \beta\cdot\frac{\partial\beta'_u}{\partial x}$$
$$= (\beta_s\beta'_{us',x} + \beta_x\beta'_{ux',x})\cos\Delta\theta + (-\beta_s\beta'_{ux',x} + \beta_x\beta'_{us',x})\sin\Delta\theta + \beta_y\beta'_{uy',x}, \tag{E24}$$

$$\beta'_{u\lambda',x} \equiv \frac{\partial\beta'_{u\lambda'}}{\partial x} = \left(-\beta_{sc}\frac{\partial\tilde{\beta}'_{u\lambda'}}{\partial s'} + \frac{\partial\tilde{\beta}'_{u\lambda'}}{c\partial t'}\right)\frac{-\partial|\mathbf{x}'-\mathbf{x}|}{\partial x} \tag{E25}$$

where $\dot{s}_c = ds_c(t')/dt'$ is the longitudinal velocity of the bunch centroid at retarded time for an ultrarelativistic bunch, and $\beta_{sc} = \dot{s}_c/c \simeq 1$.

E.1. $\hat{\partial}\mathcal{L}^{\mathrm{col}}_{\mathrm{int}}/c\partial t$ in G_v

The near-neighbor interaction contribution to $\hat{\partial}\mathcal{L}^{\mathrm{col}}_{\mathrm{int}}/\partial t$ in G_v of Eq. (126) can be studied by analyzing

$$\left[\frac{\hat{\partial}\mathcal{L}^{\mathrm{col}}_{\mathrm{int}}}{c\partial t}\right]_{\mathrm{circle}} = \left[\frac{\hat{\partial}\mathcal{L}^{\mathrm{col}}_{\mathrm{int}}}{c\partial t}\right]^{(0)}_{\mathrm{circle}} + \left[\frac{\hat{\partial}\mathcal{L}^{\mathrm{col}}_{\mathrm{int}}}{c\partial t}\right]^{(1)}_{\mathrm{circle}}, \tag{E26}$$

where by using Eqs. (E15)-(E16),

$$\left[\frac{\hat{\partial}\mathcal{L}^{\mathrm{col}}_{\mathrm{int}}}{c\partial t}\right]^{(0)}_{\mathrm{circle}} = \int_{\mathrm{circle}} d\mathbf{x}'\,\Pi^{(0)}_t, \qquad \left[\frac{\hat{\partial}\mathcal{L}^{\mathrm{col}}_{\mathrm{int}}}{c\partial t}\right]^{(1)}_{\mathrm{circle}} = \int_{\mathrm{circle}} d\mathbf{x}'\Pi^{(1)}_t, \tag{E27}$$

with

$$\Pi^{(0)}_t = \frac{\partial\Pi^{(0)}}{c\partial t} = -e\frac{\mathcal{N}^{(0)}\dfrac{\partial\rho}{c\partial t}}{|\mathbf{x}'-\mathbf{x}|}, \tag{E28}$$

$$\Pi^{(1)}_t = \frac{\partial\Pi^{(1)}}{c\partial t} = -e\frac{\dfrac{\partial\mathcal{N}^{(1)}}{c\partial t}\rho + \mathcal{N}^{(1)}\dfrac{\partial\rho}{c\partial t}}{|\mathbf{x}'-\mathbf{x}|}. \tag{E29}$$

Here the terms $\partial\rho/c\partial t$, $\partial\mathcal{N}^{(1)}/c\partial t$ and $\partial\beta'_{u\lambda'}/c\partial t$ can be found in Eqs. (E20), (E22) and (E23). Let ς_s be the characteristic width in the bunch (such as the width of the microstructure in a bunch). Assuming the characteristic time for bunch shape variation is longer than ς_s/c, then the two terms in Eq. (E20) satisfy

$$\left\|\frac{\partial\varrho}{c\partial t'}\right\| \ll \left\|\frac{\partial\varrho}{\partial s'}\right\|, \tag{E30}$$

with $||f(x)||$ represents the amplitude of the function $f(x)$. One further assumes that the bunch centroid moving with the design longitudinal velocity $\beta_{sc} \simeq \beta_{s0} \simeq 1$, and the longitudinal distance of a particle with the bunch centroid being $z = s - \beta_{s0}ct$, therefore

$$z' = s' - s_c(t') \simeq s' - \beta_{s0}ct' = z + \Delta s' + \beta_{s0}|\mathbf{x}' - \mathbf{x}|. \tag{E31}$$

Due to $\mathcal{N}^{(0)} \simeq \Delta\theta^2/2 \ll 1$ at $|\Delta\theta| \ll 1$, significant contribution to $\left[\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}/c\partial t\right]^{(0)}_{\rm circle}$ in Eq. (E27) only comes from non-local interaction $|\Delta s'| \gg \sigma_\perp$ when we have in $\Pi^{(0)}_t$ the approximation $|\mathbf{x}' - \mathbf{x}| \simeq |\Delta s'|$. Let $\lambda(z', t')$ be the normalized bunch longitudinal density distribution for $\Delta s' \gg \sigma_\perp$

$$\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \varrho(s' - s_c(t'), x', y', t') \approx Ne\lambda(z', t - |\Delta s'|/c), \tag{E32}$$

then for $(s - s_{\rm min})/R > (\varsigma_s/R)^{1/3}$ and $\gamma_0(\varsigma_s/R)^{1/3} \gg 1$, we find from Eqs. (E27), (E30)-(E32)

$$\begin{aligned}[F^{\rm eff}_v]_{\rm circle} &= -\left[\frac{\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}}{c\partial t}\right]_{\rm circle} \simeq -\left[\frac{\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}}{c\partial t}\right]^{(0)}_{\rm circle} \\ &\simeq -\frac{2Ne^2}{(3R^2)^{1/3}} \int_0^{\Delta z(s)} \frac{d\Delta z}{(\Delta z)^{1/3}} \frac{\partial}{\partial z}\lambda\left(z - \Delta z, t - \frac{(24R^2\Delta z)^{1/3}}{c}\right),\end{aligned} \tag{E33}$$

with

$$\Delta z(s) = \frac{(s - s'_{\rm min})^3}{24R^2}. \tag{E34}$$

The contributions to the effective longitudinal force from $s' > s$ are negligible compared to those from $s' < s$ in Eq. (E33). Note that $F^{\rm eff}_v$ in Eq. (E33) reduces to the steady-state longitudinal CSR force [12, 18] when $\Delta z(s) \gg \sigma_s$ (σ_s is the rms bunch length) and when the longitudinal distribution is independent of time $\lambda(z, t) = \lambda_0(z)$.

E.2. $\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}/\partial x$ in G_x

The near-neighbor interaction in $\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}/\partial x$of Eq. (117) for a test particle on a circular orbit is studied by

$$\left[\frac{\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}}{\partial x}\right]_{\rm circle} = \left[\frac{\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}}{\partial x}\right]^{(0)}_{\rm circle} + \left[\frac{\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}}{\partial x}\right]^{(1)}_{\rm circle}, \tag{E35}$$

where by using Eqs. (E15)-(E16),

$$\left[\frac{\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}}{\partial x}\right]^{(0)}_{\rm circle} = \int_{\rm circle} d\mathbf{x}'\, \Pi^{(0)}_x, \qquad \left[\frac{\hat{\partial}\mathcal{L}^{\rm col}_{\rm int}}{\partial x}\right]^{(1)}_{\rm circle} = \int_{\rm circle} d\mathbf{x}'\, \Pi^{(1)}_x, \tag{E36}$$

with

$$\Pi_x^{(0)} = \frac{\partial \Pi^{(0)}}{\partial x} = -e\frac{\mathcal{N}^{(0)}}{|\mathbf{x}'-\mathbf{x}|}\left[\frac{\partial \rho}{\partial x} - \rho\frac{\partial|\mathbf{x}'-\mathbf{x}|/\partial x}{|\mathbf{x}'-\mathbf{x}|}\right], \tag{E37}$$

$$\Pi_x^{(1)} = \frac{\partial \Pi^{(1)}}{\partial x} = -e\frac{\frac{\partial \mathcal{N}^{(1)}}{\partial x}\rho}{|\mathbf{x}'-\mathbf{x}|} - e\frac{\mathcal{N}^{(1)}}{|\mathbf{x}'-\mathbf{x}|}\left[\frac{\partial \rho}{\partial x} - \rho\frac{\partial|\mathbf{x}'-\mathbf{x}|/\partial x}{|\mathbf{x}'-\mathbf{x}|}\right]. \tag{E38}$$

Using Eqs. (E21) and (E30), and $\beta_{sc} \simeq 1$, Eq. (E37) becomes

$$\Pi_x^{(0)} \approx -e\frac{\mathcal{N}^{(0)}}{|\mathbf{x}'-\mathbf{x}|}\left[\frac{\partial \rho}{\partial s} - \frac{\rho}{|\mathbf{x}'-\mathbf{x}|}\right]\frac{\partial|\mathbf{x}'-\mathbf{x}|}{\partial x}. \tag{E39}$$

As in Appendix E.1, due to $\mathcal{N}^{(0)} \simeq \Delta\theta^2/2 \ll 1$ at $|\Delta\theta| \ll 1$, a significant contribution to $\left[\partial\hat{\mathcal{L}}_{\rm int}^{\rm col}/c\partial t\right]^{(0)}_{\rm circle}$ only comes from non-local interaction $|\Delta s'| \gg \sigma_\perp$ when we have in $\Pi_x^{(0)}$ the approximation $|\mathbf{x}'-\mathbf{x}| \simeq |\Delta s'|$. Thus

$$\left\|\frac{\partial\rho}{\partial s}\right\| \sim \left\|\frac{\rho}{\varsigma_s}\right\| \gg \left\|\frac{\rho}{|\mathbf{x}'-\mathbf{x}|}\right\| \sim \left\|\frac{\rho}{\Delta s}\right\|. \tag{E40}$$

In addition, for a thin bunch case when $\sigma_\perp\sqrt{\sigma_\perp/R} \ll \sigma_s$,

$$\frac{\partial|\mathbf{x}'-\mathbf{x}|}{\partial x} \simeq \left|\sin\frac{\Delta\theta}{2}\right| + \frac{x-x'}{2R|\sin\Delta\theta/2|} \simeq \left|\sin\frac{\Delta\theta}{2}\right|. \tag{E41}$$

With the above approximations and Eqs. (131) and (141) for a relativistic flow, as well as $(s-s_{\rm min})/R > (\varsigma_s/R)^{1/3}$ and $\gamma_0(\varsigma_s/R)^{1/3} \gg 1$, we have

$$\begin{aligned}[F_x^{\rm eff}]_{\rm circle} &= \left[\frac{\partial\hat{\mathcal{L}}_{\rm int}^{\rm col}}{\partial x}\right]_{\rm circle} \simeq \left[\frac{\partial\hat{\mathcal{L}}_{\rm int}^{\rm col}}{\partial x}\right]^{(0)}_{\rm circle} \\ &\simeq -\frac{2Ne^2}{R}\int_0^{\Delta z(s)} d\Delta z \frac{\partial}{\partial z}\lambda\left(z-\Delta z, t-\frac{(24R^2\Delta z)^{1/3}}{c}\right)\end{aligned} \tag{E42}$$

where $\lambda(z,t)$ and $\Delta z(s)$ are given in Eqs. (E32) and (E34) respectively. For the longitudinal distribution independent of time $\lambda(z,t) = \lambda_0(z)$, Eq. (E42) is reduced to

$$[F_x^{\rm eff}]_{\rm circle} \simeq \frac{2Ne^2}{R}[\lambda_0(z-\Delta z(s)) - \lambda_0(z)], \tag{E43}$$

and for $\Delta z(s) \gg \sigma_s$, one gets the steady-state solution [5]

$$F_x^{\rm eff} = -\frac{2Ne^2\lambda_0(z)}{R}. \tag{E44}$$

It can be shown that contributions to the effective radial force from $s' > s$ are negligible compared to those from $s' < s$ in Eq. (E42).

E.3. $G_{\rm res}$

Here we estimate $G_{\rm res}$ in Eq. (125) on a circular orbit. With Eqs. (131) and (141),

$$[G_{\rm res}]_{\rm circle} \simeq e\frac{A_s^{\rm col} - \Phi^{\rm col}}{R} \simeq -\frac{e}{R}\int_{\rm circle} d\mathbf{x}'\frac{\mathcal{N}^{(0)}\rho(\mathbf{x}', t')}{|\mathbf{x}' - \mathbf{x}|}, \tag{E45}$$

with $\mathcal{N}^{(0)}$ in Eq. (E5). Due to $\mathcal{N}^{(0)} \ll 1$ for $|\Delta\theta| \ll 1$, the local interaction has negligible contribution to $G_{\rm res}$, and one finds

$$[G_{\rm res}]_{\rm circle} \simeq \frac{-2Ne^2}{R(3R^2)^{1/3}}\int_0^{\Delta z(s)} \frac{d\Delta z}{(\Delta z)^{1/3}}\lambda\left(z - \Delta z, t - \frac{(24R^2\Delta z)^{1/3}}{c}\right), \tag{E46}$$

where $\lambda(z, t)$ and $\Delta z(s)$ are given in Eqs. (E32) and (E34) respectively. Eq. (E46) reduces to the steady state results [6] when $\lambda(z, t) = \lambda_0(z)$ and $\Delta z(s) \gg \sigma_s$. It can be shown that contributions to $G_{\rm res}$ from $s' > s$ are negligible compared to those from $s' < s$ in Eq. (E46).

The ratio of integrands of $G_{\rm res}$, $F_x^{\rm eff}$ and $F_v^{\rm eff}$ in Eqs. (E46), (E42) and (E33) is

$$\frac{\lambda}{R}\left(\frac{R}{\Delta z}\right)^{1/3} : \frac{\partial\lambda}{\partial z} : \left(\frac{R}{\Delta z}\right)^{1/3}\frac{\partial\lambda}{\partial z}. \tag{E47}$$

For $|\Delta z/R| \sim \varsigma_s/R \ll 1$ and $||\partial\lambda/\partial z|| \sim \lambda/\varsigma_s$, we have the ratio of the amplitudes

$$||G_{\rm res}|| \ll ||F_x^{\rm eff}|| \ll ||F_v^{\rm eff}||. \tag{E48}$$

E.4. $dA_x^{\rm col}/dt$ in G_x

Using the Liénard-Wiechert potentials in Eq. (D1) from a source particle on a test particle,

$$A_{0x} = \mathbf{A}_0 \cdot \mathbf{e}_x = \left[\frac{\beta'_{s'}\sin\Delta\theta + \beta'_{x'}\cos\Delta\theta}{(1 - \boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{\rm ret}, \tag{E49}$$

and using Eq. (D5), the total potential $A_x^{\rm col}$ on the test particle due to its interaction with the whole charge distribution is

$$A_x^{\rm col} = \int A_{0x}(\mathbf{x}, t; \mathbf{s}')n(\mathbf{s}')d\mathbf{s}'. \tag{E50}$$

For near-neighbor interaction in an ultrarelativistic bunch satisfying Eq. (141), when both the test and source particles are on a straight path ($\Delta\theta = 0$) or on a circular orbit ($|\Delta\theta| \to 0$), Eqs. (E50) and (135) yield

$$A_{0x}(\mathbf{x}, t) \ll A_{0s}(\mathbf{x}, t), \tag{E51}$$

therefore the near-neighbor interactions have negligible contributions to $A_x^{\rm col}$ in Eq. (E50) compared to their contributions to $A_s^{\rm col}$. This can also be seen by comparing $A_s^{\rm col}$ in Eq. (143) with the retarded potential equivalent to Eq. (E50)

$$\begin{aligned} A_x^{\rm col} = \mathbf{A}^{\rm col}\cdot\mathbf{e}_x &= \int d\mathbf{x}'\frac{\rho(\mathbf{x}',t')(\beta'_{us'}\sin\Delta\theta+\beta'_{ux'}\cos\Delta\theta)}{|\mathbf{x}-\mathbf{x}'|} \\ &\approx \int d\mathbf{x}'\frac{\rho(\mathbf{x}',t')\sin\Delta\theta}{|\mathbf{x}-\mathbf{x}'|}, \end{aligned} \tag{E52}$$

with the use of Eq. (141).

We now look at the transient behavior of $A_x^{\rm col}$ on a test particle. Let us assume the test particle is ahead of the source particle. As the test particle enters from a straight path to a circular orbit, the potential $A_{0x}(\mathbf{x},t)$ on the test particle generated from the source particle on the straight path will start to change due to the change of $\mathbf{e}_x$ (while $\mathbf{e}_{s'}$ remains parallel to the straight path), causing a sudden change of $dA_x^{\rm col}/dt$. As a result, even though the near-neighbor interaction contribution to $A_x^{\rm col}$ is negligible compared that to $A_s^{\rm col}$ in $F^{\rm CSCF}$, the non-local interaction from the trailing particles will have non-negligible contributions to $dA_x^{\rm col}/dt$ in the transient regime. Transient behavior due to interactions of a test particle with both trailing and preceding source particles for entering or exiting a circular orbit can be considered in the same manner. It should be noted that $dA_x^{\rm col}/cdt$ shows up in $F_x^{\rm eff}$ of Eq. (117), and thus in the equation of motion Eq. (114), as the total time derivative. As a result, when one integrates the equation of motion over time, the evolution of the transverse phase space distribution will only depend on $A_x^{\rm col}$.

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