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USPAS Course on
4^{th} Generation Light Sources II
ERLs and Thomson Scattering

G. A. Krafft and I. V. Bazarov
Jefferson Lab and Cornell University

Introduction to Undulator Radiation

Prerequisites

- Jackson (*Classical Electrodynamics)* or Landau and Lifshitz (*The Classical Theory of Fields)* level of understanding of electrodynamics
- Facility in arguments based on the Special Theory of Relativity
- Some previous exposure to particle accelerators, and typical single particle motion calculations for relativistic particle motion
- Understanding of statistical arguments at the level of a typical undergraduate course in Thermodynamics or Statistical Mechanics

Course Outline

Day 1

1. Radiation from undulators (GK, IB)
   - Radiation from an electric dipole
   - Weak-field (short) insertion devices
   - Strong-field insertion devices
2. Scaling Rules (GK, IB)
   - Flux
   - Brilliance
3. Thomson Scattering (GK)
   - Basics

Course Outline

Day 2

4. Average Brilliance/Scaling (GK)
   - General formula for spectral characteristics
   - Weak-field scattering
   - Strong-field scattering
   - Flux
   - Brilliance
5. Thomson Scatter Sources (IB)

Course Outline

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Day 3

6. Thomson Scatter Sources and Laser Synchrotron Sources (IB)
   - Overview
   - Jefferson Lab
   - BNL
   - Berkeley
   - Duke
   - Idaho
   - NRL
   - Small Angle Thomson Scattering
   - Low Energy Storage Ring
7. ALS Short-Pulse Facility (IB)
8. RF and SRF (GK)

Course Outline

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Day 4

9. Energy Recovering Linacs (ERLs) and their properties (GK)
   - Beam Stability in ERLs (GK)
   - Design of ERLs (GK, IB)
10. ERL example (JLAB IRFEL) (GK)
11. ERL example (Cornell prototype and Phase II design) (IB)

Day 5

12. ERL examples (BNL, Berkeley, Japan, Erlangen, MARS, 4GLS) (IB)
13. Critical Future Problems to be solved (IB)

Introductory Lecture: Undulators

1. Larmor Dipole Radiation
   1. Review of Maxwell
   2. Monochromatic “Dipole” Solution
   3. Finite Pulse Solution
2. Lorentz Transformation
   1. Photon Number Invariance
   2. Wave Vector Transformation
   3. Angular Distribution Transformation
3. Undulators
   1. Parameters and Properties
   2. General Solution for Small *K*
   3. Finite *K* Effects
4. Qualitative Discussion on Angular Patterns
5. Finite Emittance Effects
6. Brilliance Scaling
7. Summary and Some Slides on Coherence

Media Free Maxwell Equations (cgs) -

$$\nabla \cdot \vec{E} = 4\pi\rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

Have wave solutions with wave velocity c

EM Momentum and Energy Density –

From Maxwell Equations one derives an exact conservation law

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = -\vec{J} \cdot \vec{E}$$

where

$$u = \frac{1}{8\pi}\left(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}\right)$$ Energy Density

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$ Energy Flux (Poynting)

Plane Wave Solutions

Source free Maxwell Equations have plane wave solutions

$$\vec{E}(x,y,z,t) = \vec{\varepsilon} E_0 \sin(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

$$\vec{B}(x,y,z,t) = \vec{\varepsilon}_\perp E_0 \sin(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

$\omega = |k|c;\quad \vec{\varepsilon},\ \vec{\varepsilon}_\perp,$ and $\vec{k}$ form a right-handed set

E_0 is the amplitude of the field ($2E_0$ is the peak to peak)

$$u = \frac{E_0^2}{8\pi}$$

$$\vec{S} = \frac{cE_0^2}{8\pi}\hat{k} \tag{1.1}$$

Monochromatic Dipole Radiation

Assume a single charge moves sinusoidally in the x direction with angular frequency ω

$$\rho(x,y,z,t) = e\delta\big(x - d\sin(\omega t)\big)\delta(y)\delta(z)$$

$$\vec{J}(x,y,z,t) = ed\omega\cos(\omega t)\hat{x}\delta\big(x - d\sin(\omega t)\big)\delta(y)\delta(z)$$

Introduce scalar and vector potential for fields.
Retarded solution to wave equation (Lorenz gauge), $R = \left|\vec{r} - \vec{r}\,'(t')\right|$

$$\Phi(\vec{r},t) = \int \frac{1}{R}\rho\left(\vec{r}\,',t-\frac{R}{c}\right)dx'\,dy'\,dz' = e\int\frac{\delta(t'-t+R/c)}{R}dt'$$

$$A_x(\vec{r},t) = \int \frac{1}{Rc}J_x\left(\vec{r}\,',t-\frac{R}{c}\right)dx'\,dy'\,dz' = ed\omega\int\frac{\cos\omega t'\,\delta(t'-t+R/c)}{Rc}dt' \qquad (1.2)$$

Dipole Radiation

Perform proper differentiations to obtain field and integrate by parts the delta function properly.

Use far field approximation, $r = |\vec{r}| >> d$ (velocity terms small)

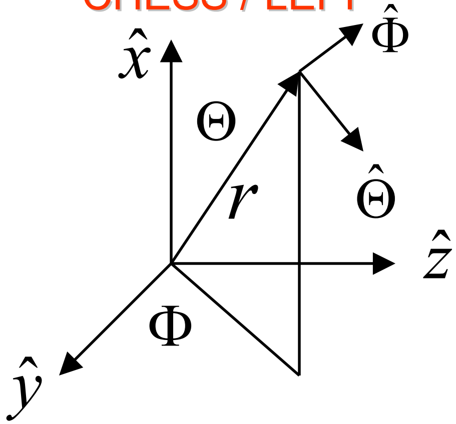
"Long" wave length approximation, $\lambda >> d$ (source smaller than λ)

Low velocity approximation, $\omega d / 2\pi << c$ (for given ω, really a limit on excitation strength)

$$B_y = \partial A_x / \partial z = \frac{ed\omega^2}{c^2} z \frac{\sin[\omega(t - r/c)]}{r^2}$$

$$B_z = -\partial A_x / \partial y = -\frac{ed\omega^2}{c^2} y \frac{\sin[\omega(t - r/c)]}{r^2}$$

Dipole Radiation



$$\vec{B} = -\frac{ed\omega^2}{c^2 r}\sin\Theta \sin[\omega(t - r/c)]\hat{\Phi}$$

$$\vec{E} = -\frac{ed\omega^2}{c^2 r}\sin\Theta \sin[\omega(t - r/c)]\hat{\Theta}$$

$$I = \frac{c}{8\pi}\frac{e^2 d^2 \omega^4}{c^4 r^2}\sin^2\Theta\hat{r} \qquad \frac{dI}{d\Omega} = \frac{1}{8\pi}\frac{e^2 d^2 \omega^4}{c^3}\sin^2\Theta$$

Blue Sky! Polarized in the plane containing $\hat{r}$ and $\hat{x}$

Dipole Radiation (Frequency Spread) - CORNELL UNIVERSITY CHESS / LEPP

Let $d(t)$ be the (one dimensional!) displacement of the charge along the x-axis. Define the Fourier Transform

$$\tilde{d}(\omega) = \int d(t) e^{-i\omega t} dt \qquad d(t) = \frac{1}{2\pi}\int \tilde{d}(\omega) e^{i\omega t} d\omega$$

What does the radiation look like? Note the DIPOLE PATTERN does not depend on frequency (within the approximations made)!

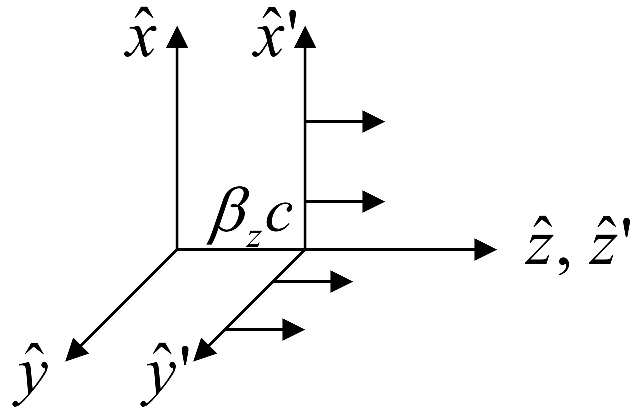
Obvious generalization (superposition) is

$$\frac{dE}{d\omega d\Omega} = \frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega)\right|^2\omega^4}{c^3}\sin^2\Theta \qquad (1.3)$$

Eqn. 1.3 does not follow the typical (see Jackson) convention that combines both positive and negative frequencies together in a single positive frequency integral. The reason is that we would like to apply Parseval's Theorem in subsequent work. By symmetry, the difference is a factor of two.

Co-moving Coordinates

- Assume radiating charge is moving with a velocity close to light in a direction taken to be the *z* axis, and the charge is on average at rest in this coordinate system
- For the remainder of the presentation, quantities referred to the moving coordinates will have primes; unprimed quantities refer to the lab system



- In the co-moving system the dipole radiation pattern applies

Frequency Spectrum: Pulsed Source

N periods of undulation at frequency f'

$$d(t') = d_0 \sin(\omega'_0 t')\left[\Theta(t' + N/2f') - \Theta(t' - N/2f')\right]$$

where $\omega'_0 = 2\pi f'$

$$\tilde{d}(\omega') = d_0 i(-1)^N \frac{\sin(\pi N\omega'/\omega'_0)}{\sin(\pi\omega'/\omega'_0)} \frac{2\omega'_0 \sin(\pi\omega'/\omega'_0)}{(\omega'_0)^2 - \omega'^2}$$

$$\equiv d_0 i(-1)^N f_N(\omega';\omega'_0) f_1(\omega';\omega'_0)$$

$$f_N(\omega';\omega'_0) \equiv \frac{\sin(\pi N\omega'/\omega'_0)}{\sin(\pi\omega'/\omega'_0)} = \sum_{n=-(N-1)/2}^{(N-1)/2} \exp(in\pi\omega'/\omega'_0) \quad N \text{ odd}, \neq 1$$

$$= 2\sum_{n=1}^{N/2} \cos((2n-1)\omega'/2\omega'_0) \quad N \text{ even}$$

Spectrum from a Pulsed Source

$$\int_{-\pi f'}^{\pi f'} f_N(\omega';\omega'_0)\,d\omega' = \omega'_0$$

Exactly for N odd, approximately for N large and even

$$\int f_N(\omega';\omega'_0) f^*{}_N(\omega';\omega'_0)\,d\omega' = \int \sum_{m=-(N-1)/2}^{(N-1)/2} \exp(im\pi\omega'/\omega'_0) \sum_{n=-(N-1)/2}^{(N-1)/2} \exp(-in\pi\omega'/\omega'_0)\,d\omega'$$

$$= \sum_{m=1}^{N}\sum_{n=1}^{N} \omega'_0\,\delta_{mn} = \sum_{n=1}^{N} \omega'_0\,\delta_{nn} = \omega'_0\,N$$

Exactly for N both even and odd

$$\therefore \lim_{N\to\infty} \left[\frac{\sin(\pi N\omega'/\omega'_0)}{\sin(\pi\omega'/\omega'_0)}\right]^2 \to \sum_{k=-\infty}^{\infty} \omega'_0\, N\,\delta(\omega' - k\omega'_0) \qquad (1.4)$$

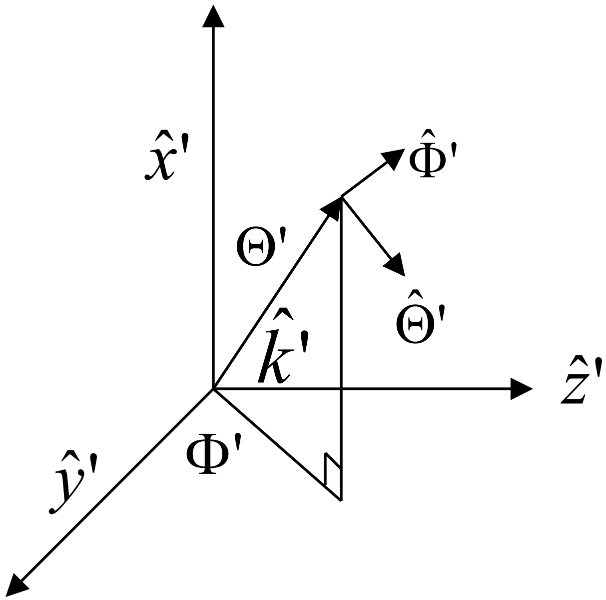
Summary

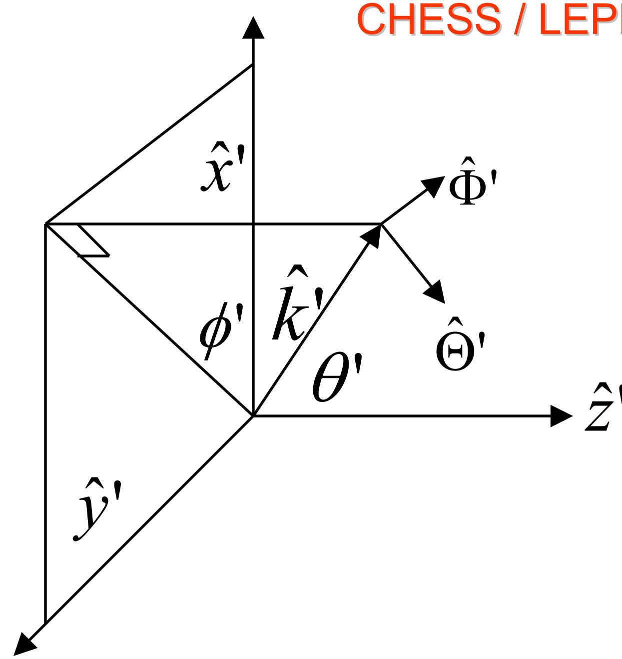
f_1 function goes to $1/(2f') = \pi/\omega'_0$ at the fundamental, and is much "wider" than f_N

$$\left|\tilde{d}(\omega')\right|^2 \rightarrow d_0^2 \frac{\pi^2}{\omega'_0} N \delta(\omega' - \omega'_0)$$

Total number of photons produced goes as *N*, in an energy distribution that narrows as 1/*N*

New Coordinates

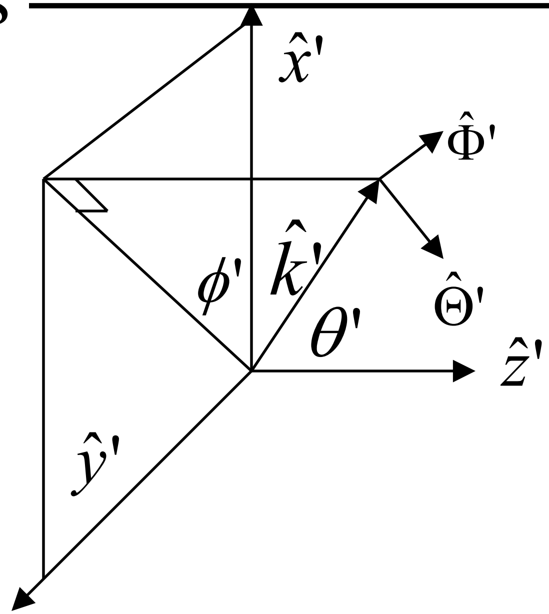




Resolve the polarization into that perpendicular (*perp)* and that parallel (*par*) to the *k-z* (scattering) plane

$$\vec{E}/\left|\vec{E}\right| = -\hat{\Theta}' = \frac{\hat{k}'\times\left(\hat{x}'\times\hat{k}'\right)}{\left|\hat{x}'\times\hat{k}'\right|}$$

New Coordinates



$$\hat{k}' = \sin\theta'\cos\phi'\,\hat{x}' + \sin\theta'\sin\phi'\,\hat{y}' + \cos\theta'\,\hat{z}'$$

$$\hat{e}_{perp} = \hat{k}'\times\hat{z}' / \left|\hat{k}'\times\hat{z}'\right| = \sin\phi'\,\hat{x}' - \cos\phi'\,\hat{y}' = -\hat{\phi}'$$

$$\hat{e}_{par} = \hat{k}'\times\hat{e}_{perp} = \cos\theta'\cos\phi'\,\hat{x}' + \cos\theta'\sin\phi'\,\hat{y}' - \sin\theta'\,\hat{z}' = \hat{\theta}'$$

Note $$-\hat{\Theta}' = \frac{\hat{k}'\times\left(\hat{x}'\times\hat{k}'\right)}{\left|\hat{x}'\times\hat{k}'\right|} = \frac{\hat{x}' - \left(\hat{k}'\cdot\hat{x}'\right)\hat{k}'}{\sin\Theta'}$$

Polarization

It follows that

$$\sin\Theta'\left(-\hat{\Theta}'\cdot\hat{e}_{perp}\right)=\sin\phi'$$
$$\sin\Theta'\left(-\hat{\Theta}'\cdot\hat{e}_{par}\right)=\cos\theta'\cos\phi'$$

So the energy into the two polarizations is

$$\frac{dE'_{perp}}{d\omega'\,d\Omega'}=\frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega')\right|^2\omega'^4}{c^3}\sin^2\phi'$$
$$\frac{dE'_{par}}{d\omega'\,d\Omega'}=\frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega')\right|^2\omega'^4}{c^3}\cos^2\theta'\cos^2\phi' \qquad (1.5)$$

Comments

- There is no radiation parallel or anti-parallel to the *x*-axis
- In the forward direction $\theta' \to 0$, the radiation polarization is parallel to the *x*-axis
- One may integrate over all angles to obtain the total energy radiated

$$\frac{dE'_{perp}}{d\omega'} = \frac{1}{8\pi^2} \frac{e^2 \left|\tilde{d}(\omega')\right|^2 \omega'^4}{c^3} 2\pi$$

$$\frac{dE'_{par}}{d\omega'} = \frac{1}{8\pi^2} \frac{e^2 \left|\tilde{d}(\omega')\right|^2 \omega'^4}{c^3} \frac{2\pi}{3}$$

$$\frac{dE'_{tot}}{d\omega'} = \frac{1}{8\pi^2} \frac{e^2 \left|\tilde{d}(\omega')\right|^2 \omega'^4}{c^3} \frac{8\pi}{3}$$

Generalized Larmor

Relativistic Invariances

To determine the radiation pattern for a "moving" oscillating charge we use this solution plus transformation formulas from relativity theory. For future reference, we note photon number invariance: The total number of photons emitted must be independent of the frame where the calculation is done. In particular,

$$N_{tot} = \frac{1}{3\pi} \int_{-\infty}^{\infty} \frac{e^2 \left| \tilde{d}(\omega') \right|^2 \omega'^4}{\hbar |\omega'| c^3} d\omega' \qquad (1.6)$$

must be frame independent. Rewriting formulas in terms of relativistically invariant quantities tends to simplify formulas.

Wave Vector Transformation Law

Follows from relativistic invariance of wave phase, which implies $k^\mu = \left(\omega / c, k_x, k_y, k_z\right)$ is a four vector

$$\omega' / c = \gamma\omega / c - \beta\gamma k \cos\theta$$

$$k' \sin\theta' \cos\phi' = k \sin\theta \cos\phi$$

$$k' \sin\theta' \sin\phi' = k \sin\theta \sin\phi$$

$$k' \cos\theta' = -\beta\gamma\omega / c + \gamma k \cos\theta$$

and $k = \omega / c$ and $k' = \omega' / c$ are the magnitudes of the wave propagation vectors

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \qquad \phi = \phi'$$

Invert by reversing the sign of β

Solid Angle Transformation

$$d\cos\theta' \wedge d\phi' = d\left(\frac{\cos\theta - \beta}{1 - \beta\cos\theta}\right) \wedge d\phi$$

$$= \left(\frac{1 - \beta\cos\theta + \beta\cos\theta - \beta^2}{(1 - \beta\cos\theta)^2}\right) d\cos\theta \wedge d\phi$$

$$= \left(\frac{1}{\gamma^2(1 - \beta\cos\theta)^2}\right) d\cos\theta \wedge d\phi$$

$$d\Omega' = \left(\frac{1}{\gamma^2(1 - \beta\cos\theta)^2}\right) d\Omega$$

Photon Distribution in Beam Frame

$$\frac{dN_{perp}}{d\omega' \, d\Omega'} = \frac{1}{8\pi^2} \frac{e^2 \left| \tilde{d}(\omega') \right|^2 \omega'^4}{\hbar |\omega'| c^3} \sin^2 \phi'$$

$$\frac{dN_{par}}{d\omega' \, d\Omega'} = \frac{1}{8\pi^2} \frac{e^2 \left| \tilde{d}(\omega') \right|^2 \omega'^4}{\hbar |\omega'| c^3} \cos^2 \theta' \cos^2 \phi'$$

Photon Distribution in Lab Frame

$$\frac{dN_{perp}}{d\omega d\Omega} = \frac{d\omega'}{d\omega}\frac{d\Omega'}{d\Omega}\frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega')\right|^2\omega'^4}{\hbar|\omega'|c^3}\sin^2\phi'$$

$$\frac{dN_{par}}{d\omega d\Omega} = \frac{d\omega'}{d\omega}\frac{d\Omega'}{d\Omega}\frac{1}{8\pi^2}\frac{e^2\left|\tilde{d}(\omega')\right|^2\omega'^4}{\hbar|\omega'|c^3}\cos^2\theta'\cos^2\phi'$$

Where the expression for the Doppler shifted frequency and angles are placed in these expressions

Photon Distribution in Lab Frame

$$\frac{dN_{perp}}{d\omega d\Omega} = \frac{1}{8\pi^2} \frac{e^2 \left|\tilde{d}(\omega')\right|^2 \omega'^4}{\hbar|\omega'|c^3} \frac{1}{\gamma(1-\beta\cos\theta)} \sin^2\phi$$

$$\frac{dN_{par}}{d\omega d\Omega} = \frac{1}{8\pi^2} \frac{e^2 \left|\tilde{d}(\omega')\right|^2 \omega'^4}{\hbar|\omega'|c^3} \frac{1}{\gamma(1-\beta\cos\theta)} \left(\frac{\cos\theta - \beta}{1-\beta\cos\theta}\right)^2 \cos^2\phi \tag{1.7}$$

$$\omega' = \gamma(1-\beta\cos\theta)\omega$$

$$(1-\beta\cos\theta)(1+\beta) \approx \frac{1}{\gamma^2} + \theta^2 + \ldots \approx \frac{1+\gamma^2\theta^2}{\gamma^2}$$

Photon Distribution in Lab Frame

$$\frac{dN_{perp}}{d\omega d\Omega} = \frac{\alpha}{8\pi^2}\frac{\left|\tilde{d}(\omega')\right|^2\omega'^4}{|\omega'|c^2}\frac{1}{\gamma\left(1-\beta_z\cos\theta\right)}\sin^2\phi$$

$$\frac{dN_{par}}{d\omega d\Omega} = \frac{\alpha}{8\pi^2}\frac{\left|\tilde{d}(\omega')\right|^2\omega'^4}{|\omega'|c^2}\frac{1}{\gamma\left(1-\beta_z\cos\theta\right)}\left(\frac{\cos\theta-\beta_z}{1-\beta_z\cos\theta}\right)^2\cos^2\phi \qquad (2.1)$$

$\omega' = \gamma\left(1-\beta_z\cos\theta\right)\omega \approx \omega/2\gamma$ for $\theta \to 0$ Doppler

$$\left(1-\beta_z\cos\theta\right)\left(1+\beta_z\right) \approx \frac{1}{\gamma^2}+\theta^2+\ldots \approx \frac{1+\gamma^2\theta^2}{\gamma^2}$$

Undulator Radiation: Single Electron -

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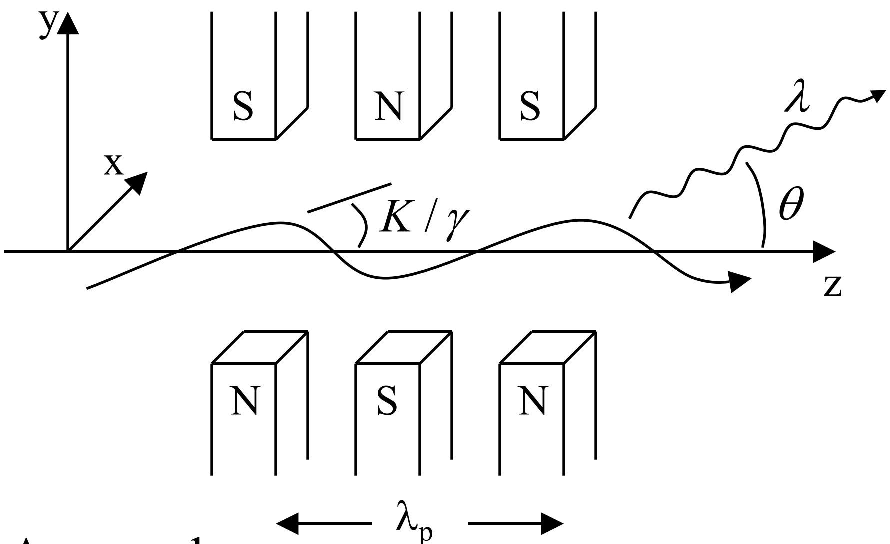


$$B_y = B_0 \sin k_p z$$

$$K = 9.34 B_0[\mathrm{T}]\lambda_p[\mathrm{m}]$$

Halbach permanent magnet undulator:

$$B_0[\mathrm{T}] \approx 3.33 \exp[-\kappa(5.47 - 1.8\kappa)]$$

for $SmCo_5$, here $\kappa = gap/\lambda_p$

Approaches:

1. Solve equation of motion* (trivial), grab Jackson and calculate retarded potentials (not so trivial – usually done in the far field approximation). Fourier Transform the field seen by the observer to get the spectrum.

More intuitively in the electron rest frame:

2. Doppler shift to the lab frame (nearly) simple harmonic oscillator radiation.

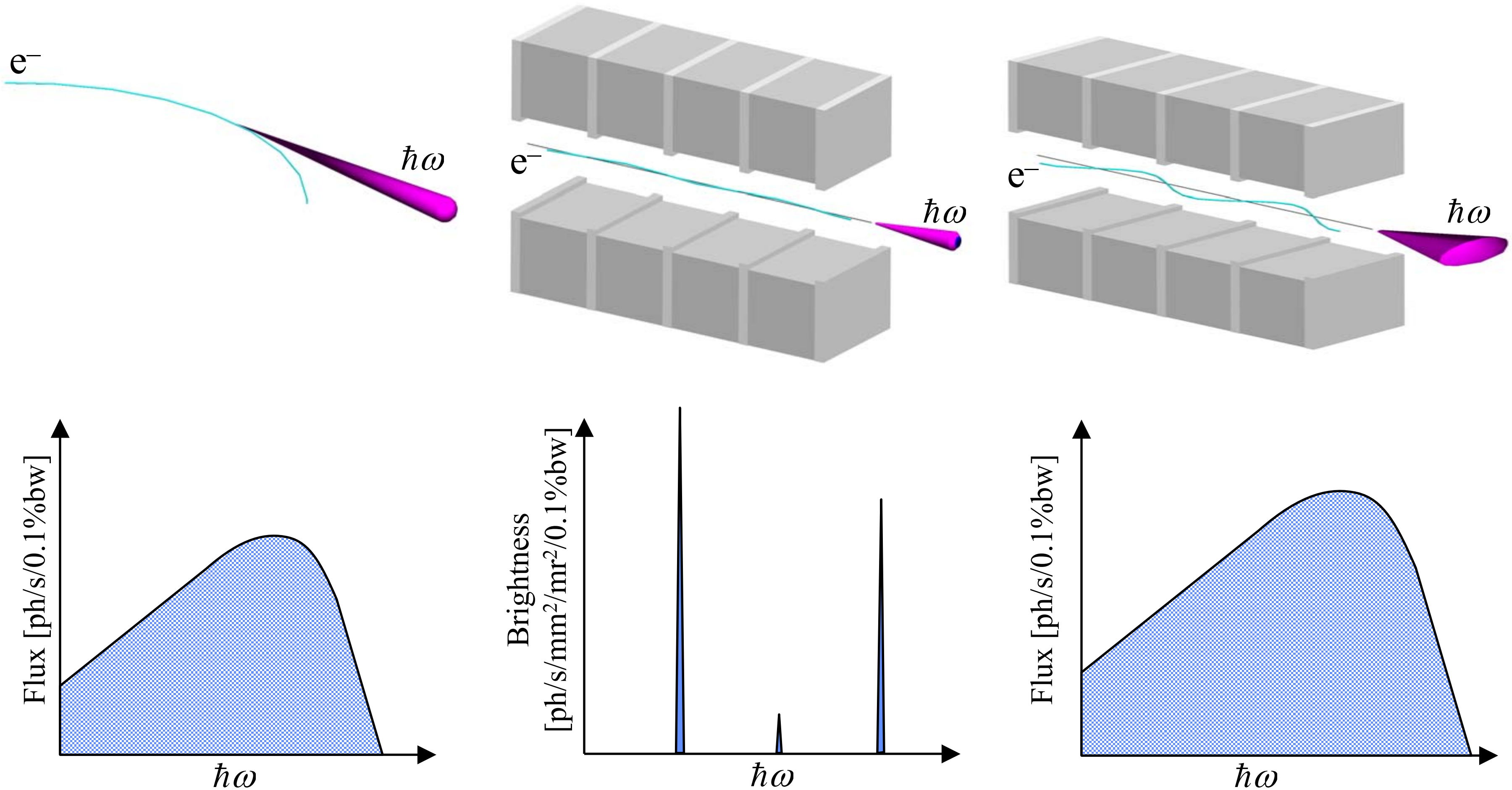
3. Doppler shift Compton back-scattered undulator field “photons”. *

Or simply

4. Write interference condition of wavefront emitted by the electron.*

* means home problem

Bend — Undulator — Wiggler



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Undulator Orbits

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- Will proceed by first computing emission for small field strength, then discuss the generalizations for field strengths typical in real undulators
- Need to calculate $d(t')$
- Can do whole calculation in the beam frame (easy for Thomson scatter calculations) or calculate the orbit in the lab frame, and Lorentz transform to the beam frame. Most references on undulators calculate the undulator orbit in the lab frame, as shall we.
- NB: Most references also do the electrodynamics in the lab frame too, using general (more complicated!) formulas for the emission.

Equations of Motion (Lab Frame)

$$\frac{d}{dt}\gamma = 0 \tag{2.2}$$

$$\frac{d}{dt}\gamma m\vec{\beta}c = -e\vec{\beta}\times\vec{B} \tag{2.3}$$

$$\vec{B} = B_0\,\frac{B(z)}{B_0}\,\hat{y}$$

$$\vec{\beta} = \beta_z\hat{z} + \beta_x\hat{x} \qquad \beta_x << \beta_z \approx 1$$

e is the fundamental charge, -*e* the electron charge

Approximate Solution

$$\frac{d\beta_x}{dt} = \frac{e\beta_z}{\gamma mc} B(z(t))$$

$$\beta_x(z) = \frac{e}{\gamma mc^2} \int_{-\infty}^{z} B(z')dz' \tag{2.4}$$

$$x(z) = \frac{e}{\beta_z \gamma mc^2} \int_{-\infty}^{z} \int_{-\infty}^{z'} B(z'')dz''\, dz'$$

Fourier Transformed

$$\tilde{x}(k) = \int x(z) e^{-ikz} dz \qquad x(z) = \frac{1}{2\pi} \int \tilde{x}(k) e^{ikz} dk$$

$$\tilde{x}(k) = -\frac{e\tilde{B}(k)}{\beta_z \gamma m c^2 k^2} \tag{2.5}$$

$$x(z(t)) = x(\beta_z ct) = -\frac{e}{2\pi\beta_z \gamma m c^2} \int_{-\infty}^{\infty} \frac{\tilde{B}(k)}{k^2} e^{ik\beta_z ct} dk$$

Eqn. 2.5 is strictly valid only if the electron is undeflected and unmoved by the undulator. In practical undulators, these conditions are approximately achieved by choosing an anti-symmetrical magnetic field, and by proper design of the two end cells of the undulator.

Lorentz Transformed

$$x(z(t)) = x(\beta_z ct) = -\frac{e}{2\pi\beta_z \gamma mc^2}\int_{-\infty}^{\infty}\frac{\tilde{B}(k)}{k^2}e^{ik\beta_z ct}dk$$

$$ct = \gamma ct' + \beta_z \gamma z'$$

$$x = x'$$

$$y = y'$$

$$z = \beta_z \gamma ct' + \gamma z'$$

$$x'(t') = -\frac{e}{2\pi\beta_z \gamma mc^2}\int_{-\infty}^{\infty}\frac{\tilde{B}(k)}{k^2}e^{ik\gamma\beta_z ct'}dk \qquad (z' = 0) \qquad (2.6)$$

Undulator period Lorentz contracted

Beam Frame Displacement Spectrum - CORNELL UNIVERSITY CHESS / LEPP

$$\tilde{d}(\omega') = \int d(t')e^{-i\omega' t'}dt'$$

$$\tilde{d}(\omega') = \int_{-\infty}^{\infty} x'(t')e^{-i\omega' t'}dt' = -\frac{e}{2\pi\beta_z\gamma mc^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{B}(k)}{k^2}e^{ik\gamma\beta_z ct'}dk e^{-i\omega' t'}dt'$$

$$\tilde{d}(\omega') = -\frac{e}{\beta_z\gamma mc^2}\int_{-\infty}^{\infty}\frac{\tilde{B}(k)}{k^2}\frac{\delta(k-\omega'/c\beta_z\gamma)}{c\beta_z\gamma}dk$$

$$\tilde{d}(\omega') = -\frac{ec\beta_z\gamma}{\beta_z\gamma mc^2}\frac{\tilde{B}(\omega'/c\beta_z\gamma)}{\omega'^2} \qquad (2.7)$$

Weak Field Undulator Spectrum

Combining previous results and, e. g., $\dfrac{dE_{perp}}{d\omega d\Omega} = \dfrac{dN_{perp}}{d\omega d\Omega}\hbar|\omega|$

$$\frac{dE_{perp}}{d\omega d\Omega} = \frac{1}{8\pi^2}\frac{e^4}{m^2c^5}\frac{\left|\widetilde{B}\left(\omega\left(1-\beta_z\cos\theta\right)/c\beta_z\right)\right|^2}{\gamma^2\left(1-\beta_z\cos\theta\right)^2}\sin^2\phi$$

$$\frac{dE_{par}}{d\omega d\Omega} = \frac{1}{8\pi^2}\frac{e^4}{m^2c^5}\frac{\left|\widetilde{B}\left(\omega\left(1-\beta_z\cos\theta\right)/c\beta_z\right)\right|^2}{\gamma^2\left(1-\beta_z\cos\theta\right)^2}\left(\frac{\cos\theta-\beta_z}{1-\beta_z\cos\theta}\right)^2\cos^2\phi$$

$$r_e^2 \equiv \frac{e^4}{m^2c^4} \qquad \lambda = \frac{\lambda_0}{2\gamma^2}$$

Frequency Integrated Angular Dist.

Recall Parseval's Theorem

$$\int_{-\infty}^{\infty} \left|B(z)\right|^2 dz = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widetilde{B}(k)\right|^2 dk = \frac{B_0^2 L}{2}$$

$$\frac{dE_{perp}}{d\Omega} = \frac{\beta_z r_e^2}{4\pi}\frac{\int_{-\infty}^{\infty}\left|B(z)\right|^2 dz}{\gamma^2\left(1-\beta_z\cos\theta\right)^3}\sin^2\phi$$

$$\frac{dE_{par}}{d\Omega} = \frac{\beta_z r_e^2}{4\pi}\frac{\int_{-\infty}^{\infty}\left|B(z)\right|^2 dz}{\gamma^2\left(1-\beta_z\cos\theta\right)^3}\left(\frac{\cos\theta-\beta_z}{1-\beta_z\cos\theta}\right)^2\cos^2\phi \qquad (2.8)$$

Frequency Integrated Number Dist.

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$$\frac{dN_{perp}}{d\Omega} = \frac{1}{8\pi^2}\frac{e^4}{m^2c^4\hbar c}\frac{\int_{-\infty}^{\infty}\left|\tilde{B}\left(\omega\left(1-\beta_z\cos\theta\right)/c\beta_z\right)\right|^2\frac{d\omega}{|\omega|}}{\gamma^2\left(1-\beta_z\cos\theta\right)^2}\sin^2\phi$$

$$\frac{dN_{par}}{d\Omega} = \frac{1}{8\pi^2}\frac{e^4}{m^2c^4\hbar c}\frac{\int_{-\infty}^{\infty}\left|\tilde{B}\left(\omega\left(1-\beta_z\cos\theta\right)/c\beta_z\right)\right|^2\frac{d\omega}{|\omega|}}{\gamma^2\left(1-\beta_z\cos\theta\right)^2}\left(\frac{\cos\theta-\beta_z}{1-\beta_z\cos\theta}\right)^2\cos^2\phi$$

Frequency Integrated Number Dist.

$$\frac{dN_{perp}}{d\Omega} \approx \frac{\alpha}{8\pi^2}\frac{e^2}{m^2c^4}\lambda_0 \frac{\int_{-\infty}^{\infty}\left|B(z)\right|^2 dz}{\gamma^2\left(1-\beta_z\cos\theta\right)^2}\sin^2\phi$$

$$\frac{dN_{par}}{d\Omega} \approx \frac{\alpha}{8\pi^2}\frac{e^2}{m^2c^4}\lambda_0 \frac{\int_{-\infty}^{\infty}\left|B(z)\right|^2 dz}{\gamma^2\left(1-\beta_z\cos\theta\right)^2}\left(\frac{\cos\theta-\beta_z}{1-\beta_z\cos\theta}\right)^2\cos^2\phi$$

Undulator Parameter

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$$K = \frac{eB_0\lambda_0}{2\pi mc^2} \tag{2.9}$$

$$\frac{dN_{perp}}{d\Omega} = \frac{\alpha}{4}\frac{NK^2}{\gamma^2(1-\beta_z\cos\theta)^2}\sin^2\phi$$

$$\frac{dN_{par}}{d\Omega} = \frac{\alpha}{4}\frac{NK^2}{\gamma^2(1-\beta_z\cos\theta)^2}\left(\frac{\cos\theta-\beta_z}{1-\beta_z\cos\theta}\right)^2\cos^2\phi \tag{2.10}$$

Strong Field Case

$$\frac{d}{dt}\gamma = 0$$

$$\frac{d}{dt}\gamma m\vec{\beta}c = -e\vec{\beta}\times\vec{B}$$

$$\beta_x(z) = \frac{e}{\gamma mc^2}\int_{-\infty}^{z} B(z')dz'$$

High K

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2(z)}$$

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \left(\frac{e}{\gamma mc^2}\int_{-\infty}^{z} B(z')dz'\right)^2}$$

$$\beta_z(z) \approx 1 - \frac{1}{2\gamma^2} - \frac{1}{2}\left(\frac{e}{\gamma mc^2}\int_{-\infty}^{z} B(z')dz'\right)^2 = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) - \frac{K^2}{4\gamma^2}\cos(2k_0 z)$$

High K

Inside the insertion device the average (z) velocity is

$$\beta^*_z = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) \tag{2.11}$$

with corresponding

$$\gamma^* = \frac{1}{\sqrt{1-\beta^{*2}_z}} = \frac{\gamma}{\sqrt{1+K^2/2}} \tag{2.12}$$

To apply dipole distributions, must transform into this frame

Orbit in the Beam Frame

Assume orbit turning point event at z_0, t_0
Next one at $z_0 + \lambda_0$, $t_0 + \lambda_0/\beta^*_z c$

By Lorentz Transformation formula

$$\Delta z' = 0, \qquad c\Delta t' = \lambda_0 / \gamma^* \beta^*_z$$

$$\omega'_0 = 2\pi\gamma^* \beta^*_z \, c / \lambda_0 = \gamma^* \beta^*_z \, \omega_0$$

$$x' = x = \frac{K}{\gamma} \frac{c}{\beta^*_z \, \omega_0} \left[\sin\left(\beta^*_z \, \omega_0 t \right) - 1 \right]$$

$$z' = \gamma^* \left(-\beta^*_z \, ct + z \right) = \gamma^* \left(-\beta^*_z \, ct + \beta^*_z \, ct - \frac{K^2 c}{8\gamma^2 \beta^2 \omega_0} \sin\left(2\beta^*_z \, \omega_0 t \right) \right)$$

Figure Eight

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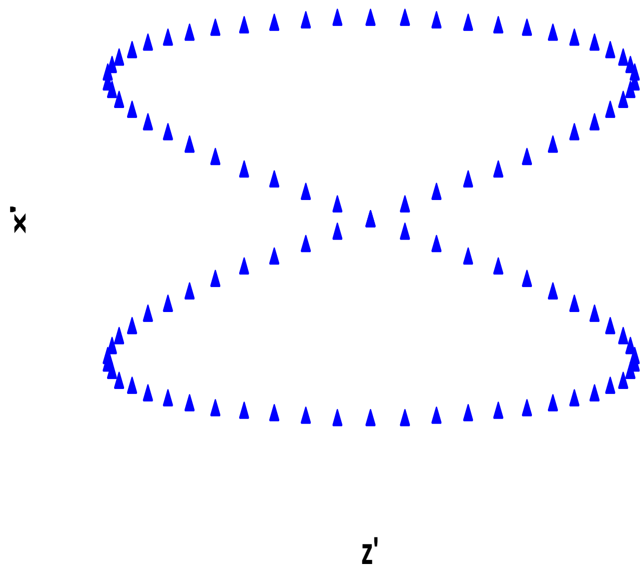
$$\frac{K}{\gamma}\,\frac{\lambda_0}{2\pi\beta^*_z}$$



$$\frac{\gamma^* K^2 \lambda_0}{8\gamma^2\, 2\pi\beta^*_z}$$

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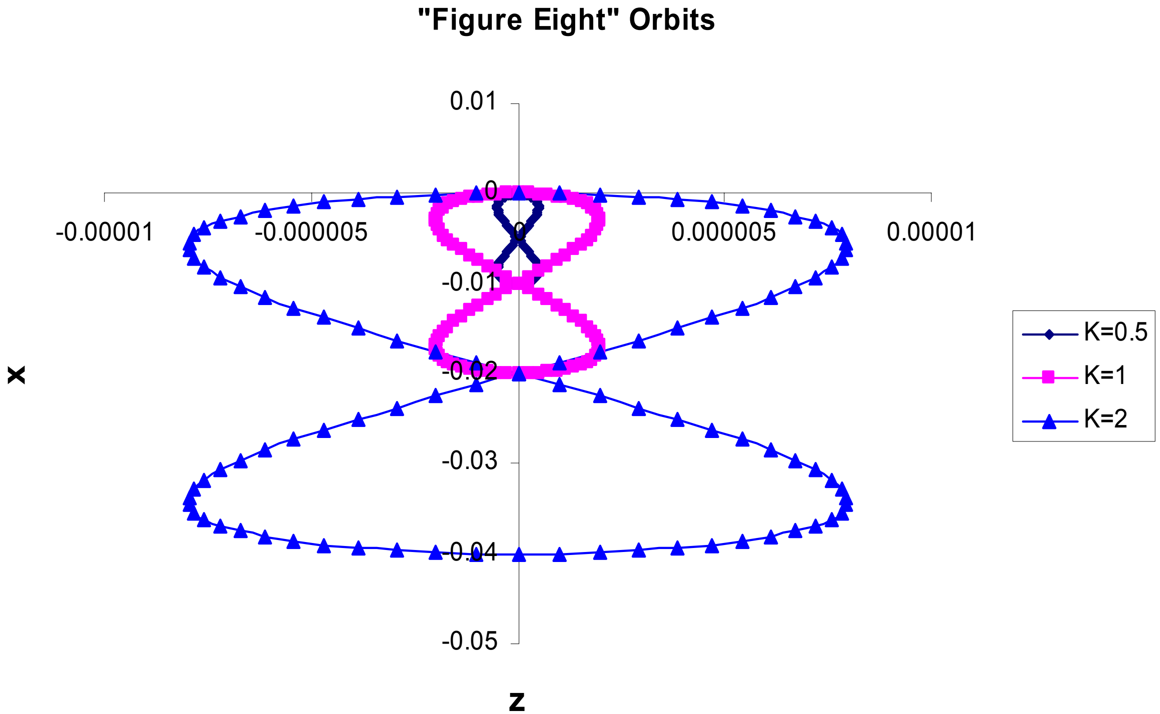

$\gamma = 100$, distances are normalized by $\lambda_0 / 2\pi$

More on Beam Frame Electron Orbit –

Even though the *x-z* orbit is easily specified in terms of trigonometric functions, the TIME dependence of the beam orbit in the beam frame is very complicated. However, by the following trick, we don't need to know time dependence explicitly.

$$d\tau = \sqrt{1-\beta^2}\,dt$$

The magnitude of the velocity doesn't vary, so the (invariant!) proper time is directly proportional to the "lab time"

$$\tau = \frac{t}{\gamma}$$

The orbit in the beam frame is expressed simply in terms of the proper time

$$x'(\tau)=\frac{K}{\gamma}\frac{c}{\beta^*_z\,\omega_0}\left[\sin\left(\beta^*_z\,\omega_0\gamma\tau\right)-1\right]$$

$$z'(\tau)=-\frac{\gamma^* K^2 c}{8\gamma^2\beta^2\omega_0}\sin\left(2\beta^*_z\,\omega_0\gamma\tau\right)$$

and most importantly of all

$$ct'=\gamma^*\left(ct-\beta^*_z\,z\right)=\frac{\gamma c\tau}{\gamma^*}+\frac{\gamma^*\beta^*_z K^2 c}{8\gamma^2\beta^2\omega_0}\sin\left(2\beta^*_z\,\omega_0\gamma\tau\right)$$

High K "Monochromatic" Solution

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Single electron moves "sinusoidally" with angular frequency Ω'_0 in x and $2\Omega'_0$ in z where

$$\Omega'_0 = \gamma\beta^*_z\,\omega_0 = \frac{\gamma}{\gamma^*}\omega'_0$$

$$\rho'(x',y',z',t') = -e\delta\left(x'-d_x\sin\left(\Omega'_0\,\tau(t')\right)\right)\delta(y')\delta\left(z'+d_z\sin\left(2\Omega'_0\,\tau(t')\right)\right)$$

$$\vec{J}'(x',y',z',t') =$$
$$-ed_x\Omega'_0\cos\left(\Omega'_0\,\tau(t')\right)\left(d\tau/dt'\right)\hat{x}\delta\left(x'+d_x\sin\left(\Omega'_0\,\tau(t')\right)\right)\delta(y')\delta\left(z'+d_z\sin\left(2\Omega'_0\,\tau(t')\right)\right)$$
$$+ed_z 2\Omega'_0\cos\left(\Omega'_0\,\tau(t')\right)\left(d\tau/dt'\right)\hat{z}\delta\left(x'+d_x\sin\left(\Omega'_0\,\tau(t')\right)\right)\delta(y')\delta\left(z'+d_z\sin\left(2\Omega'_0\,\tau(t')\right)\right)$$

EM Potentials (Beam Frame)

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$$\Phi'(\vec{r}',t')=\int\frac{1}{R'}\rho'\left(\vec{r}'',t'-\frac{R'}{c}\right)dx''dy''dz''=-\frac{e}{2\pi}\int\frac{e^{i\omega'(t''-t'+R'/c)}}{R'}dt''d\omega'$$

$$=-\frac{e}{2\pi}\int\frac{e^{i\omega'\left(\frac{\gamma}{\gamma^*}\tau(t'')+\beta^*_z(d_z/c)\sin(2\Omega'_0\tau(t''))-t'+R'/c\right)}}{R'}dt''d\omega'$$

$$A'_x(\vec{r}',t')=\int\frac{1}{R'c}J'_x\left(\vec{r}'',t'-\frac{R'}{c}\right)dx''dy''dz''=-\frac{e}{2\pi}\int\frac{v'_x(t'')e^{i\omega'(t''-t'+R'/c)}}{R'c}dt''d\omega'$$

$$=-\frac{ed_x\Omega'_0}{2\pi c}\int\frac{\cos(\Omega'_0\tau)e^{i\omega'\left(\frac{\gamma}{\gamma^*}\tau+\beta^*_z(d_z/c)\sin(2\Omega'_0\tau)-t'+R'/c\right)}}{R'}d\tau d\omega'$$

$$A'_z(\vec{r}',t')=\int\frac{1}{R'c}J'_z\left(\vec{r}'',t'-\frac{R'}{c}\right)dx''dy''dz''=\frac{e}{2\pi}\int\frac{v'_z(t'')e^{i\omega'(t''-t'+R'/c)}}{R'c}dt''d\omega'$$

$$=\frac{ed_z2\Omega'_0}{2\pi c}\int\frac{\cos(2\Omega'_0\tau)e^{i\omega'\left(\frac{\gamma}{\gamma^*}\tau+\beta^*_z(d_z/c)\sin(2\Omega'_0\tau)-t'+R'/c\right)}}{R'}d\tau d\omega'$$

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A'_z term gives no contribution at all in the forward direction! Why?

Dipole doesn't radiate in z direction if motion in z!

Contribution off axis small also, because $d_z \sim o(Kd_x/\gamma)$

As before, space differentiate the potentials (cannot time integrate by parts this time because R' depends on time!) to obtain, e. g.,

$$\vec{B}' \approx \frac{ed_x \Omega'_0}{2\pi c^2 r'} \sin\Theta' \hat{\Phi}' \int \cos(\Omega'_0 \tau) i\omega' e^{i\omega'\left(\frac{\gamma}{\gamma^*}\tau + \beta^*_z (d_z/c)\sin(2\Omega'_0 \tau) - t' + R'/c\right)} d\tau d\omega'$$

$$-\frac{ed_z 2\Omega'_0}{2\pi c^2 r'} \sin\theta' \hat{\phi}' \int \cos(2\Omega'_0 \tau) i\omega' e^{i\omega'\left(\frac{\gamma}{\gamma^*}\tau + \beta^*_z (d_z/c)\sin(2\Omega'_0 \tau) - t' + R'/c\right)} d\tau d\omega'$$

Now include the fact that emission phase depends on retarded time

$$R' = \sqrt{\left(x' - d_x \sin\left(\Omega'_0 \tau\right)\right)^2 + y'^2 + \left(z' + d_z \sin\left(2\Omega'_0 \tau\right)\right)^2}$$

$$\approx r'\left(1 - \frac{x' d_x}{r'^2}\sin\left(\Omega'_0 \tau\right) + \frac{z' d_z}{r'^2}\sin\left(2\Omega'_0 \tau\right)\right)$$

Use

$$e^{\pm iz\sin\theta} = \sum_{k=-\infty}^{\infty} e^{\pm ik\theta} J_k\left(z\right)$$

To show for the x-dipole term

$$\delta(\omega'\pm\omega'_0)\rightarrow\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\delta(\omega'\pm\omega'_0-k\omega'_0+k'2\omega'_0)J_k(\sin\theta'\cos\phi' d_x\omega'/c)$$
$$\cdot J_{k'}((\beta^*_z+\cos\theta')d_z\omega'/c)$$

and for the z-dipole term

$$\delta(\omega'\pm\omega'_0)\rightarrow\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\delta(\omega'\pm 2\omega'_0-k\omega'_0+k'2\omega'_0)J_k(\sin\theta'\cos\phi' d_x\omega'/c)$$
$$\cdot J_{k'}((\beta^*_z+\cos\theta')d_z\omega'/c)$$

Resum using

$$J_{k-1}(z)+J_{k+1}(z)=\frac{2k}{z}J_k(z)$$

x-dipole

$$\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\delta\left(\omega'-k\omega'_0+k'2\omega'_0\right)\frac{kc}{\sin\theta'\cos\phi' d_x\omega'}J_k\left(\sin\theta'\cos\phi' d_x\omega'/c\right)$$
$$\cdot J_{k'}\left(\left(\beta^*_z+\cos\theta'\right)d_z\omega'/c\right)$$

z-dipole

$$\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\delta\left(\omega'-k\omega'_0+k'2\omega'_0\right)\frac{k'c}{\left(\beta^*_z+\cos\theta'\right)d_z\omega'}J_k\left(\sin\theta'\cos\phi' d_x\omega'/c\right)$$
$$\cdot J_{k'}\left(\left(\beta^*_z+\cos\theta'\right)d_z\omega'/c\right)$$

Harmonic Number

Define harmonic number n

$$n = k - 2k', \qquad k = n + 2k'$$

$$\sum_{n=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\delta(\omega' - n\omega'_0)\frac{(n+2k')c}{n\sin\theta'\cos\phi' d_x\omega'_0} J_{n+2k'}(n\sin\theta'\cos\phi' d_x\omega'_0 / c) \cdot J_{k'}(n(\beta^*_z + \cos\theta')d_z\omega'_0 / c)$$

$$\sum_{n=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\delta(\omega' - n\omega'_0)\frac{2k'c}{n(\beta^*_z + \cos\theta')d_z\omega'_0} J_{n+2k'}(n\sin\theta'\cos\phi' d_x\omega'_0 / c) \cdot J_{k'}(n(\beta^*_z + \cos\theta')d_z\omega'_0 / c)$$

$$\vec{B}'_n = \frac{e\omega'_0}{r'c} ine^{-in\omega'_0(t'-r'/c)}\left[\Gamma_{xn}\sin\Theta'\hat{\Phi}' - \Gamma_{zn}\sin\theta'\hat{\phi}'\right] \qquad (2.13)$$

$$\vec{B}'_n + \vec{B}'_{-n} = \frac{e\omega'_0}{r'c} 2n\sin\left(n\omega'_0\left(t'-r'/c\right)\right)\left[\Gamma_{xn}\sin\Theta'\hat{\Phi}' - \Gamma_{zn}\sin\theta'\hat{\phi}'\right]$$

$$\Gamma_{xn} \equiv \sum_{k'=-\infty}^{\infty} \frac{(n+2k')}{n\sin\theta'\cos\phi'} J_{n+2k'}\left(n\sin\theta'\cos\phi' d_x\omega'_0/c\right) \cdot J_{k'}\left(n\left(\beta^*_z + \cos\theta'\right)d_z\omega'_0/c\right)$$

$$\Gamma_{zn} \equiv \sum_{k'=-\infty}^{\infty} \frac{2k'}{n\left(\beta^*_z + \cos\theta'\right)} J_{n+2k'}\left(n\sin\theta'\cos\phi' d_x\omega'_0/c\right) \cdot J_{k'}\left(n\left(\beta^*_z + \cos\theta'\right)d_z\omega'_0/c\right)$$

$$\frac{dI'_n}{d\Omega'} = \frac{4e^2\omega'^2_0}{8\pi c} n^2 \left[\Gamma_{xn} \sin\Theta' \hat{\Phi}' - \Gamma_{zn} \sin\theta' \hat{\phi}'\right]^2$$

Resolved into the two polarization states

$$\frac{dI'_{perp,n}}{d\Omega'} = \frac{e^2\omega'^2_0}{2\pi c} n^2 \Gamma^2_{xn} \sin^2\phi'$$

$$(2.14)$$

$$\frac{dI'_{par,n}}{d\Omega'} = \frac{e^2\omega'^2_0}{2\pi c} n^2 \left[\Gamma_{xn} \cos\theta' \cos\phi' + \Gamma_{zn} \sin\theta'\right]^2$$

The *par* component may be further manipulated into separate Bessel function sums

$$\frac{dI'_{par,n}}{d\Omega'} = \frac{e^2 \omega'^2_0}{2\pi c} n^2 \left[\begin{array}{c} S_{1n} \dfrac{\cos\theta'}{\sin\theta'} + \\ \dfrac{S_{2n}}{n} \left(\dfrac{\cos\theta'}{\sin\theta'} + \dfrac{\sin\theta'}{(\beta^*_z + \cos\theta')} \right) \end{array} \right]^2 \qquad (2.15)$$

$$S_{1n} \equiv \sum_{k'=-\infty}^{\infty} J_{n+2k'}(n\sin\theta'\cos\phi' d_x \omega'_0 / c) J_{k'}(n(\beta^*_z + \cos\theta') d_z \omega'_0 / c)$$

$$S_{2n} \equiv \sum_{k'=-\infty}^{\infty} 2k' J_{n+2k'}(n\sin\theta'\cos\phi' d_x \omega'_0 / c) J_{k'}(n(\beta^*_z + \cos\theta') d_z \omega'_0 / c)$$

Comment

Unless one knows, in detail, the spectral distribution of the x-motion in some frame (and by implication the z-motion!), it is difficult to push this solution as before to get a general form for the energy distribution by superposition, including full details of the magnetic field spectrum, because for high K the rest-frame velocity depends on the field strength at that frequency. If the motions ARE known in some frame (as they will be for Thomson Scattering!), superposition starting with Eqns. 2.14 and 2.15 WILL give the general solution. Because for an undulator d_x is highly peaked at a single frequency Eqns. 2.14 and 2.15 still apply, and an argument the same as before plus a simple Fourier analysis of the magnetic field (Eqn. 2.13) yields:

High Field Spectral Distribution

In the beam frame

$$\frac{dE'_{perp,n}}{d\omega' d\Omega'} = \frac{e^2 \omega'^2_0}{2\pi^2 c} n^2 \Gamma^2_{xn} \sin^2\phi' \sigma'^2_n\left(\omega';\omega'_0\right)$$

$$\frac{dE'_{par,n}}{d\omega' d\Omega'} = \frac{e^2 \omega'^2_0}{2\pi^2 c} n^2 \left[\begin{array}{c} S_{1n}\dfrac{\cos\theta'}{\sin\theta'} + \\ \dfrac{S_{2n}}{n}\left(\dfrac{\cos\theta'}{\sin\theta'} + \dfrac{\sin\theta'}{\left(\beta^*_z + \cos\theta'\right)}\right) \end{array} \right]^2 \sigma'^2_n\left(\omega';\omega'_0\right) \qquad (2.16)$$

where

$$\sigma'_n\left(\omega';\omega'_0\right) = f_{nN}\left(\omega';n\omega'_0\right) f_1\left(\omega';n\omega'_0\right) \approx \frac{\sin\left(\pi n N \omega'/n\omega'_0\right)}{\sin\left(\pi\omega'/n\omega'_0\right)} \frac{\pi}{n\omega'_0}$$

In the lab frame

$$\frac{dE_{perp,n}}{d\omega d\Omega} = \frac{e^2}{2c}\frac{1}{\gamma^{*2}\left(1-\beta^*_z\cos\theta\right)^2}\frac{\gamma^{*2}\left(1-\beta^*_z\cos\theta\right)^2}{\sin^2\theta\cos^2\phi}$$

$$\cdot\begin{bmatrix} S_{1n} + \\ S_{2n}/n \end{bmatrix}^2 \sin^2\phi f_{nN}^2\left(\omega;n\omega(\theta)\right)$$

$$\frac{dE_{par,n}}{d\omega d\Omega} = \frac{e^2}{2c}\frac{1}{\gamma^{*2}\left(1-\beta^*_z\cos\theta\right)^2}\begin{bmatrix} S_{1n}\dfrac{\gamma^*\left(\cos\theta-\beta^*_z\right)}{\sin\theta}+ \\ \dfrac{S_{2n}}{n}\dfrac{\gamma^*\left(1-\beta^*_z\cos\theta\right)}{\sin\theta\cos\theta} \end{bmatrix}^2 f_{nN}^2\left(\omega;n\omega(\theta)\right)$$

$$f_{nN}\left(\omega;n\omega(\theta)\right) \approx \frac{\sin\left(\pi nN\omega\left(1-\beta^*_z\cos\theta\right)/\beta^*_z n\omega_0\right)}{\sin\left(\pi\omega\left(1-\beta^*_z\cos\theta\right)/\beta^*_z n\omega_0\right)}$$

$$\frac{dE_{perp,n}}{d\omega d\Omega} = \frac{e^2}{2c}\left[S_{1n} + S_{2n}/n\right]^2 \frac{\sin^2\phi}{\sin^2\theta\cos^2\phi} f_{nN}^2\left(\omega; n\omega(\theta)\right)$$

$$\frac{dE_{par,n}}{d\omega d\Omega} = \frac{e^2}{2c}\left[\begin{array}{c}\dfrac{S_{1n}\left(\cos\theta - \beta^*_z\right)}{\left(1-\beta^*_z\cos\theta\right)\sin\theta} \\ +\dfrac{S_{2n}}{n\sin\theta\cos\theta}\end{array}\right]^2 f_{nN}^2\left(\omega; n\omega(\theta)\right)$$

f_{nN} is highly peaked, with peak value *nN*, around angular frequency

$$n\omega(\theta) = \frac{\beta^*_z n\omega_0}{\left(1-\beta^*_z\cos\theta\right)} \rightarrow 2\gamma^{*2}\beta^*_z n\omega_0 \approx \frac{2\gamma^2}{1+K^2/2} n\omega_0 \ \text{ as } \ \theta \rightarrow 0$$

Energy Distribution in Lab Frame

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$$\frac{dE_{perp,n}}{d\omega d\Omega}=\frac{e^2}{2c}\left[S_{1n}+S_{2n}/n\right]^2\frac{\sin^2\phi}{\sin^2\theta\cos^2\phi}f_{nN}^2(\omega;n\omega(\theta))$$

$$\frac{dE_{par,n}}{d\omega d\Omega}=\frac{e^2}{2c}\left[\begin{array}{c}\dfrac{S_{1n}\left(\cos\theta-\beta^*_z\right)}{\left(1-\beta^*_z\cos\theta\right)\sin\theta}\\ +\dfrac{S_{2n}}{n\sin\theta\cos\theta}\end{array}\right]^2 f_{nN}^2(\omega;n\omega(\theta)) \qquad (2.17)$$

The arguments of the Bessel Functions are now

$$\xi_x\equiv n\sin\theta'\cos\phi' d_x\omega'_0/c=n\frac{\sin\theta\cos\phi}{\left(1-\beta^*_z\cos\theta\right)}\frac{K}{\gamma}$$

$$\xi_z\equiv n\left(\beta^*_z+\cos\theta'\right)d_z\omega'_0/c=n\frac{\cos\theta}{\left(1-\beta^*_z\cos\theta\right)}\frac{\beta^*_z K^2}{8\gamma^2\beta^2}$$

In the Forward Direction

In the forward direction even harmonics vanish (n+2k' term vanishes when "x" Bessel function non-zero at zero argument, and all other terms in sum vanish with a power higher than 2 as the argument goes to zero), and for odd harmonics only n+2k'=1,-1 contribute to the sum

$$\frac{dE_{perp,n}}{d\omega d\Omega} = \frac{e^2}{2c}\gamma^2\left(\frac{F_n(K)}{n^2}\right)\sin^2\phi\, f_{nN}^2(\omega; n\omega(\theta=0))$$

$$\frac{dE_{par,n}}{d\omega d\Omega} = \frac{e^2}{2c}\gamma^2\left(\frac{F_n(K)}{n^2}\right)\cos^2\phi\, f_{nN}^2(\omega; n\omega(\theta=0))$$

$$F_n(K) \approx \frac{1}{\gamma^2}\frac{n^2}{4(1-\beta^*_z)^2}\frac{K^2}{\gamma^2}\left[J_{\frac{n-1}{2}}\left(\frac{nK^2}{4(1+K^2/2)}\right) - J_{\frac{n+1}{2}}\left(\frac{nK^2}{4(1+K^2/2)}\right)\right]^2$$

Number Spectral Angular Density

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Converting the energy density into an number density by dividing by the photon energy (don't forget both signs of frequency!)

$$\frac{dN_{perp,n}}{(d\omega/\omega)d\Omega} = \alpha\gamma^2\left(\frac{F_n(K)}{n^2}\right)\sin^2\phi\, f_{nN}^2\left(\omega; n\omega(\theta=0)\right)$$

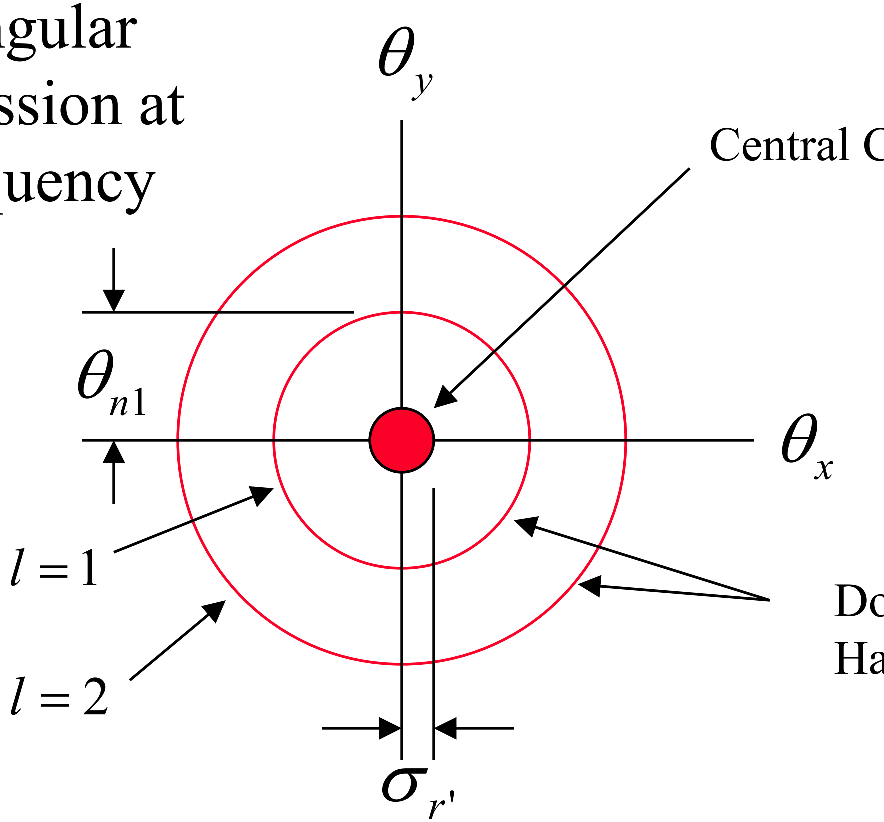
$$\frac{dN_{par,n}}{(d\omega/\omega)d\Omega} = \alpha\gamma^2\left(\frac{F_n(K)}{n^2}\right)\cos^2\phi\, f_{nN}^2\left(\omega; n\omega(\theta=0)\right)$$

Peak value in the forward direction

$$\frac{dN_{tot,n}}{(d\omega/\omega)d\Omega} = \alpha\gamma^2 N^2 F_n(K)$$

Radiation Pattern: Qualitatively

- Non-zero Angular Density Emission at a Given Frequency



Central cone: high angular density region around forward direction

Dimension Estimates

Harmonic bands at

$$\theta_{nl} = \frac{1}{\gamma}\sqrt{\frac{l}{n}\left(1 + K^2/2\right)}$$

Central cone size estimated by requiring Gaussian distribution with correct peak value integrate over solid angle to the same number of total photons as integrating *f*

$$\sigma_{r'} = \frac{1}{2\gamma}\sqrt{\frac{\left(1 + K^2/2\right)}{nN}} = \sqrt{\frac{\lambda_n}{2L}} \qquad \lambda_n = c/n\omega\left(\theta = 0\right)$$

Much narrower than typical opening angle for bend

Number Spectral Density (Flux)

The flux in the central cone is obtained by estimating solid angle integral by the peak angular density multiplied by the Gaussian integral

$$\mathrm{F}^{\mathrm{n}} = \left.\frac{dN_{tot,n}}{d\Omega}\right|_{\theta=0} 2\pi\sigma_{r'}^2$$

$$\mathrm{F}^{\mathrm{n}} = \pi\alpha N \frac{\Delta\omega}{\omega}\frac{I}{e} g_n(K)$$

$$g_n(K) = \left(1 + K^2/2\right) F_n(K)/n$$

Power Angular Density

$$\frac{dE_{perp,n}}{d\Omega} = \alpha N n \hbar \omega(\theta) \left[S_{1n} + S_{2n} / n \right]^2 \frac{\sin^2 \phi}{\sin^2 \theta \cos^2 \phi}$$

$$\frac{dE_{par,n}}{d\Omega} = \alpha N n \hbar \omega(\theta) \left[\frac{S_{1n}\left(\cos\theta - \beta^*_z\right)}{\left(1 - \beta^*_z \cos\theta\right)\sin\theta} + \frac{S_{2n}}{n \sin\theta \cos\theta} \right]^2$$

Don't forget both signs of frequency!

For K less than or of order one

$$\frac{dN_{n,perp}}{d\Omega} \approx \frac{\alpha}{4} \frac{NF_n(K)}{\gamma^2 \left(1-\beta^*_z \cos\theta\right)^2} \sin^2\phi$$

$$\frac{dN_{n,par}}{d\Omega} \approx \frac{\alpha}{4} \frac{NF_n(K)}{\gamma^2 \left(1-\beta^*_z \cos\theta\right)^2} \left(\frac{\cos\theta-\beta^*_z}{1-\beta^*_z \cos\theta}\right)^2 \cos^2\phi$$

$$F_n(K) = \frac{n^2K^2}{\left(1+K^2/2\right)^2} \left\{ J_{\frac{n-1}{2}}\left(\frac{nK^2}{4\left(1+K^2/2\right)}\right) - J_{\frac{n+1}{2}}\left(\frac{nK^2}{4\left(1+K^2/2\right)}\right)\right\}^2$$

Compare with (2.10)

Homework Problem

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Up to now, we have done the electrodynamics in the beam frame. In most of the literature, the calculation is done in the lab frame. Reproduce Eqns. 2.17, working entirely in the lab frame, starting from the standard formula from Jackson's *Classical Electrodynamics:*

$$\frac{d^2E}{d\omega d\Omega}=\frac{e^2\omega^2}{8\pi^2 c}\left|\int_{-\infty}^{\infty}\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)e^{i\omega\left(t-\vec{n}\cdot\vec{r}(t)/c\right)}dt\right|^2$$

Show that the final results have the proper symmetry with respect to the sign of the angular frequency ω. An early reference to calculations of this type, is Alferov, D. F., Bashmakov, Yu. A., and Bessonov, E. G., *Sov. Phys. Tech. Phys.*, **18,** 1336, (1974).

Conclusions

- Emission (in forward direction) is at ODD harmonics of the fundamental frequency, in addition to the fundamental frequency emission. The strength of the emission at the harmonics is quantified by a Bessel function factor.
- All kinematic parameters, including the angular distribution functions and frequency distributions, are just the same as before except unstarred quantities should be replaced by starred quantities
- In particular, the (FEL) resonance condition becomes

$$\lambda_n = \frac{n\lambda_0}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

Steps We Followed in the Lecture

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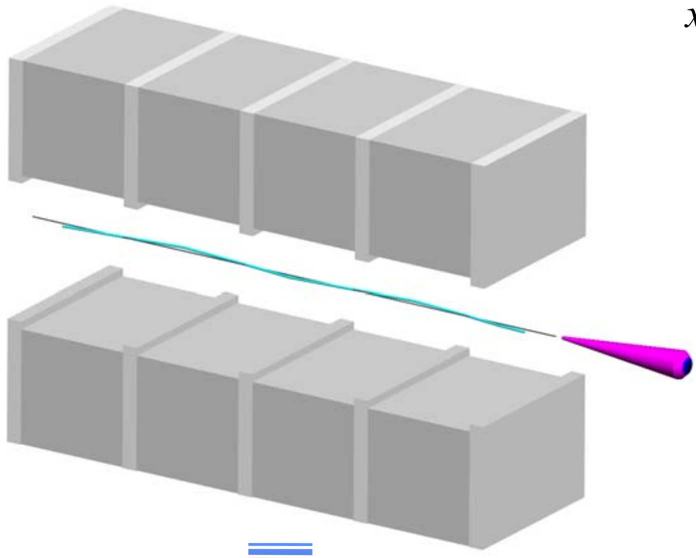
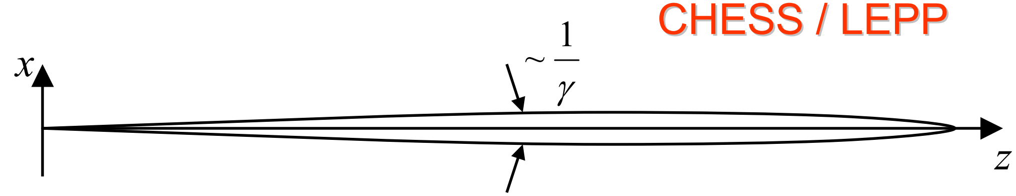
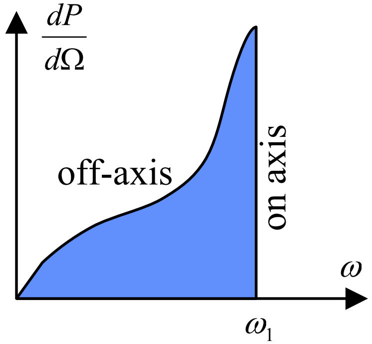
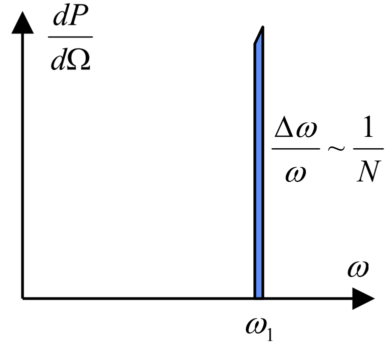




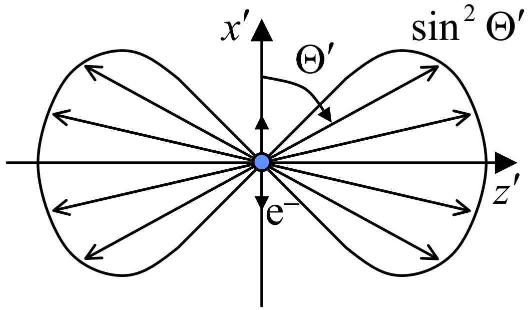


back to lab frame

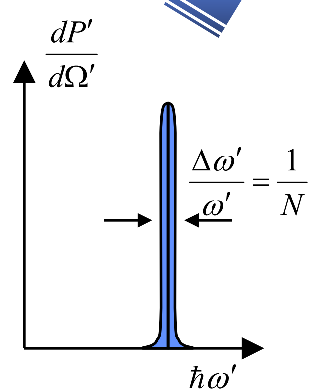


after pin-hole aperture

$\lambda_p' = \lambda_p / \bar{\gamma}$



in e⁻ frame



$$\boxed{\lambda_n = \frac{\lambda_p}{2\gamma^2 n}\left(1 + \tfrac{1}{2}K^2 + \gamma^2\theta^2\right)}$$

$$\boxed{\frac{\Delta\lambda}{\lambda_n} \sim \frac{1}{nN_p}}$$ (for fixed θ only!)

Higher Harmonics / Wiggler

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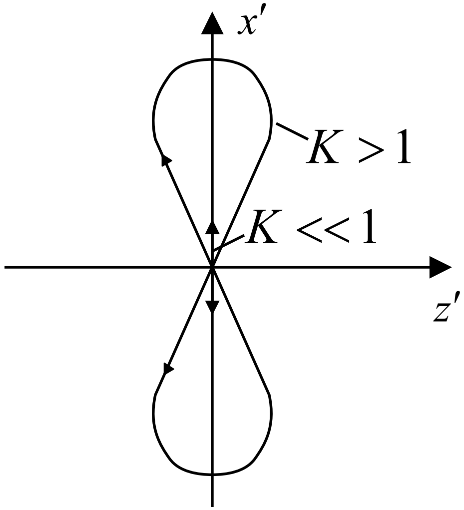
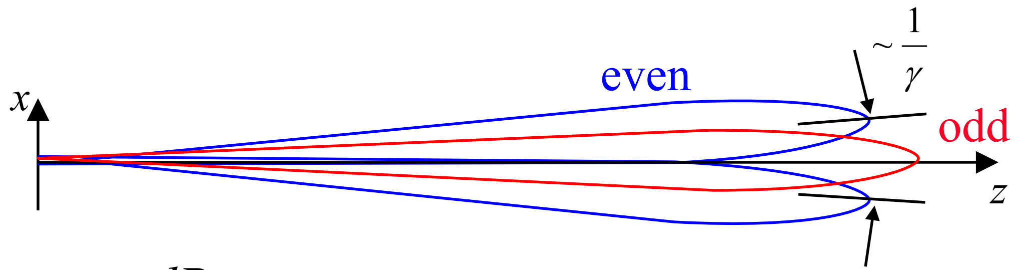


motion in e⁻ frame

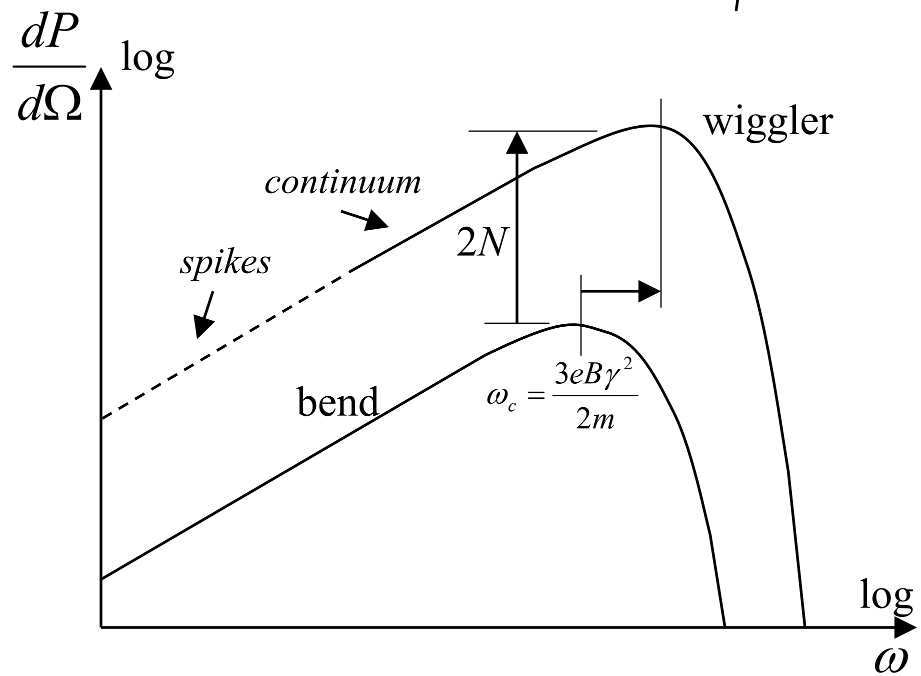


$K \le 1$ undulator
$K > 1$ wiggler

$$n_c = \frac{3K}{4}\left(1 + \frac{K^2}{2}\right)$$

K	n_c
1	1
2	4
4	27
8	198
16	1548

critical harmonic number for wiggler
(in analogy to ω_c of bending magnet)



wiggler and bend spectra after pin-hole aperture

Total Radiation Power

$$\boxed{P_{tot} = \frac{\pi}{3}\alpha\hbar\omega_1 K^2 (1+\tfrac{1}{2}K^2) N \frac{I}{e}}$$

or

$$P_{tot}[\mathrm{W}] = 726 \frac{E[\mathrm{GeV}]^2 K^2}{\lambda_p[\mathrm{cm}]^2} L[\mathrm{m}] I[\mathrm{A}]$$

e.g. about 1 photon from each electron in a 100-pole undulator, or
1 kW c.w. power from 1 m insertion device for beam current of
100 mA @ 5 GeV, $K = 1.5$, $\lambda_p = 2$ cm

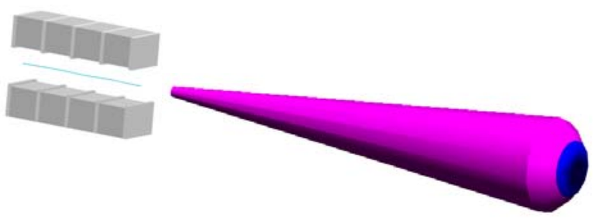
Note: the radiated power is independent from electron beam energy **if** one can keep $B_0\gamma \cong$ const, while $\lambda_p \sim \gamma^2$ to provide the same radiation wavelength. (e.g. low energy synchrotron and Compton back-scattering light sources)

However, most of this power is discarded (bw ~ 1). Only a small fraction is used.

Radiation Needed

wavelength	0.1 – 2 Å (if a hard x-ray source)	
bw	$10^{-2} - 10^{-4}$	← *temporal coherence*
small source size & divergence		← *spatial coherence*

Undulator Central Cone



Select with a pin-hole aperture the cone:

$$\theta_{cen} = \frac{1}{2\gamma}\sqrt{\frac{1+\frac{1}{2}K^2}{nN}} = \sqrt{\frac{\lambda_n}{2L}}$$

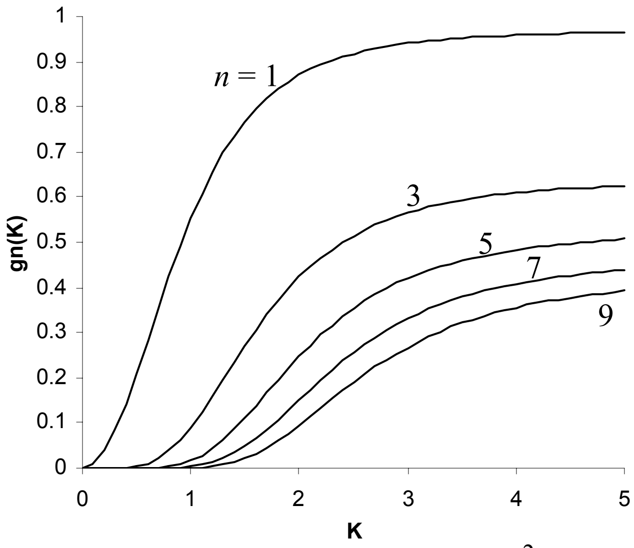
to get bw: $\dfrac{\Delta\omega}{\omega_n} \sim \dfrac{1}{nN}$

Flux in the central cone from n^{th} harmonic in bw $\Delta\omega/\omega_n$:

$$\dot{N}_{ph}\Big|_n = \pi\alpha N \frac{\Delta\omega}{\omega_n}\frac{I}{e} g_n(K) \le \boxed{\pi\alpha \frac{I}{e}\frac{g_n(K)}{n}}$$

Note: the number of photons in bw ~ $1/N$ is about 2 % max of the number of e⁻ for any-length undulator.

Undulator "efficiency": $\dfrac{P_{cen}}{P_{tot}} \le \dfrac{3g_n(K)}{K^2(1+\frac{1}{2}K^2)}\dfrac{1}{N_p}$



Function $g_n(K) = \dfrac{nK^2[JJ]}{(1+\frac{1}{2}K^2)}$

Coherent or Incoherent Radiation From Many Electrons?

Radiation field from a single k^{th} electron in a bunch:

$$E_k = E_0 \exp(i\omega t_k)$$

Radiation field from the whole bunch $\propto$ bunching factor (*b.f.)*

$$b.f. = \frac{1}{N_e} \sum_{k=1}^{N_e} \exp(i\omega t_k)$$

Radiation Intensity: $I = I_0 \left| b.f. \right|^2 N_e^2$

single electron

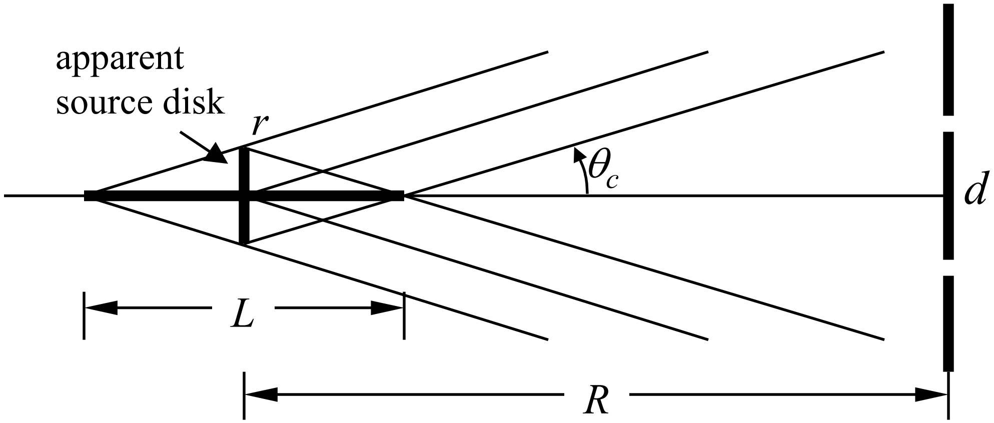
1) "long bunch": $\left| b.f. \right|^2 \sim 1/N_e$ => $I = I_0 N_e$ *incoherent (conventional) SR*

2) "short bunch" or μ-bunching: $\left| b.f. \right| \leq 1$ => $I \sim I_0 N_e^2$ *coherent (FELs) SR*

In this course we are dealing mostly with spontaneous (non-FEL) SR.

A Word on Coherence of Undulator

Radiation contained in the central cone is transversely coherent (no beam emittance!)



Young's double-slit interference condition:

$$\frac{rd}{R} \sim \lambda$$

in Fraunhofer limit:

$$r \sim \theta_c L \qquad \theta_c \sim r / R \quad \Rightarrow \theta_c \sim \sqrt{\lambda / L}$$

same as central cone

Spatial coherence (rms): $r \cdot \theta_c = \lambda / 4\pi$

Temporal coherence: $l_c = \lambda^2 / (2\Delta\lambda)$, $t_c = l_c / c$

Photon degeneracy*: $\Delta_c = \dot{N}_{ph,c} t_c$

X-ray source	Δ_c
Storage rings	<1
ERLs	>1
XFEL	>>1

Next, we will study the effect of finite beam 6D emittance on undulator radiation.

More on Synchrotron Radiation

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1. K.J. Kim, Characteristics of Synchrotron Radiation, AIP Conference Proceedings **189** (1989) pp.565-632

2. R.P. Walker, Insertion Devices: Undulators and Wigglers, CERN Accelerator School **98-04** (1998) pp.129-190, and references therein. Available on the Internet at http://preprints.cern.ch/cernrep/1998/98-04/98-04.html

3. B. Lengeler, Coherence in X-ray physics, Naturwissenschaften **88** (2001) pp. 249-260, and references therein.

4. D. Attwood, Soft X-rays and Extreme UV Radiation: Principles and Applications, Cambridge University Press, 1999. Chapters 5 (Synchrotron Radiation) and 8 (Coherence at Short Wavelength) and references therein.

Conclusions

- We've discussed dipole solutions to the Maxwell Equations, and how they may be used to obtain the radiation distribution from undulators by Lorentz transformation
- We've given an introduction to undulator radiation calculations and a general formulas for obtaining the spectral brilliance
- We've investigated how flux and brilliance scales with various parameters

Lecture: Introduction to ThomsonScattering

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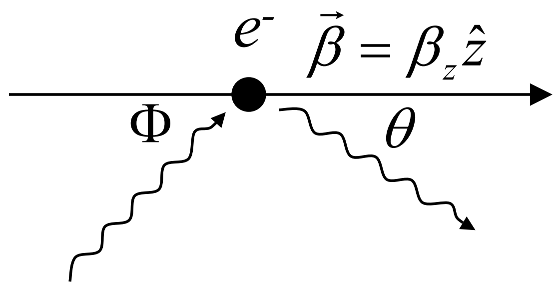
1. Thomson Scattering
    1. Process
    2. Simple Kinematics
    3. Finite Pulse Effects
2. Hamilton-Jacobi Solution of an Electron in a Plane Wave
    1. Hamilton-Jacobi Method
    2. Application to Orbits
    3. Exact Solution for Classical Electron in a Plane Wave
3. Applications to Scattered Spectrum
    1. Displacement Spectrum
    2. General Solution for Small K
    3. Finite K Effects
4. Qualitative Discussion on Angular Patterns
5. Finite Emittance Effects
6. Brilliance Scaling

Thomson Scattering

- Purely "classical" scattering of photons by electrons
- Thomson regime defined by the photon energy in the electron rest frame being small compared to the rest energy of the electron
- In this case electron radiates at the same frequency as incident photon for small field strengths
- Dipole radiation pattern is generated in beam frame, as for undulators
- Therefore radiation patterns can be largely copied from our previous undulator work

- Note on terminology: Some authors call any scattering of photons by free electrons Compton Scattering. Compton observed (the so-called Compton effect) frequency shifts in X-ray scattering off (resting!) electrons that depended on scattering angle. Such frequency shifts arise only when the energy of the photon in the rest frame becomes comparable with 0.511 MeV. We will reserve the words "Compton Scattering", only for such higher energy scattering. We will talk about only one experiment in the "Compton regime".

Simple Kinematics



Beam Frame

$$p'_{e\mu} = \left(mc^2, 0\right)$$

$$p'_{p\mu} = \left(E'_L, \vec{E}'_L\right)$$

Lab Frame

$$p_{e\mu} = mc^2\left(\gamma, \gamma\beta_z \hat{z}\right)$$

$$p_{p\mu} = E_L\left(1, \sin\Phi\hat{x} + \cos\Phi\hat{z}\right)$$

$$p_e \cdot p_p = mc^2 E'_L = mc^2 E_L \gamma\left(1 - \beta_z \cos\Phi\right) \qquad (3.1)$$

$$E'_L = E_L \gamma \left(1 - \beta_z \cos\Phi\right)$$

In beam frame scattered photon radiated with wave vector

$$k'_\mu = \frac{E'_L}{c}\left(1, \sin\theta' \cos\phi', \sin\theta' \sin\phi', \cos\theta'\right)$$

Back in the lab frame, the scattered photon energy E_s is

$$E_s = E'_L \, \gamma \left(1 + \beta_z \cos\theta\,'\right) = \frac{E'_L}{\gamma\left(1 - \beta_z \cos\theta\right)}$$

$$E_s = E_L \frac{\left(1 - \beta_z \cos\Phi\right)}{\left(1 - \beta_z \cos\theta\right)} \qquad (3.2)$$

Cases explored

Backscattered

$$\Phi = \pi$$

$$E_s = E_L \frac{(1+\beta_z)}{(1-\beta_z \cos\theta)} \approx 4\gamma^2 E_L \qquad \text{at} \quad \theta = 0$$

Provides highest energy photons for a given beam energy, or alternatively, the lowest beam energy to obtain a given photon wavelength. Pulse length roughly the ELECTRON bunch length

Cases explored, contd.

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Ninety degree scattering

$$\Phi = \pi / 2$$

$$E_s = E_L \frac{1}{\left(1 - \beta_z \cos\theta\right)} \approx 2\gamma^2 E_L \qquad \text{at} \quad \theta = 0$$

Provides factor of two lower energy photons for a given beam energy than the equivalent Backscattered situation. However, very useful for making short X-ray pulse lengths. Pulse length a complicated function of electron bunch length and transverse size.

Cases explored, contd.

Small angle scattered (SATS)

$$\Phi << 1$$

$$E_s = E_L \frac{\Phi^2}{2(1-\beta_z \cos\theta)} \approx \Phi^2 \gamma^2 E_L \qquad \text{at} \quad \theta = 0$$

Provides much lower energy photons for a given beam energy than the equivalent Backscattered situation. Alternatively, need greater beam energy to obtain a given photon wavelength. Pulse length roughly the PHOTON pulse length.

Transformation of Photon Field

Photon field for *x*-polarized plane wave traveling in the $-z$ direction (i.e., for the backscattered case!)

$$A_x(t,x,y,z) = A(z+ct)e^{i(k_z z+\omega t)}$$

$$A'_x(t',x',y',z') = A(\gamma(1+\beta_z)(z'+ct'))e^{i(k'_z z'+\omega' t')}$$

because $z+ct = \gamma(1+\beta_z)(z'+ct')$

$$\omega' = \gamma(1+\beta_z)\omega \tag{3.3}$$

$$k'_z = \gamma(1+\beta_z)k_z$$

$$E'_x = \gamma(1+\beta_z)E_x$$

$$B'_y = -E'_x = \gamma(1+\beta_z)B_y$$