Ponderomotive Broadening

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Outline

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Ancient History

- Kibble and Brown (1964): Earliest definition of *K/a* in the literature that I'm aware of

$$a = \frac{eE_0\lambda_0}{2\pi mc^2} \text{ Thomson Sources} \qquad K = \frac{eB_0\lambda_0}{2\pi mc^2} \text{ Undulators}$$

  Interpreted frequency shifts that occur at high fields as a "relativistic mass shift".
- Sarachik and Schappert (1970): Power into harmonics at high *K/a* . Full calculation for CW (monochromatic) laser. Later referenced, corrected, and extended by workers in fusion plasma diagnostics.
- Alferov, Bashmakov, and Bessonov (1974): Undulator/Insertion Device theories developed under the assumption of constant field strength. Numerical codes developed to calculate "real" fields in undulators.
- Coisson (1979): Simplified undulator theory, which works at low *K/a,* developed to understand the frequency distribution of "edge" emission, or emission from "short" magnets, i.e., including pulse effects

Coisson's Spectrum from a Short Magnet

Coisson low-field strength undulator spectrum*

$$\frac{dE}{d\nu d\Omega} = \frac{r_e^2 c}{\pi}\gamma^2 f^2 \left(1+\gamma^2\theta^2\right)^2 \left|\tilde{B}\left(\nu\left(1+\gamma^2\theta^2\right)/\,2\gamma^2\right)\right|^2$$

$$f^2 = f_\sigma^2 + f_\pi^2$$

$$f_\sigma = \frac{1}{\left(1+\gamma^2\theta^2\right)^2}\sin\phi$$

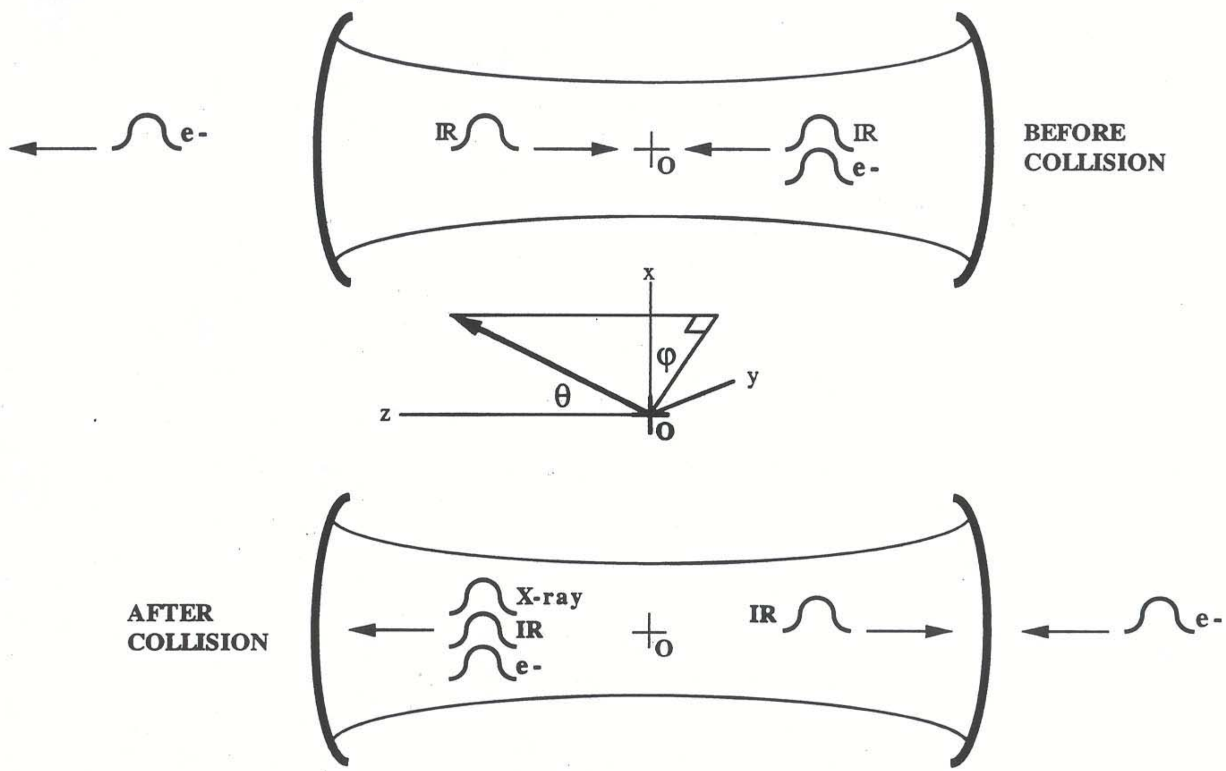
$$f_\pi = \frac{1}{\left(1+\gamma^2\theta^2\right)^2}\left(\frac{1-\gamma^2\theta^2}{1+\gamma^2\theta^2}\right)\cos\phi$$

*R. Coisson, Phys. Rev. A **20**, 524 (1979)

What's New in this Work

- Many of the the newer Thomson Sources are based on a PULSED Laser (e.g. the Free Electron Laser Thomson Sources are of this type, many of the high-field single-pulse lasers must be pulsed by their very nature)
- Have developed a general theory to cover the calculations in the general case of a pulsed, high field strength laser interacting with electrons.
- This theory shows that in many situations the estimates people do to calculate flux and brilliance, based on a constant amplitude models, are just plain wrong.
- The theory is general enough to cover all the previous cases mentioned.
- The main "new physics" that these calculations include properly is the fact that the electron longitudinal motion changes based on the local value of the field strength. Such ponderomotive forces (i.e., forces proportional to the field strength squared), lead to a detuning of the emission that this theory can calculate.

Thomson Source Scattering Geometry



SCATTERING GEOMETRY

Media Free Maxwell Equations (cgs)

$$\nabla \cdot \vec{E} = 4\pi\rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

Have wave solutions with wave velocity c

EM Momentum and Energy Density

From Maxwell Equations one derives an exact conservation law

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = -\vec{J} \cdot \vec{E}$$

where

$$u = \frac{1}{8\pi}\left(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}\right)$$ Energy Density

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$ Energy Flux (Poynting)

Plane Wave Solutions

Source free Maxwell Equations have plane wave solutions

$$\vec{E}(x,y,z,t) = \vec{\varepsilon} E_0 \sin(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

$$\vec{B}(x,y,z,t) = \vec{\varepsilon}_\perp E_0 \sin(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

$\omega = |k| c;\quad \vec{\varepsilon},\ \vec{\varepsilon}_\perp,$ and $\vec{k}$ form a right-handed set

E_0 is the amplitude of the field ($2E_0$ is the peak to peak)

$$u = \frac{E_0^2}{8\pi}$$

$$\vec{S} = \frac{cE_0^2}{8\pi}\hat{k} \tag{1.1}$$

Monochromatic Dipole Radiation

Assume a single charge moves sinusoidally in the x direction with angular frequency ω

$$\rho(x,y,z,t) = e\delta\left(x - d\sin(\omega t)\right)\delta(y)\delta(z)$$

$$\vec{J}(x,y,z,t) = ed\omega\cos(\omega t)\hat{x}\delta\left(x - d\sin(\omega t)\right)\delta(y)\delta(z)$$

Introduce scalar and vector potential for fields.
Retarded solution to wave equation (Lorenz gauge), $R = \left|\vec{r} - \vec{r}\,'(t')\right|$

$$\Phi(\vec{r},t) = \int \frac{1}{R}\rho\left(\vec{r}\,',t-\frac{R}{c}\right)dx'dy'dz' = e\int\frac{\delta(t'-t+R/c)}{R}dt'$$

$$A_x(\vec{r},t) = \int \frac{1}{Rc}J_x\left(\vec{r}\,',t-\frac{R}{c}\right)dx'dy'dz' = ed\omega\int\frac{\cos\omega t'\,\delta(t'-t+R/c)}{Rc}dt' \qquad (1.2)$$

Dipole Radiation

Perform proper differentiations to obtain field and integrate by parts the delta function properly.

Use far field approximation, $r = |\vec{r}| >> d$ (velocity terms small)

"Long" wave length approximation, $\lambda >> d$ (source smaller than λ)

Low velocity approximation, $\omega d / 2\pi << c$ (for given ω, really a limit on excitation strength)

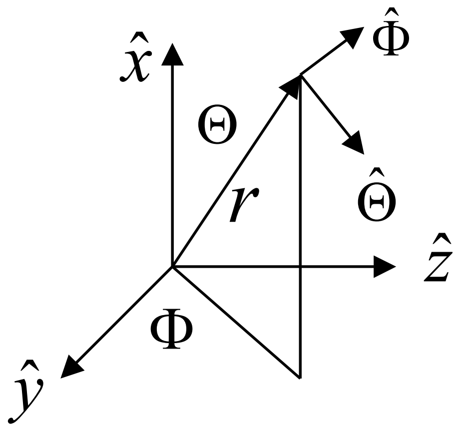
$$B_y = \partial A_x / \partial z = \frac{ed\omega^2}{c^2} z \frac{\sin[\omega(t - r/c)]}{r^2}$$

$$B_z = -\partial A_x / \partial y = -\frac{ed\omega^2}{c^2} y \frac{\sin[\omega(t - r/c)]}{r^2}$$

Dipole Radiation

$$\vec{B} = -\frac{ed\omega^2}{c^2 r}\sin\Theta\sin[\omega(t - r/c)]\hat{\Phi}$$

$$\vec{E} = -\frac{ed\omega^2}{c^2 r}\sin\Theta\sin[\omega(t - r/c)]\hat{\Theta}$$



$$I = \frac{c}{8\pi}\frac{e^2 d^2 \omega^4}{c^4 r^2}\sin^2\Theta\hat{r} \qquad \frac{dI}{d\Omega} = \frac{1}{8\pi}\frac{e^2 d^2 \omega^4}{c^3}\sin^2\Theta$$

Blue Sky! Polarized in the plane containing $\hat{r}$ and $\hat{x}$

Dipole Radiation (Frequency Spread)

Let $d(t)$ be the (one dimensional!) displacement of the charge along the x-axis. Define the Fourier Transform

$$\tilde{d}(\omega) = \int d(t) e^{-i\omega t} dt \qquad\qquad d(t) = \frac{1}{2\pi}\int \tilde{d}(\omega) e^{i\omega t} d\omega$$

What does the radiation look like? Note the DIPOLE PATTERN does not depend on frequency (within the approximations made)!

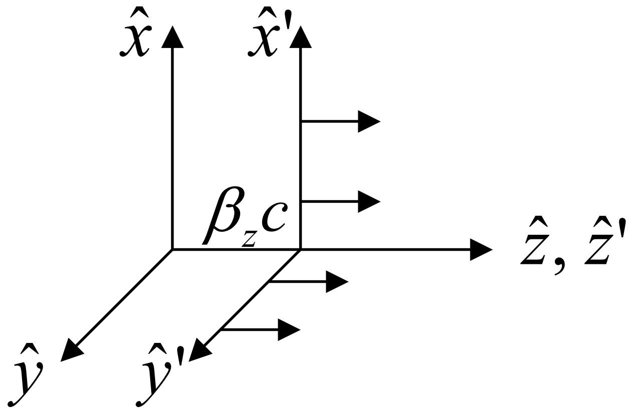
Obvious generalization (superposition) is

$$\frac{dE}{d\omega d\Omega} = \frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega)\right|^2\omega^4}{c^3}\sin^2\Theta \qquad (1.3)$$

Eqn. 1.3 does not follow the typical (see Jackson) convention that combines both positive and negative frequencies together in a single positive frequency integral. The reason is that we would like to apply Parseval's Theorem in subsequent work. By symmetry, the difference is a factor of two.

Co-moving Coordinates

- Assume radiating charge is moving with a velocity close to light in a direction taken to be the *z* axis, and the charge is on average at rest in this coordinate system
- For the remainder of the presentation, quantities referred to the moving coordinates will have primes; unprimed quantities refer to the lab system



- In the co-moving system the dipole radiation pattern applies

Frequency Spectrum: Pulsed Source

N periods of undulation at frequency f'

$$d(t') = d_0 \sin(\omega'_0 t')\left[\Theta(t' + N/2f') - \Theta(t' - N/2f')\right]$$

where $\omega'_0 = 2\pi f'$

$$\tilde{d}(\omega') = d_0 i(-1)^N \frac{\sin(\pi N \omega'/\omega'_0)}{\sin(\pi \omega'/\omega'_0)} \frac{2\omega'_0 \sin(\pi \omega'/\omega'_0)}{(\omega'_0)^2 - \omega'^2}$$

$$\equiv d_0 i(-1)^N f_N(\omega';\omega'_0) f_1(\omega';\omega'_0)$$

$$f_N(\omega';\omega'_0) \equiv \frac{\sin(\pi N \omega'/\omega'_0)}{\sin(\pi \omega'/\omega'_0)} = \sum_{n=-(N-1)/2}^{(N-1)/2} \exp(in\pi\omega'/\omega'_0) \quad N \text{ odd}, \neq 1$$

$$= 2\sum_{n=1}^{N/2} \cos((2n-1)\omega'/2\omega'_0) \quad N \text{ even}$$

Spectrum from a Pulsed Source

$$\int_{-\pi f'}^{\pi f'} f_N(\omega';\omega'_0)\,d\omega' = \omega'_0$$

Exactly for N odd, approximately for N large and even

$$\int f_N(\omega';\omega'_0) f^*_N(\omega';\omega'_0)\,d\omega' = \int \sum_{m=-(N-1)/2}^{(N-1)/2} \exp(im\pi\omega'/\omega'_0) \sum_{n=-(N-1)/2}^{(N-1)/2} \exp(-in\pi\omega'/\omega'_0)\,d\omega'$$

$$= \sum_{m=1}^{N}\sum_{n=1}^{N} \omega'_0\,\delta_{mn} = \sum_{n=1}^{N} \omega'_0\,\delta_{nn} = \omega'_0 N$$

Exactly for N both even and odd

$$\therefore \lim_{N\to\infty} \left[\frac{\sin(\pi N\omega'/\omega'_0)}{\sin(\pi\omega'/\omega'_0)}\right]^2 \to \sum_{k=-\infty}^{\infty} \omega'_0 N \delta(\omega' - k\omega'_0) \qquad (1.4)$$

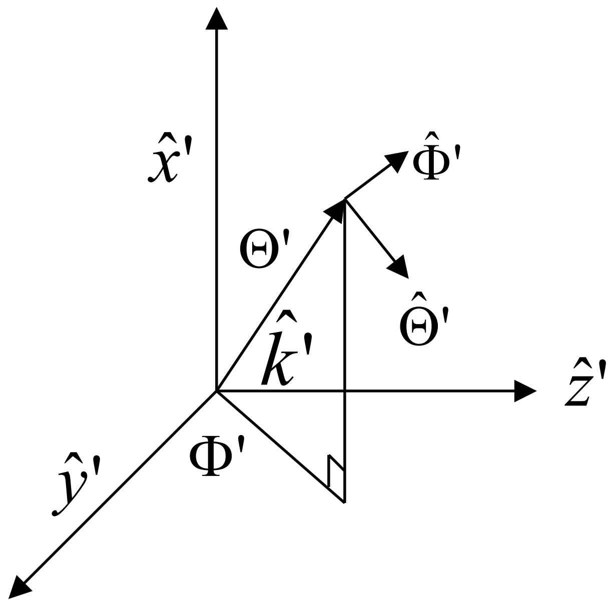
Summary

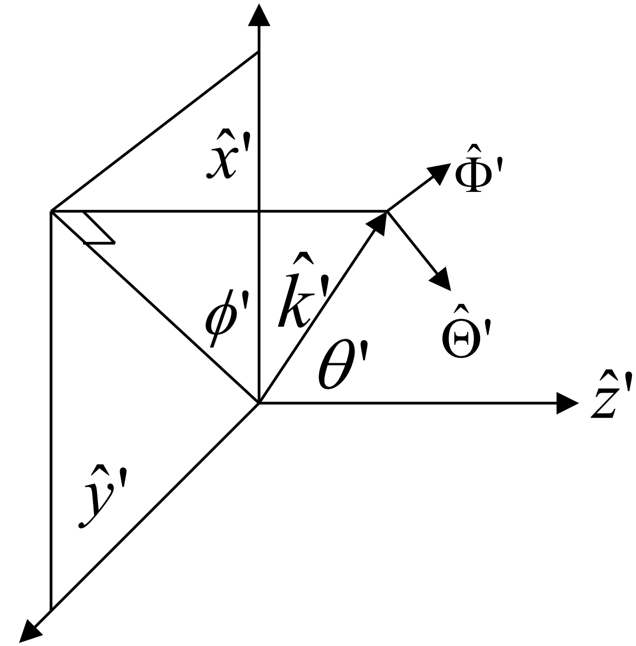
f_1 function goes to $1/(2f') = \pi/\omega'_0$ at the fundamental, and is much "wider" than f_N

$$\left|\tilde{d}(\omega')\right|^2 \rightarrow d_0^2 \frac{\pi^2}{\omega'_0} N \delta(\omega' - \omega'_0)$$

Total number of photons produced goes as *N*, in an energy distribution that narrows as 1/*N*

New Coordinates

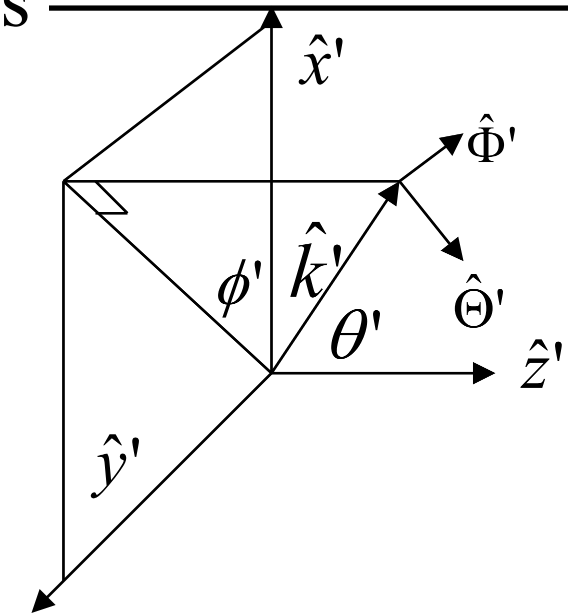




Resolve the polarization into that perpendicular (*perp)* and that parallel (*par*) to the *k-z* (scattering) plane

$$\vec{E}/\left|\vec{E}\right| = -\hat{\Theta}' = \frac{\hat{k}'\times\left(\hat{x}'\times\hat{k}'\right)}{\left|\hat{x}'\times\hat{k}'\right|}$$

New Coordinates



$$\hat{k}' = \sin\theta'\cos\phi'\,\hat{x}' + \sin\theta'\sin\phi'\,\hat{y}' + \cos\theta'\,\hat{z}'$$

$$\hat{e}_{perp} = \hat{k}'\times\hat{z}' / \left|\hat{k}'\times\hat{z}'\right| = \sin\phi'\,\hat{x}' - \cos\phi'\,\hat{y}' = -\hat{\phi}'$$

$$\hat{e}_{par} = \hat{k}'\times\hat{e}_{perp} = \cos\theta'\cos\phi'\,\hat{x}' + \cos\theta'\sin\phi'\,\hat{y}' - \sin\theta'\,\hat{z}' = \hat{\theta}'$$

Note $$-\hat{\Theta}' = \frac{\hat{k}'\times\left(\hat{x}'\times\hat{k}'\right)}{\left|\hat{x}'\times\hat{k}'\right|} = \frac{\hat{x}' - \left(\hat{k}'\cdot\hat{x}'\right)\hat{k}'}{\sin\Theta'}$$

Polarization

It follows that

$$\sin\Theta'\left(-\hat{\Theta}'\cdot\hat{e}_{perp}\right)=\sin\phi'$$
$$\sin\Theta'\left(-\hat{\Theta}'\cdot\hat{e}_{par}\right)=\cos\theta'\cos\phi'$$

So the energy into the two polarizations is

$$\frac{dE'_{perp}}{d\omega'\,d\Omega'}=\frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega')\right|^2\omega'^4}{c^3}\sin^2\phi'$$
$$\frac{dE'_{par}}{d\omega'\,d\Omega'}=\frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega')\right|^2\omega'^4}{c^3}\cos^2\theta'\cos^2\phi' \qquad (1.5)$$

Comments

- There is no radiation parallel or anti-parallel to the *x*-axis
- In the forward direction $\theta' \to 0$, the radiation polarization is parallel to the *x*-axis
- One may integrate over all angles to obtain the total energy radiated

$$\frac{dE'_{perp}}{d\omega'} = \frac{1}{8\pi^2} \frac{e^2 \left| \tilde{d}(\omega') \right|^2 \omega'^4}{c^3} 2\pi$$

$$\frac{dE'_{par}}{d\omega'} = \frac{1}{8\pi^2} \frac{e^2 \left| \tilde{d}(\omega') \right|^2 \omega'^4}{c^3} \frac{2\pi}{3}$$

$$\frac{dE'_{tot}}{d\omega'} = \frac{1}{8\pi^2} \frac{e^2 \left| \tilde{d}(\omega') \right|^2 \omega'^4}{c^3} \frac{8\pi}{3}$$

Generalized Larmor

Relativistic Invariances

To determine the radiation pattern for a "moving" oscillating charge we use this solution plus transformation formulas from relativity theory. For future reference, we note photon number invariance: The total number of photons emitted must be independent of the frame where the calculation is done. In particular,

$$N_{tot} = \frac{1}{3\pi} \int_{-\infty}^{\infty} \frac{e^2 \left| \tilde{d}(\omega') \right|^2 \omega'^4}{\hbar |\omega'| c^3} d\omega' \qquad (1.6)$$

must be frame independent. Rewriting formulas in terms of relativistically invariant quantities tends to simplify formulas.

Wave Vector Transformation Law

Follows from relativistic invariance of wave phase, which implies $k^\mu = \left(\omega / c, k_x, k_y, k_z\right)$ is a four vector

$$\omega' / c = \gamma\omega / c - \beta\gamma k \cos\theta$$

$$k' \sin\theta' \cos\phi' = k \sin\theta \cos\phi$$

$$k' \sin\theta' \sin\phi' = k \sin\theta \sin\phi$$

$$k' \cos\theta' = -\beta\gamma\omega / c + \gamma k \cos\theta$$

and $k = \omega / c$ and $k' = \omega' / c$ are the magnitudes of the wave propagation vectors

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \qquad \phi = \phi'$$

Invert by reversing the sign of β

Solid Angle Transformation

$$d\cos\theta' \wedge d\phi' = d\left(\frac{\cos\theta - \beta}{1-\beta\cos\theta}\right)\wedge d\phi$$

$$= \left(\frac{1-\beta\cos\theta + \beta\cos\theta - \beta^2}{(1-\beta\cos\theta)^2}\right) d\cos\theta \wedge d\phi$$

$$= \left(\frac{1}{\gamma^2(1-\beta\cos\theta)^2}\right) d\cos\theta \wedge d\phi$$

$$d\Omega' = \left(\frac{1}{\gamma^2(1-\beta\cos\theta)^2}\right) d\Omega$$

Photon Distribution in Beam Frame

$$\frac{dN_{perp}}{d\omega' d\Omega'} = \frac{1}{8\pi^2} \frac{e^2 \left|\tilde{d}(\omega')\right|^2 \omega'^4}{\hbar|\omega'|c^3} \sin^2 \phi'$$

$$\frac{dN_{par}}{d\omega' d\Omega'} = \frac{1}{8\pi^2} \frac{e^2 \left|\tilde{d}(\omega')\right|^2 \omega'^4}{\hbar|\omega'|c^3} \cos^2 \theta' \cos^2 \phi'$$

Photon Distribution in Lab Frame

$$\frac{dN_{perp}}{d\omega d\Omega} = \frac{d\omega'}{d\omega}\frac{d\Omega'}{d\Omega}\frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega')\right|^2\omega'^4}{\hbar|\omega'|c^3}\sin^2\phi'$$

$$\frac{dN_{par}}{d\omega d\Omega} = \frac{d\omega'}{d\omega}\frac{d\Omega'}{d\Omega}\frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega')\right|^2\omega'^4}{\hbar|\omega'|c^3}\cos^2\theta'\cos^2\phi'$$

Where the expression for the Doppler shifted frequency and angles are placed in these expressions

Photon Distribution in Lab Frame

$$\frac{dN_{perp}}{d\omega d\Omega} = \frac{1}{8\pi^2} \frac{e^2 \left|\tilde{d}(\omega')\right|^2 \omega'^4}{\hbar |\omega'| c^3} \frac{1}{\gamma(1-\beta\cos\theta)} \sin^2\phi$$

$$\frac{dN_{par}}{d\omega d\Omega} = \frac{1}{8\pi^2} \frac{e^2 \left|\tilde{d}(\omega')\right|^2 \omega'^4}{\hbar |\omega'| c^3} \frac{1}{\gamma(1-\beta\cos\theta)} \left(\frac{\cos\theta - \beta}{1-\beta\cos\theta}\right)^2 \cos^2\phi \qquad (1.7)$$

$$\omega' = \gamma(1-\beta\cos\theta)\omega$$

$$(1-\beta\cos\theta)(1+\beta) \approx \frac{1}{\gamma^2} + \theta^2 + \ldots \approx \frac{1+\gamma^2\theta^2}{\gamma^2}$$

Photon Distribution in Lab Frame

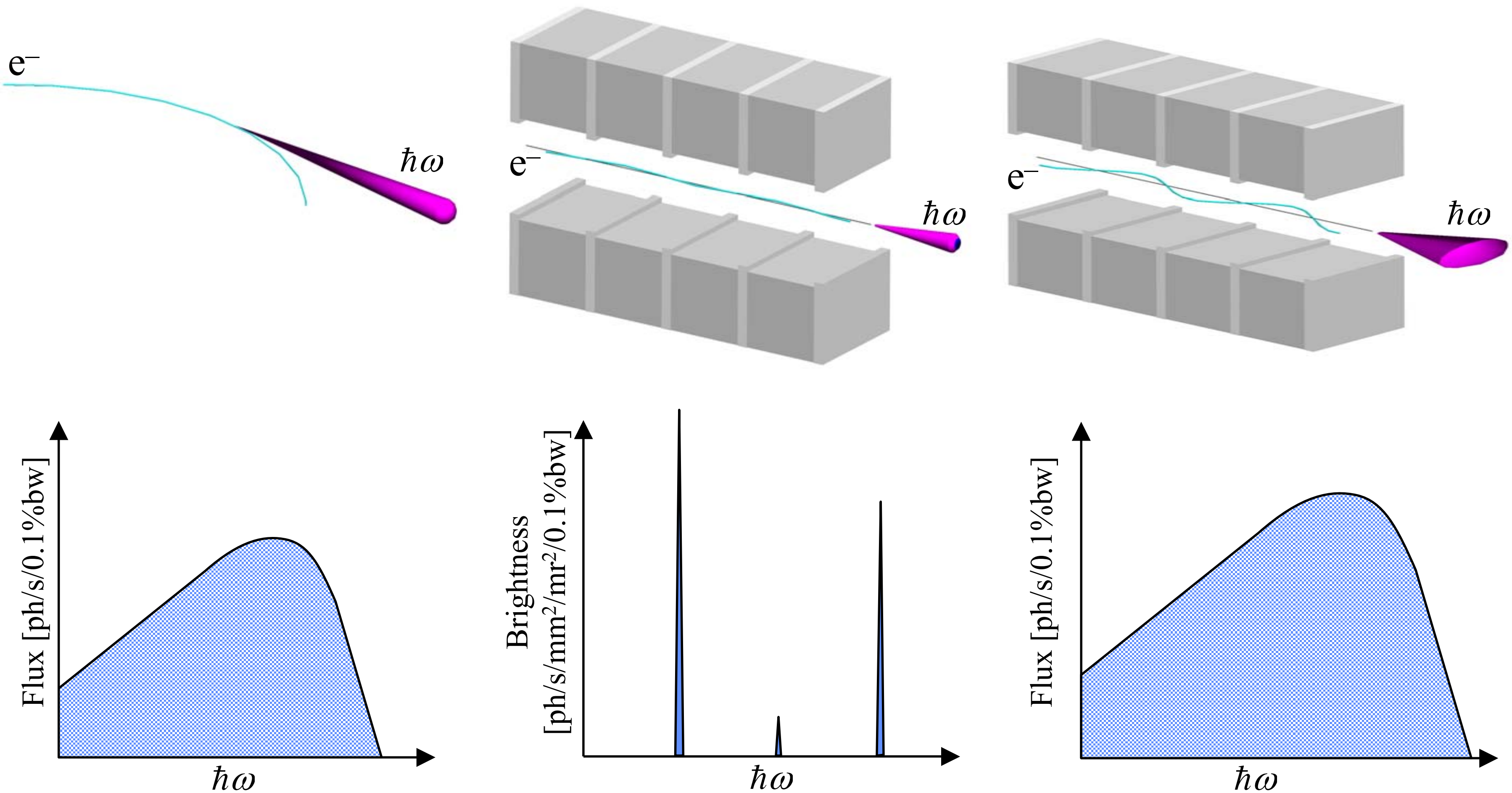
$$\frac{dN_{perp}}{d\omega d\Omega} = \frac{\alpha}{8\pi^2} \frac{\left|\tilde{d}(\omega')\right|^2 \omega'^4}{|\omega'|c^2} \frac{1}{\gamma(1-\beta_z \cos\theta)} \sin^2\phi$$

$$\frac{dN_{par}}{d\omega d\Omega} = \frac{\alpha}{8\pi^2} \frac{\left|\tilde{d}(\omega')\right|^2 \omega'^4}{|\omega'|c^2} \frac{1}{\gamma(1-\beta_z \cos\theta)} \left(\frac{\cos\theta - \beta_z}{1-\beta_z \cos\theta}\right)^2 \cos^2\phi \qquad (2.1)$$

$\omega' = \gamma(1-\beta_z \cos\theta)\omega \approx \omega / 2\gamma$ for $\theta \to 0$ Doppler

$$(1-\beta_z \cos\theta)(1+\beta_z) \approx \frac{1}{\gamma^2} + \theta^2 + \ldots \approx \frac{1+\gamma^2\theta^2}{\gamma^2}$$

Bend — Undulator — Wiggler



white source — partially coherent source — powerful white source

Undulator Orbits

- Will proceed by computing emission for small field strength, see USPAS lectures for the generalizations for field strengths typical in real undulators
- Need to calculate $d(t')$
- Can do whole calculation in the beam frame (easy for Thomson scatter calculations) or calculate the orbit in the lab frame, and Lorentz transform to the beam frame. Most references on undulators calculate the undulator orbit in the lab frame, as shall we.
- NB: Most references also do the electrodynamics in the lab frame too, using general (more complicated!) formulas for the emission.

Equations of Motion (Lab Frame)

$$\frac{d}{dt}\gamma = 0 \qquad (2.2)$$

$$\frac{d}{dt}\gamma m\vec{\beta}c = -e\vec{\beta}\times\vec{B} \qquad (2.3)$$

$$\vec{B} = B_0\frac{B(z)}{B_0}\hat{y}$$

$$\vec{\beta} = \beta_z\hat{z} + \beta_x\hat{x} \qquad \beta_x << \beta_z \approx 1$$

e is the fundamental charge, -*e* the electron charge

Approximate Solution

$$\frac{d\beta_x}{dt} = \frac{e\beta_z}{\gamma mc} B(z(t))$$

$$\beta_x(z) = \frac{e}{\gamma mc^2} \int_{-\infty}^{z} B(z')dz' \qquad (2.4)$$

$$x(z) = \frac{e}{\beta_z \gamma mc^2} \int_{-\infty}^{z} \int_{-\infty}^{z'} B(z'')dz''\,dz'$$

Fourier Transformed

$$\tilde{x}(k) = \int x(z) e^{-ikz} dz \qquad x(z) = \frac{1}{2\pi} \int \tilde{x}(k) e^{ikz} dk$$

$$\tilde{x}(k) = -\frac{e\tilde{B}(k)}{\beta_z \gamma m c^2 k^2} \qquad (2.5)$$

$$x(z(t)) = x(\beta_z c t) = -\frac{e}{2\pi \beta_z \gamma m c^2} \int_{-\infty}^{\infty} \frac{\tilde{B}(k)}{k^2} e^{ik\beta_z ct} dk$$

Eqn. 2.5 is strictly valid only if the electron is undeflected and unmoved by the undulator. In practical undulators, these conditions are approximately achieved by choosing an anti-symmetrical magnetic field, and by proper design of the two end cells of the undulator.

Lorentz Transformed

$$x(z(t)) = x(\beta_z ct) = -\frac{e}{2\pi\beta_z \gamma mc^2} \int_{-\infty}^{\infty} \frac{\widetilde{B}(k)}{k^2} e^{ik\beta_z ct} dk$$

$$ct = \gamma ct' + \beta_z \gamma z'$$

$$x = x'$$

$$y = y'$$

$$z = \beta_z \gamma ct' + \gamma z'$$

$$x'(t') = -\frac{e}{2\pi\beta_z \gamma mc^2} \int_{-\infty}^{\infty} \frac{\widetilde{B}(k)}{k^2} e^{ik\gamma\beta_z ct'} dk \qquad (z' = 0) \qquad (2.6)$$

Undulator period Lorentz contracted

Beam Frame Displacement Spectrum

$$\tilde{d}(\omega') = \int d(t')e^{-i\omega' t'}dt'$$

$$\tilde{d}(\omega') = \int_{-\infty}^{\infty} x'(t')e^{-i\omega' t'}dt' = -\frac{e}{2\pi\beta_z\gamma mc^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{B}(k)}{k^2}e^{ik\gamma\beta_z ct'}dke^{-i\omega' t'}dt'$$

$$\tilde{d}(\omega') = -\frac{e}{\beta_z\gamma mc^2}\int_{-\infty}^{\infty}\frac{\tilde{B}(k)}{k^2}\frac{\delta(k-\omega'/c\beta_z\gamma)}{c\beta_z\gamma}dk$$

$$\tilde{d}(\omega') = -\frac{ec\beta_z\gamma}{\beta_z\gamma mc^2}\frac{\tilde{B}(\omega'/c\beta_z\gamma)}{\omega'^2} \qquad (2.7)$$

Weak Field Undulator Spectrum

Combining previous results and, e. g.,
$$\frac{dE_{perp}}{d\omega d\Omega}=\frac{dN_{perp}}{d\omega d\Omega}\hbar|\omega|$$

$$\frac{dE_{perp}}{d\omega d\Omega}=\frac{1}{8\pi^2}\frac{e^4}{m^2c^5}\frac{\left|\tilde{B}\left(\omega\left(1-\beta_z\cos\theta\right)/c\beta_z\right)\right|^2}{\gamma^2\left(1-\beta_z\cos\theta\right)^2}\sin^2\phi$$

$$\frac{dE_{par}}{d\omega d\Omega}=\frac{1}{8\pi^2}\frac{e^4}{m^2c^5}\frac{\left|\tilde{B}\left(\omega\left(1-\beta_z\cos\theta\right)/c\beta_z\right)\right|^2}{\gamma^2\left(1-\beta_z\cos\theta\right)^2}\left(\frac{\cos\theta-\beta_z}{1-\beta_z\cos\theta}\right)^2\cos^2\phi$$

$$r_e^2\equiv\frac{e^4}{m^2c^4}\qquad\qquad\lambda=\frac{\lambda_0}{2\gamma^2}$$

Strong Field Case

$$\frac{d}{dt}\gamma = 0$$

$$\frac{d}{dt}\gamma m\vec{\beta}c = -e\vec{\beta}\times\vec{B}$$

$$\beta_x(z) = \frac{e}{\gamma mc^2}\int_{-\infty}^{z} B(z')dz'$$

High K

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2(z)}$$

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \left(\frac{e}{\gamma mc^2}\int_{-\infty}^{z} B(z')dz'\right)^2}$$

$$\beta_z(z) \approx 1 - \frac{1}{2\gamma^2} - \frac{1}{2}\left(\frac{e}{\gamma mc^2}\int_{-\infty}^{z} B(z')dz'\right)^2 = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) - \frac{K^2}{4\gamma^2}\cos(2k_0 z)$$

High K

Inside the insertion device the average (z) velocity is

$$\beta^*_z = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) \tag{2.11}$$

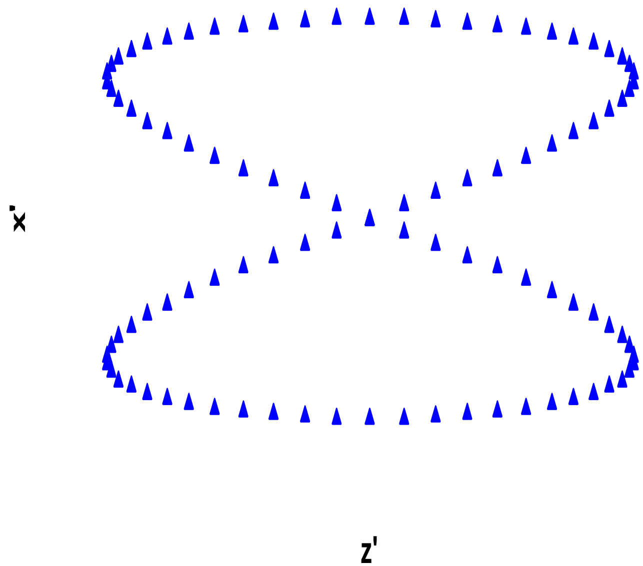
with corresponding

$$\gamma^* = \frac{1}{\sqrt{1-\beta^{*2}_z}} = \frac{\gamma}{\sqrt{1+K^2/2}} \tag{2.12}$$

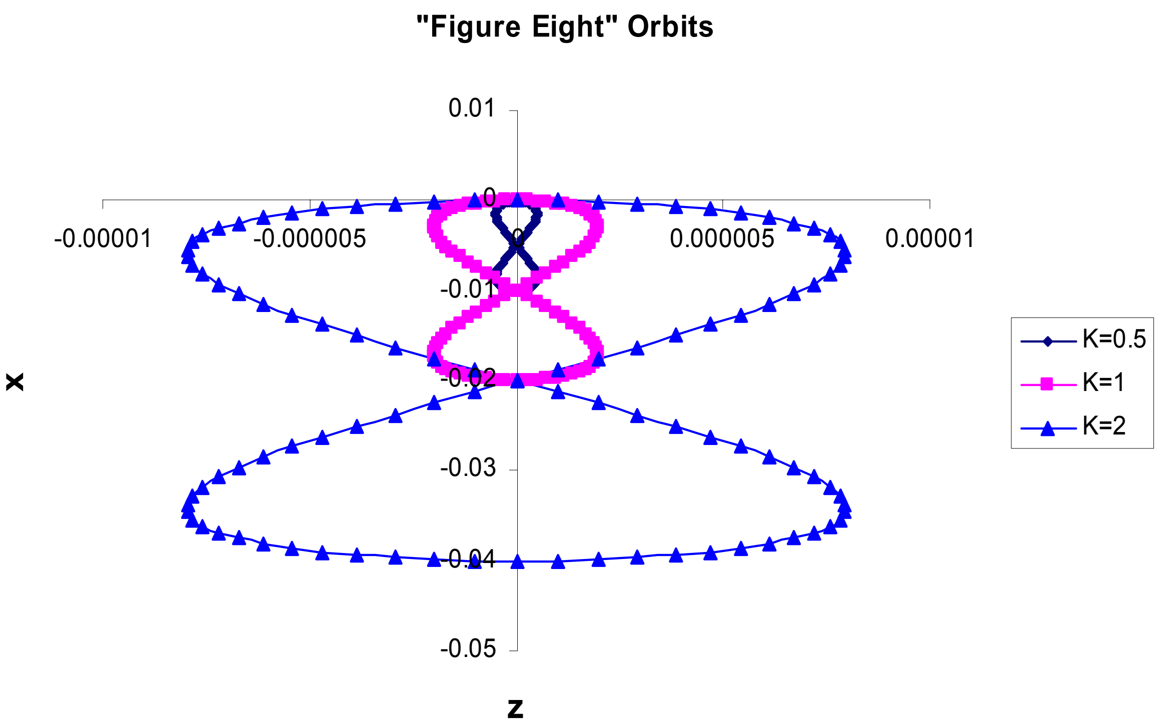
To apply dipole distributions, must be in this frame to begin with

Figure Eight

$$\frac{K}{\gamma}\frac{\lambda_0}{2\pi\beta^*_z}$$



$$\frac{\gamma^* K^2 \lambda_0}{8\gamma^2 2\pi\beta^*_z}$$



$\gamma = 100$, distances are normalized by $\lambda_0 / 2\pi$

$$\frac{dE_{perp,n}}{d\omega d\Omega} = \frac{e^2}{2c}\left[S_{1n} + S_{2n}/n\right]^2 \frac{\sin^2\phi}{\sin^2\theta\cos^2\phi} f_{nN}^2(\omega; n\omega(\theta))$$

$$\frac{dE_{par,n}}{d\omega d\Omega} = \frac{e^2}{2c}\left[\frac{S_{1n}(\cos\theta - \beta^*_z)}{(1-\beta^*_z\cos\theta)\sin\theta} + \frac{S_{2n}}{n\sin\theta\cos\theta}\right]^2 f_{nN}^2(\omega; n\omega(\theta))$$

f_{nN} is highly peaked, with peak value *nN*, around angular frequency

$$n\omega(\theta) = \frac{\beta^*_z n\omega_0}{(1-\beta^*_z\cos\theta)} \rightarrow 2\gamma^{*2}\beta^*_z n\omega_0 \approx \frac{2\gamma^2}{1+K^2/2} n\omega_0 \text{ as } \theta \rightarrow 0$$

Energy Distribution in Lab Frame

$$\frac{dE_{perp,n}}{d\omega d\Omega} = \frac{e^2}{2c}\left[S_{1n} + S_{2n}/n\right]^2 \frac{\sin^2\phi}{\sin^2\theta\cos^2\phi} f_{nN}^2\left(\omega; n\omega(\theta)\right)$$

$$\frac{dE_{par,n}}{d\omega d\Omega} = \frac{e^2}{2c}\left[\frac{S_{1n}\left(\cos\theta - \beta^*_z\right)}{\left(1-\beta^*_z\cos\theta\right)\sin\theta} + \frac{S_{2n}}{n\sin\theta\cos\theta}\right]^2 f_{nN}^2\left(\omega; n\omega(\theta)\right) \qquad (2.17)$$

The arguments of the Bessel Functions are now

$$\xi_x \equiv n\sin\theta'\cos\phi' d_x\omega'_0/c = n\frac{\sin\theta\cos\phi}{\left(1-\beta^*_z\cos\theta\right)}\frac{K}{\gamma}$$

$$\xi_z \equiv n\left(\beta^*_z + \cos\theta'\right)d_z\omega'_0/c = n\frac{\cos\theta}{\left(1-\beta^*_z\cos\theta\right)}\frac{\beta^*_z K^2}{8\gamma^2\beta^2}$$

In the Forward Direction

In the forward direction even harmonics vanish ($n+2k'$ term vanishes when "x" Bessel function non-zero at zero argument, and all other terms in sum vanish with a power higher than 2 as the argument goes to zero), and for odd harmonics only $n+2k'=1,-1$ contribute to the sum

$$\frac{dE_{perp,n}}{d\omega d\Omega} = \frac{e^2}{2c}\gamma^2\left(\frac{F_n(K)}{n^2}\right)\sin^2\phi\, f_{nN}^2(\omega; n\omega(\theta=0))$$

$$\frac{dE_{par,n}}{d\omega d\Omega} = \frac{e^2}{2c}\gamma^2\left(\frac{F_n(K)}{n^2}\right)\cos^2\phi\, f_{nN}^2(\omega; n\omega(\theta=0))$$

$$F_n(K) \approx \frac{1}{\gamma^2}\frac{n^2}{4(1-\beta^*_z)^2}\frac{K^2}{\gamma^2}\left[J_{\frac{n-1}{2}}\left(\frac{nK^2}{4(1+K^2/2)}\right) - J_{\frac{n+1}{2}}\left(\frac{nK^2}{4(1+K^2/2)}\right)\right]^2$$

Summary

- Coisson's Theory may be generalized to arbitrary observation angles by using the proper polarization decomposition
- Emission (in forward direction) is at ODD harmonics of the fundamental frequency, in addition to the fundamental frequency emission. The strength of the emission at the harmonics is quantified by a Bessel function factor.
- All kinematic parameters, including the angular distribution functions and frequency distributions, are just the same as before except unstarred quantities should be replaced by starred quantities
- In particular, the (FEL) resonance condition becomes

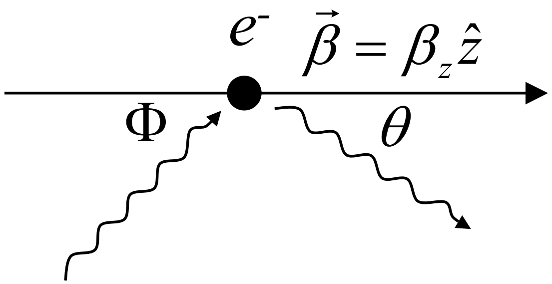
$$\lambda_n = \frac{n\lambda_0}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$

Thomson Scattering

- Purely "classical" scattering of photons by electrons
- Thomson regime defined by the photon energy in the electron rest frame being small compared to the rest energy of the electron
- In this case electron radiates at the same frequency as incident photon for small field strengths
- Dipole radiation pattern is generated in beam frame, as for undulators
- Therefore radiation patterns can be largely copied from our previous undulator work

- Note on terminology: Some authors call any scattering of photons by free electrons Compton Scattering. Compton observed (the so-called Compton effect) frequency shifts in X-ray scattering off (resting!) electrons that depended on scattering angle. Such frequency shifts arise only when the energy of the photon in the rest frame becomes comparable with 0.511 MeV. We will reserve the words "Compton Scattering", only for such higher energy scattering.

Simple Kinematics



Beam Frame

$$p'_{e\mu} = \left(mc^2, 0\right)$$

$$p'_{p\mu} = \left(E'_L, \vec{E}'_L\right)$$

Lab Frame

$$p_{e\mu} = mc^2\left(\gamma, \gamma\beta_z \hat{z}\right)$$

$$p_{p\mu} = E_L\left(1, \sin\Phi\hat{x} + \cos\Phi\hat{z}\right)$$

$$p_e \cdot p_p = mc^2 E'_L = mc^2 E_L \gamma\left(1 - \beta_z \cos\Phi\right) \tag{3.1}$$

$$E'_L = E_L \gamma \left(1 - \beta_z \cos \Phi\right)$$

In beam frame scattered photon radiated with wave vector

$$k'_\mu = \frac{E'_L}{c}\left(1, \sin\theta' \cos\phi', \sin\theta' \sin\phi', \cos\theta'\right)$$

Back in the lab frame, the scattered photon energy E_s is

$$E_s = E'_L \, \gamma\left(1 + \beta_z \cos\theta'\right) = \frac{E'_L}{\gamma\left(1 - \beta_z \cos\theta\right)}$$

$$E_s = E_L \frac{\left(1 - \beta_z \cos\Phi\right)}{\left(1 - \beta_z \cos\theta\right)} \qquad (3.2)$$

Cases explored

Backscattered

$$\Phi = \pi$$

$$E_s = E_L \frac{(1+\beta_z)}{(1-\beta_z \cos\theta)} \approx 4\gamma^2 E_L \qquad \text{at} \quad \theta = 0$$

Provides highest energy photons for a given beam energy, or alternatively, the lowest beam energy to obtain a given photon wavelength. Pulse length roughly the ELECTRON bunch length

Cases explored, contd.

Ninety degree scattering

$$\Phi = \pi / 2$$

$$E_s = E_L \frac{1}{\left(1 - \beta_z \cos\theta\right)} \approx 2\gamma^2 E_L \qquad \text{at} \quad \theta = 0$$

Provides factor of two lower energy photons for a given beam energy than the equivalent Backscattered situation. However, very useful for making short X-ray pulse lengths. Pulse length a complicated function of electron bunch length and transverse size.

Cases explored, contd.

Small angle scattered (SATS)

$$\Phi << 1$$

$$E_s = E_L \frac{\Phi^2}{2(1-\beta_z \cos\theta)} \approx \Phi^2 \gamma^2 E_L \qquad \text{at} \quad \theta = 0$$

Provides much lower energy photons for a given beam energy than the equivalent Backscattered situation. Alternatively, need greater beam energy to obtain a given photon wavelength. Pulse length roughly the PHOTON pulse length.

Transformation of Photon Field

Photon field for x-polarized plane wave traveling in the $-z$ direction (i.e., for the backscattered case!)

$$A_x(t,x,y,z) = A(z+ct)e^{i(k_z z+\omega t)}$$

$$A'_x(t',x',y',z') = A(\gamma(1+\beta_z)(z'+ct'))e^{i(k'_z z'+\omega' t')}$$

because $z+ct = \gamma(1+\beta_z)(z'+ct')$

$$\omega' = \gamma(1+\beta_z)\omega \qquad (3.3)$$

$$k'_z = \gamma(1+\beta_z)k_z$$

$$E'_x = \gamma(1+\beta_z)E_x$$

$$B'_y = -E'_x = \gamma(1+\beta_z)B_y$$

Finite Pulse Effects

The main focus of the lecture today is to generalize the work done so far to cover cases with

1. High Field strength

   And

2. Finite Energy spread from the pulsed photon beam itself

   Roughly speaking, the conclusion is that the energy spectra of the scattered photons is increased by a width of order of $1/N$, where N is the number of oscillations the electron makes for weak fields, but is considerably broader for strong fields.

Hamilton-Jacobi Method

Given single particle with Hamiltonian *H,* can obtain motion by solving the (first order) Hamilton-Jacobi partial differential equation (non-relativistic!)

$$-\frac{\partial S}{\partial t} = H\left(q_i, \frac{\partial S}{\partial q_i}\right)$$

Easy example: Free Particle Motion

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\sum_i \frac{\partial S}{\partial q_i}\cdot\frac{\partial S}{\partial q_i}$$

$$S = -\alpha t + \vec{q} \cdot \vec{k}$$

$$\alpha = \frac{1}{2m}\left|\vec{k}\right|^2$$

$$\vec{\beta} = \frac{\partial S}{\partial \vec{k}} = -\frac{\vec{k}}{m}t + \vec{q}$$ are constant

$$\vec{p} = \frac{\partial S}{\partial \vec{q}} = \vec{k}$$ ✓

$$\frac{\vec{k}}{m}(t-t_0)=\vec{q}-\vec{q}_0$$

Free particle action

$$S=-\alpha t+\vec{q}\cdot\vec{k}=\frac{m}{2}\frac{\left|\vec{q}-\vec{q}_0\right|^2}{t-t_0}$$

(by addition of an (irrelevant!) constant factor)

Electron in a Plane Wave

Assume plane-polarized pulsed laser beam moving in the –*z* direction

$$\vec{A} = A_x\left(z+ct\right)\hat{x} \equiv A_x\left(\xi\right)\hat{x}$$

Electron Langrangian (electron charge is –*e*)

$$L = -mc^2\sqrt{1-\beta^2} + e\phi - \frac{e}{c}\vec{v}\cdot\vec{A}$$

Electron canonical momentum

$$\vec{P} = \frac{\partial L}{\partial \vec{v}} = \gamma m\vec{v} - \frac{e}{c}\vec{A}$$

Electron Hamiltonian

$$H = \gamma mc^2 - e\phi$$

Hamiltonian in terms of canonical momentum

$$(H + e\phi)^2 = m^2c^4 + \left(\vec{P} + \frac{e}{c}\vec{A}\right)^2 c^2$$

Relativistic Hamilton-Jacobi Equation

$$\left(\frac{\partial S}{\partial t} + e\phi\right)^2 = m^2c^4 + \left(\frac{\partial S}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)^2 c^2 \qquad (3.4)$$

Free Particle

$$S = -k_0 ct + \vec{k} \cdot \vec{r}$$

$$k_0 = \sqrt{m^2c^2 + \left|\vec{k}\right|^2}$$

$$\vec{\beta} = \frac{\partial S}{\partial \vec{k}} = -\frac{\vec{k}c}{\sqrt{m^2c^2 + \left|\vec{k}\right|^2}} t + \vec{r}$$ are constant

$$\frac{\vec{k}c}{\sqrt{m^2c^2+\left|\vec{k}\right|^2}}\left(t-t_0\right)=\vec{r}-\vec{r}_0$$

$$\vec{k}=\frac{m\dfrac{\vec{r}-\vec{r}_0}{t-t_0}}{\sqrt{1-\left(\dfrac{\vec{r}-\vec{r}_0}{t-t_0}\right)^2/c^2}}=\gamma m\vec{v}$$ the usual formula

$$\vec{p}=\frac{\partial S}{\partial \vec{r}}=\vec{k} \quad \checkmark$$

With the Field

Assume solution of full H-J equation (3.4) of the form

$$S = -k_0 ct + \vec{k} \cdot \vec{r} + F(\xi)$$

Because the propagation vector is perpendicular to the vector potential, one obtains an ODE for *F*

$$\frac{dF}{d\xi} = \frac{1}{-k_0 - k_z}\left[\frac{m^2c^2 - k_0^2 + \left|\vec{k}\right|^2}{2} + \frac{e}{c}k_x A_x + \frac{e^2}{2c^2}A_x^2\right]$$

Obtain the energy and momentum by differentiation

$$-E=\frac{\partial S}{\partial t}=-k_0 c+\frac{-c}{k_0+k_z}\left[\frac{m^2c^2-k_0^2+\left|\vec{k}\right|^2}{2}+\frac{e}{c}k_x A_x+\frac{e^2}{2c^2}A_x^2\right]=-\gamma mc^2$$

$$P_x=\frac{\partial S}{\partial x}=k_x=\gamma mv_x-\frac{eA_x}{c}$$

$$P_y=\frac{\partial S}{\partial y}=k_y=\gamma mv_y \qquad (3.5)$$

$$P_z=\frac{\partial S}{\partial z}=k_z+\frac{1}{-k_0-k_z}\left[\frac{m^2c^2-k_0^2+\left|\vec{k}\right|^2}{2}+\frac{e}{c}k_x A_x+\frac{e^2}{2c^2}A_x^2\right]=\gamma mv_z$$

Obtain constants of motion by differentiation w.r.t. k_i

$$\beta_0 = \frac{\partial S}{\partial k_0} = -ct + \frac{1}{(k_0 + k_z)^2}\int_{-\infty}^{\xi}\left[\frac{m^2c^2 - k_0^2 + |\vec{k}|^2}{2} + \frac{e}{c}k_x A_x + \frac{e^2}{2c^2}A_x^2\right]d\xi' + \frac{k_0\xi}{k_0 + k_z}$$

$$\beta_1 = \frac{\partial S}{\partial k_x} = x - \frac{1}{k_0 + k_z}\int_{-\infty}^{\xi}\frac{eA_x}{c}d\xi' - \frac{k_x\xi}{k_0 + k_z}$$

$$\beta_2 = \frac{\partial S}{\partial k_y} = y - \frac{k_y\xi}{k_0 + k_z} \qquad (3.6)$$

$$\beta_3 = \frac{\partial S}{\partial k_z} = z + \frac{1}{(k_0 + k_z)^2}\int_{-\infty}^{\xi}\left[\frac{m^2c^2 - k_0^2 + |\vec{k}|^2}{2} + \frac{e}{c}k_x A_x + \frac{e^2}{2c^2}A_x^2\right]d\xi' - \frac{k_z\xi}{k_0 + k_z}$$

Boundary Conditions for Beam Frame

$$\dot{x}' = \dot{y}' = \dot{z}' = 0 \qquad \text{as} \qquad t \to -\infty$$

$$x' = y' = z' = 0 \qquad \text{as} \qquad t \to -\infty$$

$$k'_x = 0 \quad k'_y = 0 \quad k'_z = 0 \quad k'_0 = mc$$

$$\gamma' = 1 + \frac{e^2 A'^2_x}{2m^2c^4}$$

$$\frac{v'_x}{c} = \frac{eA'_x}{\gamma' mc^2} \tag{3.7}$$

$$\frac{v'_z}{c} = \frac{-\dfrac{e^2 A'^2_x}{2m^2c^4}}{\gamma'} = \frac{1-\gamma'}{\gamma'}$$

Ponderomotive Force along -*z*

Hamilton-Jacobi solution in lab frame: exact at high intensity

$$ct = \frac{\xi}{1+\beta_z} + \frac{1}{\gamma^2(1+\beta_z)^2}\int_{-\infty}^{\xi} \frac{e^2 A_x^2(\xi')}{2m^2c^4} d\xi'$$

$$x = \frac{1}{\gamma(1+\beta_z)}\int_{-\infty}^{\xi} \frac{eA_x(\xi')}{mc^2} d\xi'$$

$$y = 0$$

Ponderomotive Deceleration

$$z = \frac{\beta_z \xi}{1+\beta_z} - \frac{1}{\gamma^2(1+\beta_z)^2}\int_{-\infty}^{\xi} \frac{e^2 A_x^2(\xi')}{2m^2c^4} d\xi'$$

$A_x(\xi \equiv z + ct)$ is the (Lab-frame) vector potential describing the linearly polarized plane-wave laser pulse

Simplification Using the Proper Time

The orbit in the beam frame is expressed simply in terms of the proper time by differentiating the time equation

$$c\frac{dt'}{d\xi'} = 1 + \frac{e^2 A'^2_x}{2m^2c^4} = \gamma' \qquad \therefore\ \xi' = \tau + \text{constant}$$

Direct differentiation of Eqn. (3.6b,d) or ratios using Eqn. (3.7) yields

$$\frac{dx'}{d\xi'} = \frac{eA'_x}{mc^2}$$

$$\frac{dz'}{d\xi'} = -\frac{e^2 A'^2_x}{2m^2c^4}$$

The electron moves on a modulated sinusoid in proper time with angular frequency Ω'_0 in x and $2\Omega'_0$ in z where

$$\Omega'_0 = \gamma\left(1+\beta_z\right)\omega_0$$

$$\rho'(x',y',z',t') = -e\delta\left(x'-x'\left(\xi'\left(t'\right)\right)\right)\delta\left(y'\right)\delta\left(z'-z'\left(\xi'\left(t'\right)\right)\right)$$

$$\begin{aligned}\vec{J}'(x',y',z',t') = \\ -e\left(dx'/d\xi'\right)\left(d\xi'/dt'\right)\hat{x}\delta\left(x'-x'\left(\xi'\left(t'\right)\right)\right)\delta\left(y'\right)\delta\left(z'-z'\left(\xi'\left(t'\right)\right)\right) \\ -e\left(dz'/d\xi'\right)\left(d\xi'/dt'\right)\hat{z}\delta\left(x'-x'\left(\xi'\left(t'\right)\right)\right)\delta\left(y'\right)\delta\left(z'-z'\left(\xi'\left(t'\right)\right)\right)\end{aligned}$$

Radiation Potentials

$$\Phi'(\vec{r}',t')=\int\frac{1}{R'}\rho'\left(\vec{r}'',t'-\frac{R'}{c}\right)dx''dy''dz''=-\frac{e}{2\pi}\int\frac{e^{i\omega'(t''-t'+R'/c)}}{R'}dt''d\omega'$$

$$=-\frac{e}{2\pi c}\int\left[1+\frac{e^2A'^2_x(\xi')}{2m^2c^4}\right]\frac{e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''-t'+R'/c\right)}}{R'}d\xi'd\omega'$$

$$A'_x(\vec{r}',t')=\int\frac{1}{R'c}J'_x\left(\vec{r}'',t'-\frac{R'}{c}\right)dx''dy''dz''=-\frac{e}{2\pi}\int\frac{v'_x(t'')e^{i\omega'(t''-t'+R'/c)}}{R'c}dt''d\omega'$$

$$=-\frac{e}{2\pi c}\int\frac{eA'_x(\xi')}{mc^2}\frac{e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''-t'+R'/c\right)}}{R'}d\xi'd\omega'$$

$$A'_z(\vec{r}',t')=\int\frac{1}{R'c}J'_z\left(\vec{r}'',t'-\frac{R'}{c}\right)dx''dy''dz''=-\frac{e}{2\pi}\int\frac{v'_z(t'')e^{i\omega'(t''-t'+R'/c)}}{R'c}dt''d\omega'$$

$$=\frac{e}{2\pi c}\int\frac{e^2A'^2_x(\xi')}{2m^2c^4}\frac{e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''-t'+R'/c\right)}}{R'}d\xi'd\omega'$$

Magnetic Field

Space differentiate the potentials to obtain the magnetic field

$$\vec{B}'\approx\frac{e}{2\pi c^2 r'}\sin\Theta'\hat{\Phi}'\int\frac{eA'_x(\xi')}{mc^2}i\omega' e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''-t'+R'/c\right)}d\xi' d\omega'$$

$$-\frac{e}{2\pi c^2 r'}\sin\theta'\hat{\phi}'\int\frac{e^2A'^2_x(\xi')}{2m^2c^4}i\omega' e^{i\omega'\left(\frac{\xi'}{c}+\frac{1}{c}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''-t'+R'/c\right)}d\xi' d\omega'$$

Far field

$$R'=\sqrt{\left(x'-x'(\xi')\right)^2+y'^2+\left(z'-z'(\xi')\right)^2}$$

$$\approx r'\left(1-\frac{x'}{r'^2}\int_{-\infty}^{\xi'}\frac{eA'_x(\xi'')}{mc^2}d\xi''+\frac{z'}{r'^2}\int_{-\infty}^{\xi'}\frac{e^2A'^2_x(\xi'')}{2m^2c^4}d\xi''\right)$$

Fourier Transformed

$$\vec{B}'(\vec{r}\,',t') \approx \frac{e}{2\pi c^2 r'} \sin\Theta' \hat{\Phi}' \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi'' - t' + r'/c\right)} d\xi' d\omega'$$

$$-\frac{e}{2\pi c^2 r'} \sin\theta' \hat{\phi}' \int \frac{e^2 A'^2_x(\xi')}{2m^2c^4} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi'' - t' + r'/c\right)} d\xi' d\omega'$$

In Thomson limit, spectral content of the emitted radiation *depends "simply" on emission angle in the beam frame*

$$\tilde{\vec{B}}'(\vec{r}\,',\omega') \approx \frac{e}{c^2 r'} \sin\Theta' \hat{\Phi}' \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi'$$

$$-\frac{e}{c^2 r'} \sin\theta' \hat{\phi}' \int \frac{e^2 A'^2_x(\xi')}{2m^2c^4} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi'$$

Equivalent Dipole Moment Spectra

So the emission is calculated (exactly, in the far-field limit) using an equivalent *x*-dipole spectrum

$$D'_x(\omega') = \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi'$$

and an equivalent *z*-dipole spectrum

$$D'_z(\omega') = \int \frac{e^2 A'^2_x(\xi')}{2m^2c^4} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi'$$

Given the photon beam vector potential, one obtains the spectral distribution of the radiation by performing these “simple” integrals

Polarization Energy Distributions

Following through on previous calculations

$$\frac{dE'_{perp}}{d\omega' d\Omega'} = \frac{e^2}{8\pi^2 c^3} \left| D'_x(\omega') \right|^2 \sin^2 \phi'$$

$$\frac{dE'_{par}}{d\omega' d\Omega'} = \frac{e^2}{8\pi^2 c^3} \left| D'_x(\omega') \cos\theta' \cos\phi' + D'_z(\omega') \sin\theta' \right|^2$$

Results "exact" for electrons in a plane-wave and in the far-field limit. They can be used to reproduce what we previously have obtained!

Case I: Low Field Limit

$$\frac{eA'_x(\xi')}{mc^2} << 1$$

$$D'_x(\omega') = \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi' \approx -c\int \frac{eE'_x(\xi')}{mc^2} e^{i\omega'\frac{\xi'}{c}} d\xi'$$

$$D'_z(\omega') << D'_x(\omega')$$

Therefore, for low fields the spectral content of the emitted radiation *is independent of the emission angle in the beam frame.* The electron re-radiates at the same frequency as it moves.

Energy Spectral Distributions

$$\frac{dE'_{perp}}{d\omega' d\Omega'} = \frac{e^2}{8\pi^2 c} \frac{e^2 \left| \tilde{E}'_x \left(\omega'/c \right) \right|^2}{m^2 c^4} \sin^2 \phi'$$

$$\frac{dE'_{par}}{d\omega' d\Omega'} = \frac{e^2}{8\pi^2 c} \frac{e^2 \left| \tilde{E}'_x \left(\omega'/c \right) \right|^2}{m^2 c^4} \cos^2 \theta' \cos^2 \phi'$$

Or in the lab frame (see my first paper in PAC'97)

$$\frac{dE_{perp}}{d\omega d\Omega} = \frac{e^2}{8\pi^2 c} \frac{e^2 \left| \tilde{E}_x \left(\omega \left(1 - \beta_z \cos\theta \right) / \left(c \left(1 + \beta_z \right) \right) \right) \right|^2}{m^2 c^4 \gamma^2 \left(1 - \beta_z \cos\theta \right)^2} \sin^2 \phi$$

(3.8)

$$\frac{dE_{par}}{d\omega d\Omega} = \frac{e^2}{8\pi^2 c} \frac{e^2 \left| \tilde{E}_x \left(\omega \left(1 - \beta_z \cos\theta \right) / \left(c \left(1 + \beta_z \right) \right) \right) \right|^2}{m^2 c^4 \gamma^2 \left(1 - \beta_z \cos\theta \right)^2} \left(\frac{\cos\theta - \beta_z}{1 - \beta_z \cos\theta} \right)^2 \cos^2 \phi$$

Comparison to Weak Field Undulator

This result is identical to the weak field undulator result with the replacement of the magnetic field Fourier transform by the electric field Fourier transform

	Undulator	Thomson Backscatter
Driving Field	$\tilde{B}_y\left(\omega\left(1-\beta_z\cos\theta\right)/c\beta_z\right)$	$\tilde{E}_x\left(\omega\left(1-\beta_z\cos\theta\right)/\left(c\left(1+\beta_z\right)\right)\right)$
Forward Frequency	$\lambda \approx \dfrac{\lambda_0}{2\gamma^2}$	$\lambda \approx \dfrac{\lambda_0}{4\gamma^2}$
	Lorentz contract + Doppler	Double Doppler

Case II: Longitudinally Flat Photon Pulse

Suppose (unrealistically!), that the photon pulse is hard-edge and flat with N_p periods

$$A_x(\xi) = A_0 \cos\left(\frac{2\pi}{\lambda_0}\xi\right)\left[\Theta(\xi) - \Theta(\xi - N_p\lambda_0)\right] \qquad \text{define } a \equiv \frac{eA_0}{mc^2}$$

$$D'_x(\omega') = \int \frac{eA'_x(\xi')}{mc^2} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi' =$$

$$i\omega' a \int \cos(k'_z\xi') e^{i\omega'\left(\frac{\xi'}{c} - \frac{a\sin\theta'\cos\phi'}{k'_z c}\sin(k'_z\xi') + \frac{a^2(1+\cos\theta')}{4c}\left(\xi' + \frac{\sin(2k'_z\xi')}{2k'_z}\right)\right)} d\xi'$$

$$D'_z(\omega') = \int \frac{e^2 A'^2_x(\xi')}{2m^2c^4} i\omega' e^{i\omega'\left(\frac{\xi'}{c} - \frac{\sin\theta'\cos\phi'}{c}\int_{-\infty}^{\xi'} \frac{eA'_x(\xi'')}{mc^2} d\xi'' + \frac{(1+\cos\theta')}{c}\int_{-\infty}^{\xi'} \frac{e^2 A'^2_x(\xi'')}{2m^2c^4} d\xi''\right)} d\xi' =$$

$$\frac{i\omega' a^2}{2} \int \cos^2(k'_z\xi') e^{i\omega'\left(\frac{\xi'}{c} - \frac{a\sin\theta'\cos\phi'}{k'_z c}\sin(k'_z\xi') + \frac{a^2(1+\cos\theta')}{4c}\left(\xi' + \frac{\sin(2k'_z\xi')}{2k'_z}\right)\right)} d\xi'$$

Energy Spectral Distributions: Lab Frame

$$\frac{dE_{perp,n}}{d\omega d\Omega}=\frac{e^2}{2c}\frac{\sin^2\phi}{\sin^2\theta\cos^2\phi}\left[S_{1n}+S_{2n}/n\right]^2 f_{nN}^2\left(\omega;n\omega(\theta)\right)$$

(3.9)

$$\frac{dE_{par,n}}{d\omega d\Omega}=\frac{e^2}{2c}\left[S_{1n}\left(\frac{\cos\theta-\beta_z}{\left(1-\beta_z\cos\theta\right)\sin\theta}+\frac{1}{\gamma^2\left(1+\beta_z\right)}\frac{a^2}{4}\frac{\omega}{nk_zc}\frac{\sin\theta}{\left(1-\beta_z\cos\theta\right)}\right)+\frac{S_{2n}}{n}\left(\frac{1}{\sin\theta}\right)\right]^2 f_{nN}^2$$

$$S_{1n}\equiv\sum_{k'=-\infty}^{\infty}J_{n+2k'}\left(\frac{a\omega}{k_zc}\frac{\sin\theta\cos\phi}{\gamma\left(1+\beta_z\right)}\right)J_{k'}\left(\frac{a^2}{8}\frac{\omega}{k_zc}\frac{\left(1+\cos\theta\right)}{\gamma^2\left(1+\beta_z\right)^2}\right)$$

$$S_{2n}\equiv\sum_{k'=-\infty}^{\infty}2k'J_{n+2k'}\left(\frac{a\omega}{k_zc}\frac{\sin\theta\cos\phi}{\gamma\left(1+\beta_z\right)}\right)J_{k'}\left(\frac{a^2}{8}\frac{\omega}{k_zc}\frac{\left(1+\cos\theta\right)}{\gamma^2\left(1+\beta_z\right)^2}\right)$$

$$\omega(\theta)=\frac{\left(1+\beta_z\right)\left(2\pi c/\lambda_0\right)}{1-\beta_z\cos\theta+\left(a^2/4\right)\left(1-\beta_z\right)\left(1+\cos\theta\right)}$$

Comparison to High K Undulators

The main results are very similar to those from undulators with the following correspondences

	Undulator	Thomson Backscatter
Field Strength	K	a
Forward Frequency	$\lambda \approx \frac{\lambda_0}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$	$\lambda \approx \frac{\lambda_0}{4\gamma^2}\left(1+\frac{a^2}{2}\right)$
Transverse Pattern	$\beta^*_z + \cos\theta'$	$1+\cos\theta'$

NB, be careful with the radiation pattern, it is the same at small angles, but quite a bit different at large angles

Case III: Realistic Pulse Distribution at High *a*

In general, it's easiest to just numerically integrate the lab-frame expression for the spectrum in terms of D_x and D_z. A 10^5 to 10^6 point Simpson integration is adequate for most purposes. I've done two types of pulses, flat pulses to reproduce the previous results and to evaluate numerical error, and Gaussian Laser pulses.

One may utilize a two-timing approximation (i.e., the laser pulse is a slowly varying sinusoid with amplitude $a(\xi)$), and the fundamental expressions for D_x and D_z, to write the energy distribution at any angle in terms of the Bessel function expansions and a ξ integral over the modulation amplitude. This approach actually has a limited domain of applicability ($K, a < 0.1$)

Scattering of a high intensity pulsed laser

$$\frac{dE_\sigma}{d\omega d\Omega}=\frac{e^2\omega^2\left|D_x(\omega)\right|^2}{8\pi^2c^3}\sin^2\phi$$

$$\frac{dE_\pi}{d\omega d\Omega}=\frac{e^2\omega^2}{8\pi^2c^3}\left|D_x(\omega)\left(\frac{\cos\theta-\beta_z}{1-\beta_z\cos\theta}\right)\cos\phi-D_z(\omega)\sin\theta\right|^2$$

$$D_x(\omega)=\int\frac{1}{\gamma(1+\beta_z)}\frac{eA_x(\xi)}{mc^2}e^{i\omega\left(\frac{\xi(1-\beta_z\cos\theta)}{c(1+\beta_z)}-\frac{\sin\theta\cos\phi}{c\gamma(1+\beta_z)}\int_{-\infty}^{\xi}\frac{eA_x(\xi')}{mc^2}d\xi'+\frac{(1+\cos\theta)}{c\gamma^2(1+\beta_z)^2}\int_{-\infty}^{\xi}\frac{e^2A_x^2(\xi')}{2m^2c^4}d\xi'\right)}d\xi,$$

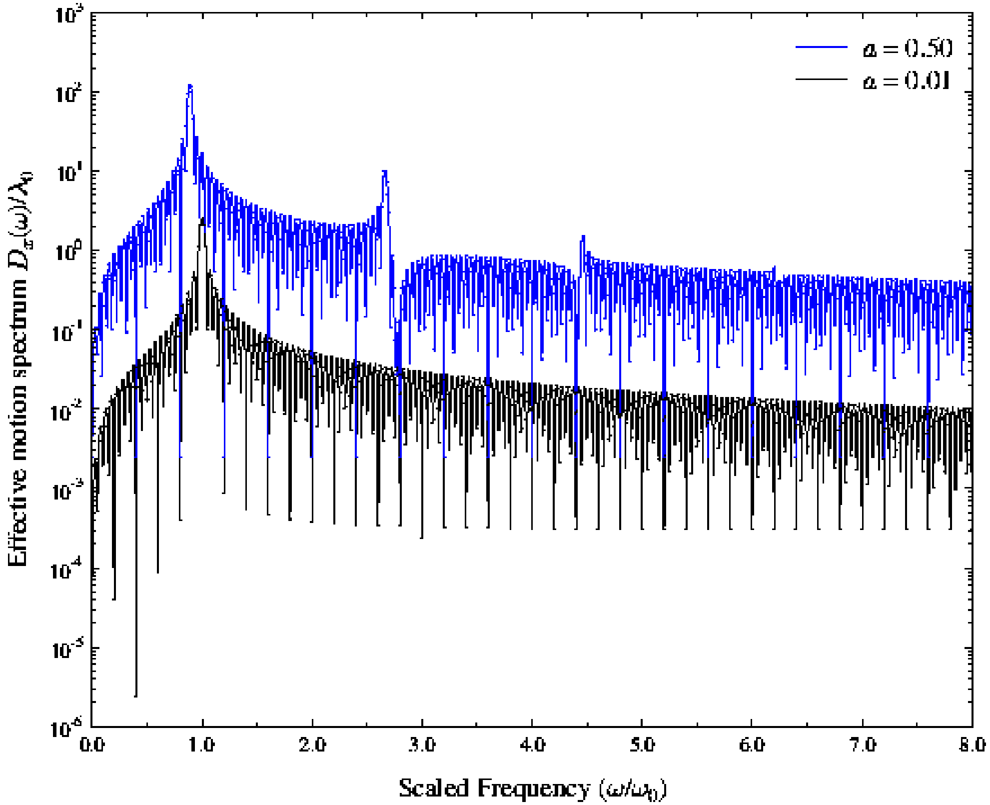
$$D_z(\omega)=-\frac{1+\beta_z}{(1-\beta_z\cos\theta)}\int\frac{1}{\gamma^2(1+\beta_z)^2}\frac{e^2A_x^2(\xi)}{2m^2c^4}e^{i\omega\left(\frac{\xi(1-\beta_z\cos\theta)}{c(1+\beta_z)}-\frac{\sin\theta\cos\phi}{c\gamma(1+\beta_z)}\int_{-\infty}^{\xi}\frac{eA_x(\xi')}{mc^2}d\xi'+\frac{(1+\cos\theta)}{c\gamma^2(1+\beta_z)^2}\int_{-\infty}^{\xi}\frac{e^2A_x^2(\xi')}{2m^2c^4}d\xi'\right)}d\xi$$

In low intensity limit D_x is “just” the Fourier Transform of the electron transverse velocity

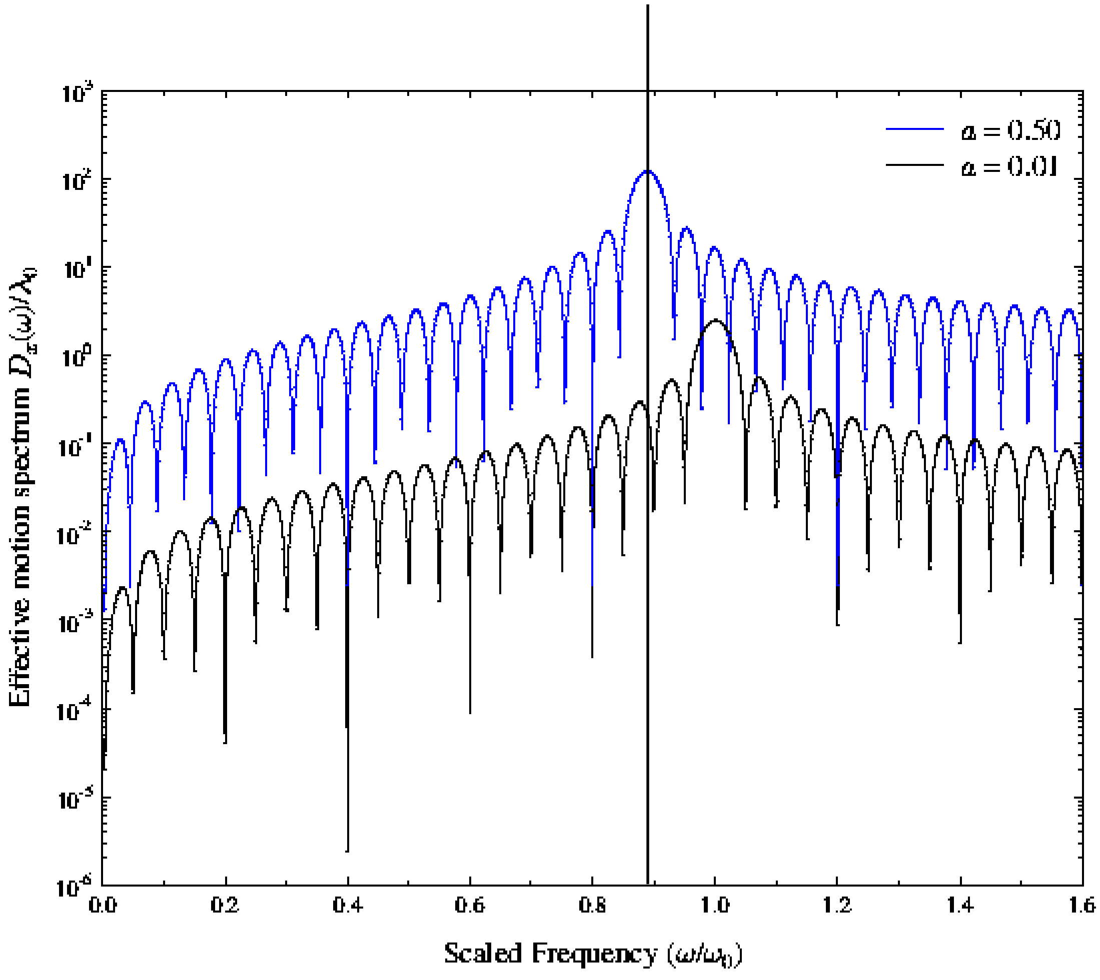
Forward Direction: Flat, Undulator-like Pulse

20-period
equivalent undulator: $A_x(\xi) = A_0 \cos(2\pi\xi/\lambda_0)[\Theta(\xi) - \Theta(\xi - 20\lambda_0)]$

$$\omega_0 \equiv (1+\beta_z)^2 \gamma^2 2\pi c/\lambda_0 \approx 4\gamma^2 2\pi c/\lambda_0, \;\; a = eA_0/mc^2$$



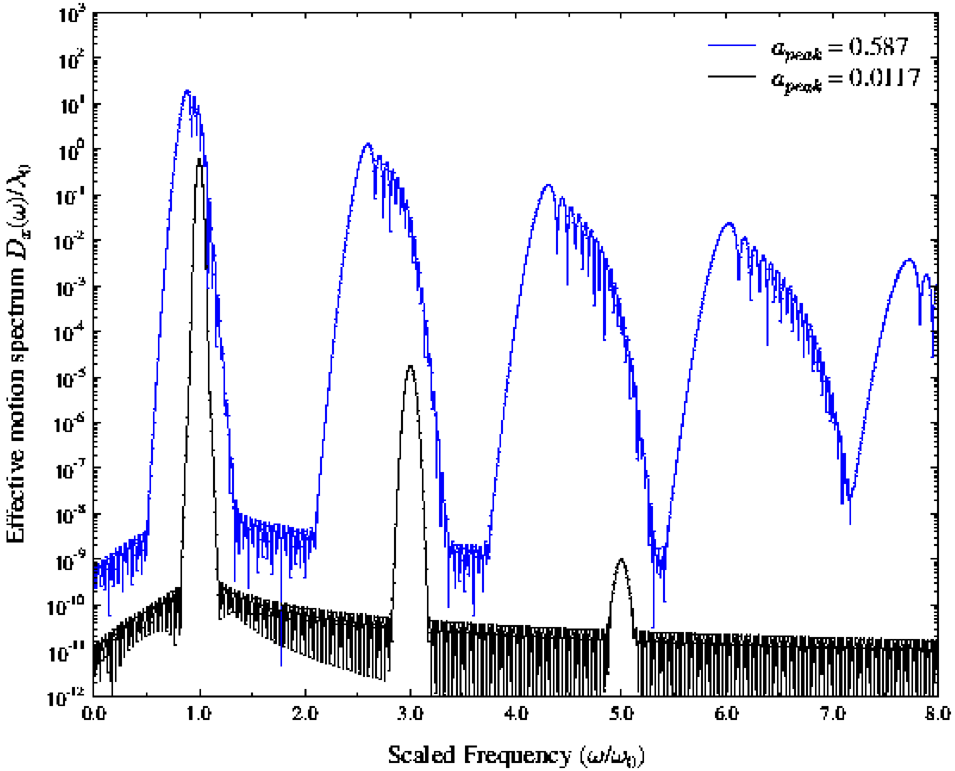
$$1/\left(1+a^2/2\right)$$



Forward Direction: Gaussian Pulse

$$A_x(\xi) = A_{peak}\exp\left(-z^2/2(8.156\lambda_0)^2\right)\cos\left(2\pi\xi/\lambda_0\right) \qquad a_{peak} = eA_{peak}/mc^2$$

A_{peak} and λ_0 chosen for same intensity and same *rms* pulse length as previous slide



Conclusions

- An introduction to Thomson Scatter source radiation calculations and a general formula for obtaining the spectral brilliance has been given
- I've shown how dipole solutions to the Maxwell Equations can be used to obtain very general expressions for the spectral angular energy distributions for Insertion Devices and Thomson Scattering photon sources
- A "new" calculation scheme for high intensity pulsed laser Thomson Scattering has been developed. This same scheme can be applied to calculate spectral properties of "short", high-*K* wigglers.
- Due to ponderomotive broadening, it is simply wrong to use single-frequency estimates of flux and brilliance in situations where the square of the field strength parameter becomes comparable to or exceeds the (1/*N*) spectral width of the induced electron wiggle
- The new theory is especially useful when considering Thomson scattering of Table-Top terawatt lasers, which have exceedingly high field and short pulses. Any calculation that does not include ponderomotive broadening is pure fiction.