USPAS Course on Recirculated and Energy Recovered Linear Accelerators

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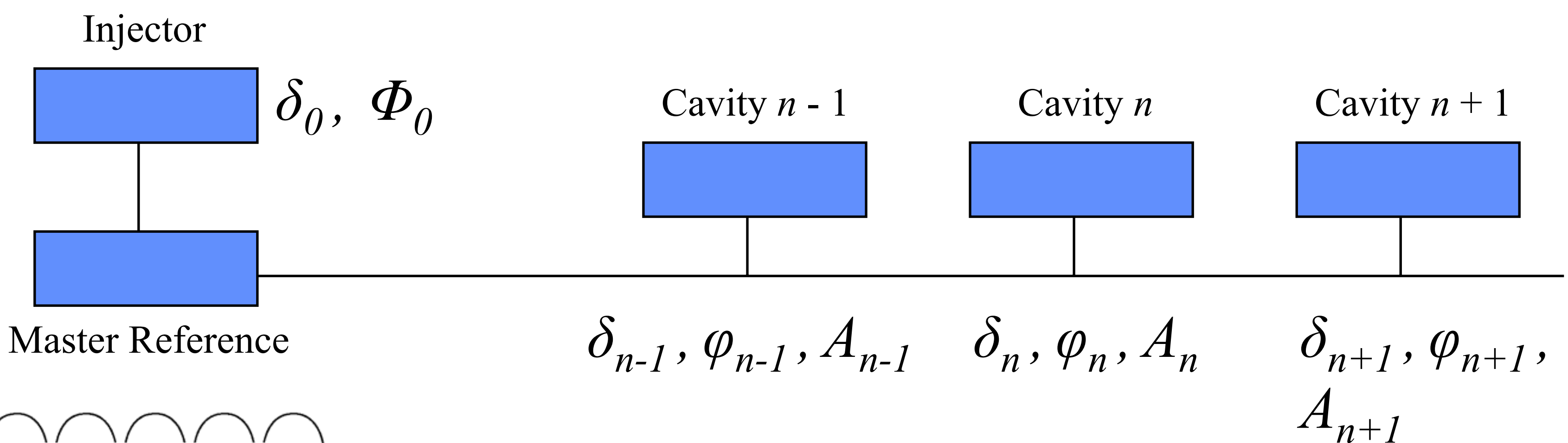
Lecture 8

Energy Spread Estimate

In the following calculation we estimate the spread of beam energy in a beam emerging from an imperfectly phased linac. Begin by considering some possible energy error sources.

A. Injected energy spread

B. From imperfectly controlled RF



Error Definitions

δ_i	$i = 0, 1, 2, \ldots, N$	Fast phase error at *N*th cavity
A_i	$i = 1, 2, \ldots, N$	Fast amplitude error at *N*th cavity
φ_i	$i = 1, 2, \ldots, N$	Slow phase error at *N*th cavity
Φ_0	$i = 1, 2, \ldots, N$	Offset phase from vernier

Statistical Considerations

Ideal Phasing Case

$$\overline{\Phi}_n = \int \Phi f_n(E, \Phi) dE d\Phi = 0$$

where f_n is the single particle longitudinal distribution function entering the nth cavity, E is the energy, and Φ is the phase referenced to the crest phase of the RF with $\Phi = 0$.

Total Energy

$$T = E + \sum_{n=1}^{N} \Delta E_n$$

Energy Spread When Perfectly Phased

With

$$\Delta E_n = E_0 \cos(\Phi)$$

Write f_n in terms of the distribution function at injection f, and use f to compute the relevant statistical averages, especially the *rms* energy spread. For a centered guassian phase distribution with *rms* width σ_φ

$$\frac{\sigma_E}{E} = \sqrt{\sigma_{E,inj}^2 / E^2 + \sigma_\phi^4 / 2}$$

More General Considerations

$$T = E + \sum_{n=1}^{N} E_n \left(1 + A_n\right) \cos\left(\phi_n - \Phi_0 + \Phi + \delta_0 + \delta_n\right)$$

Goal is to compute the energy spread as a function of the slow (uncorrected!) phase drifts $\varphi_1, \varphi_2, \ldots, \varphi_n$

Optimistic Result: Assume errors are completely uncorrelated

$$F\left(E, \Phi, \delta_0, \delta_1, A_1, \delta_2, A_2, \ldots, \delta_n, A_n\right) = f\left(E, \Phi\right) \psi_i\left(\delta_0\right) \prod_{n=1}^{N} \psi\left(\delta_n\right) g\left(A_n\right)$$

Pessimistic Result: Assume errors are completely uncorrelated

$$F\left(E,\Phi,\delta_0,\delta_1,A_1,\delta_2,A_2,\ldots,\delta_n,A_n\right)=f\left(E,\Phi\right)\psi_i\left(\delta_0\right)\psi\left(\delta_1\right)g\left(A_1\right)$$
$$\times\prod_{n=2}^{N}\delta\left(\delta_1-\delta_n\right)\delta\left(A_1-A_n\right)$$

With either choice of distribution function

$$\overline{T}=\int TF\left(E,\Phi,\delta_0,\delta_1,A_1,\delta_2,A_2,\ldots,\delta_n,A_n\right)dEd\Phi d\delta_0 d\delta_1 dA_1\ldots d\delta_n dA_n$$

$$=\overline{E}+\sum_{n=1}^{N}E_n I_1\left(\phi_n,\Phi_0\right)$$

$$I_1(\phi,\Phi_0) = \int \cos(\phi - \Phi_0 + \Phi + \delta_0 + \delta) f(E,\Phi) \psi_i(\delta_0) \psi(\delta) \times dE\, d\Phi\, d\delta_0\, d\delta$$

$$\int A g(A) dA = 0$$

$$I_1(\phi,\Phi_0) = \cos(\phi - \Phi_0) I_c - \sin(\phi - \Phi_0) I_s$$

$$I_c = \int \cos(\Phi + \delta_0 + \delta) f(E,\Phi) \psi_i(\delta_0) \psi(\delta) dE\, d\Phi\, d\delta_0\, d\delta$$

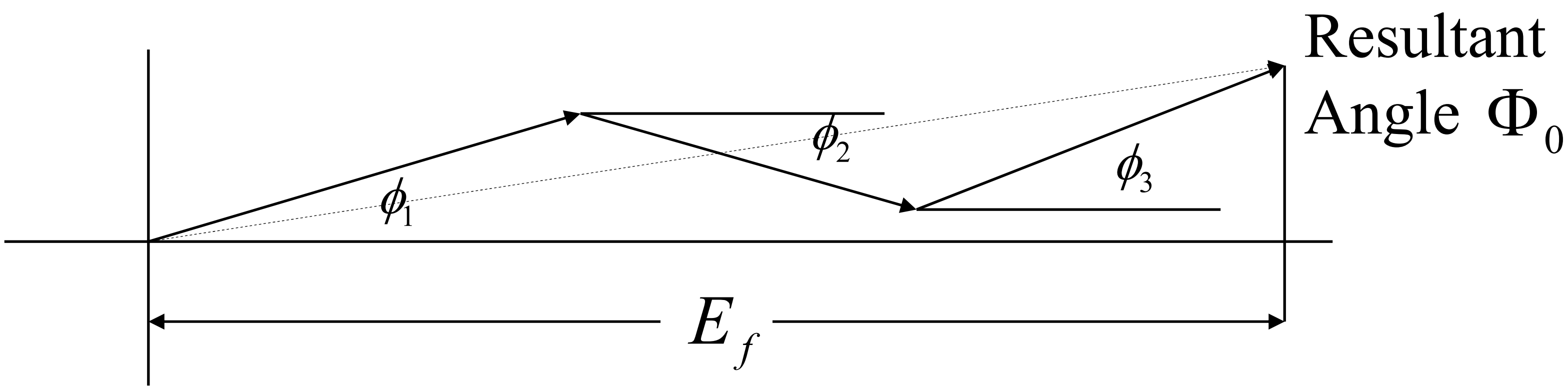
$$I_s = \int \sin(\Phi + \delta_0 + \delta) f(E,\Phi) \psi_i(\delta_0) \psi(\delta) dE\, d\Phi\, d\delta_0\, d\delta$$

Effect of the Energy Lock System

$$\overline{T} = \overline{E} + E_f$$

$$d\overline{T} / d\Phi_0 = 0$$

Phasor diagram for $N = 3$



$$\sum_{n=1}^{N} E_n \left[\cos(\phi_n - \Phi_0) I_c - \sin(\phi_n - \Phi_0) I_s\right] = E_f$$

$$\sum_{n=1}^{N} E_n \left[\sin(\phi_n - \Phi_0) I_c + \cos(\phi_n - \Phi_0) I_s\right] = 0$$

$$\sum_{n=1}^{N} E_n \cos(\phi_n - \Phi_0) = \frac{I_c E_f}{I_c^2 + I_s^2}$$

$$\sum_{n=1}^{N} E_n \sin(\phi_n - \Phi_0) = -\frac{I_s E_f}{I_c^2 + I_s^2}$$

$$\overline{T^2} = \overline{E^2} + 2\overline{E}\sum_{n=1}^{N} E_n I_1(\phi_n, \Phi_0)$$

$$+\sum_{n=1}^{N}\sum_{p\neq n}^{N} E_n E_p \left[\cos(\phi_n - \Phi_0)\cos(\phi_p - \Phi_0) I_{cc} + \sin(\phi_n - \Phi_0)\sin(\phi_p - \Phi_0) I_{ss}\right]$$

$$-\sum_{n=1}^{N}\sum_{p\neq n}^{N} E_n E_p \left[\cos(\phi_n - \Phi_0)\sin(\phi_p - \Phi_0) I_{cs} + \sin(\phi_n - \Phi_0)\cos(\phi_p - \Phi_0) I_{sc}\right]$$

$$+(1+I_A)\sum_{n=1}^{N} E_n^2 \left[\cos(\phi_n - \Phi_0)\cos(\phi_n - \Phi_0) I_{2cc} + \sin(\phi_n - \Phi_0)\sin(\phi_n - \Phi_0) I_{2ss}\right]$$

$$-(1+I_A)\sum_{n=1}^{N} E_n^2 \left[\cos(\phi_n - \Phi_0)\sin(\phi_n - \Phi_0) I_{2cs} + \sin(\phi_n - \Phi_0)\cos(\phi_n - \Phi_0) I_{2sc}\right]$$

$$I_A = \int A^2 g(A)\, dA$$

$$I_{cc} = \int \cos\left(\Phi + \delta_0 + \delta\right)\cos\left(\Phi + \delta_0 + \delta'\right) f\left(E, \Phi\right)\psi_i\left(\delta_0\right)\psi\left(\delta\right)\psi\left(\delta'\right)$$

$$\times dE d\Phi d\delta_0 d\delta d\delta'$$

$$I_{cs} = \int \cos\left(\Phi + \delta_0 + \delta\right)\sin\left(\Phi + \delta_0 + \delta'\right) f\left(E, \Phi\right)\psi_i\left(\delta_0\right)\psi\left(\delta\right)\psi\left(\delta'\right)$$

$$\times dE d\Phi d\delta_0 d\delta d\delta'$$

etc.,

$$I_{2cc} = \int \cos\left(\Phi + \delta_0 + \delta\right)\cos\left(\Phi + \delta_0 + \delta\right) f\left(E, \Phi\right)\psi_i\left(\delta_0\right)\psi\left(\delta\right)$$

$$\times dE d\Phi d\delta_0 d\delta$$

$$I_{2cs} = \int \cos\left(\Phi + \delta_0 + \delta\right)\sin\left(\Phi + \delta_0 + \delta\right) f\left(E, \Phi\right)\psi_i\left(\delta_0\right)\psi\left(\delta\right)$$

$$\times dE d\Phi d\delta_0 d\delta$$

etc.,

Relative Energy Spread (uncorrelated)

$$\frac{T_{rms}}{\bar{T}} = \sqrt{\overline{T^2} - \bar{T}^2} \,/\, \bar{T} = \frac{\sqrt{E_{rms}^2 + T_1^2 + T_2^2 + T_3^2}}{\bar{T}}$$

$$T_1^2 = \frac{E_f^2}{\left(I_c^2 + I_s^2\right)^2}\left[I_c^2\left(I_{cc} - I_c^2\right) + I_s^2\left(I_{ss} - I_s^2\right) + I_c I_s\left(I_{cs} - I_c I_s\right) + I_s I_c\left(I_{sc} - I_s I_c\right)\right]$$

$$T_2^2 = \sum_{n=1}^{N} E_n^2 \left[\begin{array}{c} \cos^2\left(\phi_n - \Phi_0\right)\left(I_{2cc} - I_{cc}\right) + \sin^2\left(\phi_n - \Phi_0\right)\left(I_{2ss} - I_{ss}\right) \\ + \cos\left(\phi_n - \Phi_0\right)\sin\left(\phi_n - \Phi_0\right)\left(I_{2cs} - I_{cs}\right) \\ + \sin\left(\phi_n - \Phi_0\right)\cos\left(\phi_n - \Phi_0\right)\left(I_{2sc} - I_{sc}\right) \end{array}\right]$$

$$T_3^2 = I_A \sum_{n=1}^{N} E_n^2 \begin{bmatrix} \cos^2(\phi_n - \Phi_0) I_{2cc} + \sin^2(\phi_n - \Phi_0) I_{2ss} \\ -\cos(\phi_n - \Phi_0)\sin(\phi_n - \Phi_0) I_{2cs} \\ -\sin(\phi_n - \Phi_0)\cos(\phi_n - \Phi_0) I_{2sc} \end{bmatrix}$$

Term 1: Injection energy spread

Term 2: Correlated errors from fast injector phase errors

Term 3: Fast phase errors including the energy lock (1/*N*)

Term 4: Uncorrected amplitude errors (1/*N*)

For Correlated Errors

$$\overline{T^2} = \overline{E^2} + 2\overline{E}\sum_{n=1}^{N} E_n I_1(\phi_n, \Phi_0)$$

$$+(1+I_A)\sum_{n=1}^{N} E_n^2\left[\cos(\phi_n-\Phi_0)\cos(\phi_n-\Phi_0)I_{2cc}+\sin(\phi_n-\Phi_0)\sin(\phi_n-\Phi_0)I_{2ss}\right]$$

$$-(1+I_A)\sum_{n=1}^{N} E_n^2\left[\cos(\phi_n-\Phi_0)\sin(\phi_n-\Phi_0)I_{2cs}+\sin(\phi_n-\Phi_0)\cos(\phi_n-\Phi_0)I_{2sc}\right]$$

But now

$$T_1^2 = \frac{E_f^2}{\left(I_c^2+I_s^2\right)^2}\left[I_c^2\left(I_{2cc}-I_c^2\right)+I_s^2\left(I_{2ss}-I_s^2\right)+I_cI_s\left(I_{2cs}-I_cI_s\right)+I_sI_c\left(I_{2sc}-I_sI_c\right)\right]$$

$$T_2^2 = 0$$

No dependence on slow phase errors because sum of cosines is cosine and energy lock takes correlated errors out

$$T_3^2 = \frac{I_A E_f^2}{\left(I_c^2 + I_s^2\right)^2}\left[I_c^2 I_{2cc} + I_s^2 I_{2ss} + I_c I_s I_{2cs} + I_s I_c I_{2sc}\right]$$

This term a factor of *N* larger than before, as is characteristic in the transition between uncorrelated errors and correlated errors

Example: Gaussian error distributions

$$f(E,\Phi)=\frac{1}{2\pi\sigma_E\sigma_\Phi}\exp\left[-\left(E-\overline{E}\right)^2/2\sigma_E^2\right]\exp\left(-\Phi^2/2\sigma_\Phi^2\right)$$

$$\psi(\delta)=\frac{1}{\sqrt{2\pi}\sigma_\delta}\exp\left(-\delta^2/2\sigma_\delta^2\right)$$

$$\psi_i(\delta)=\frac{1}{\sqrt{2\pi}\sigma_\delta}\exp\left(-\delta_0^2/2\sigma_i^2\right)$$

$$g(A)=\frac{1}{\sqrt{2\pi}\sigma_A}\exp\left(-A^2/2\sigma_A^2\right)$$

$$I_c = \exp\left(-\sigma_\Phi^2/2\right)\exp\left(-\sigma_i^2/2\right)\exp\left(-\sigma_\delta^2/2\right)$$

$$I_s = 0$$

$$I_{cc} = \left[0.5 + 0.5\exp\left(-2\sigma_\Phi^2\right)\exp\left(-2\sigma_i^2\right)\right]\exp\left(-\sigma_\delta^2\right)$$

$$I_{ss} = \left[0.5 - 0.5\exp\left(-2\sigma_\Phi^2\right)\exp\left(-2\sigma_i^2\right)\right]\exp\left(-\sigma_\delta^2\right)$$

$$I_{cs} = I_{sc} = 0$$

$$I_{2cc} = 0.5 + 0.5\exp\left(-2\sigma_\Phi^2\right)\exp\left(-2\sigma_i^2\right)\exp\left(-2\sigma_\delta^2\right)$$

$$I_{2ss} = 0.5 - 0.5\exp\left(-2\sigma_\Phi^2\right)\exp\left(-2\sigma_i^2\right)\exp\left(-2\sigma_\delta^2\right)$$

$$I_{2cs} = I_{2sc} = 0$$

$$I_A = 0$$

Effective bunch length

$$\sigma_I^2 = \sigma_\Phi^2 + \sigma_i^2$$

Therefore, little is to be gained (in energy spread!) by shortening the bunch length in phase much beyond ones ability to control the injector phase jitter, or by clamping down on the phase jitter below the single bunch phase spread.

Energy Lock in More Detail

All cavities, except one E_m, are set to the average energy gain

$$E_n = E_0 = E_f / N \qquad \text{for } \; n \neq m$$

Energy lock condition is

$$E_m \approx E_0 \left[1 + N\left(\sigma_I^2 + \sigma_\delta^2\right)/2 + N\Delta\phi^2 / 2\right]$$

$$\Delta\phi^2 \equiv \sum_{n=1}^{N} \left(\phi_n - \Phi_0\right)^2 / N$$

$$\Phi_0 \approx \sum_{n=1}^{N} \phi_n / N$$

Increase in energy gain needed to make up for the fact that bunches are not on the crest of individual cavities. Note *rms* offset appears

Energy Spread Bounds

$$\sqrt{D_1} \le \sqrt{\left(E_{rms}/E\right)^2 - \left(E_{i,rms}/E\right)^2} \le \sqrt{D_2}$$

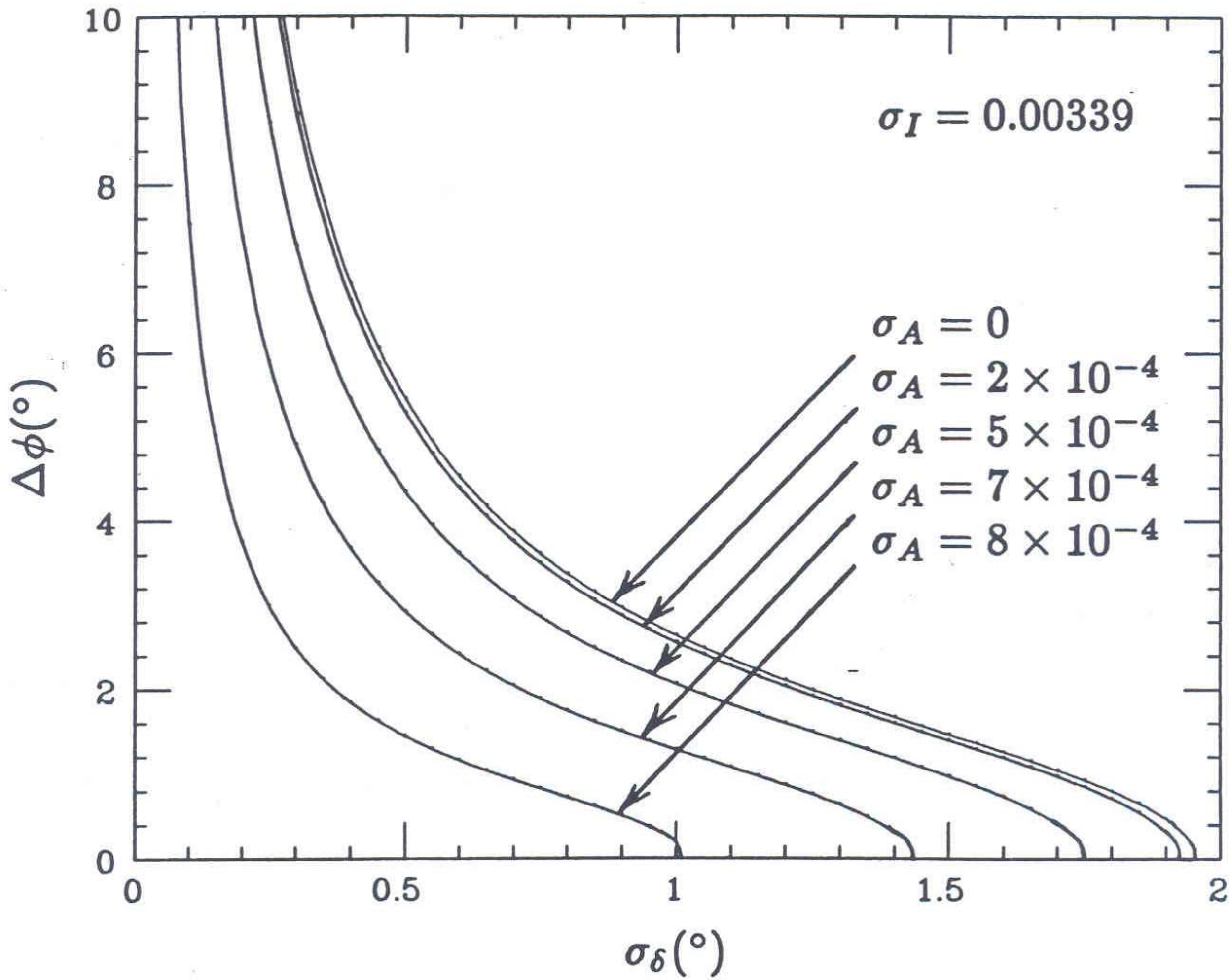
$$D_1 \approx \sigma_I^4/2 + \sigma_\delta^2 \left[\sum_{n=1}^{N} \left(\phi_n - \Phi_0\right)^2 / N \right] / N + \sigma_\delta^2 \left(\sigma_\delta^2/2 + \sigma_I^2\right) / N$$

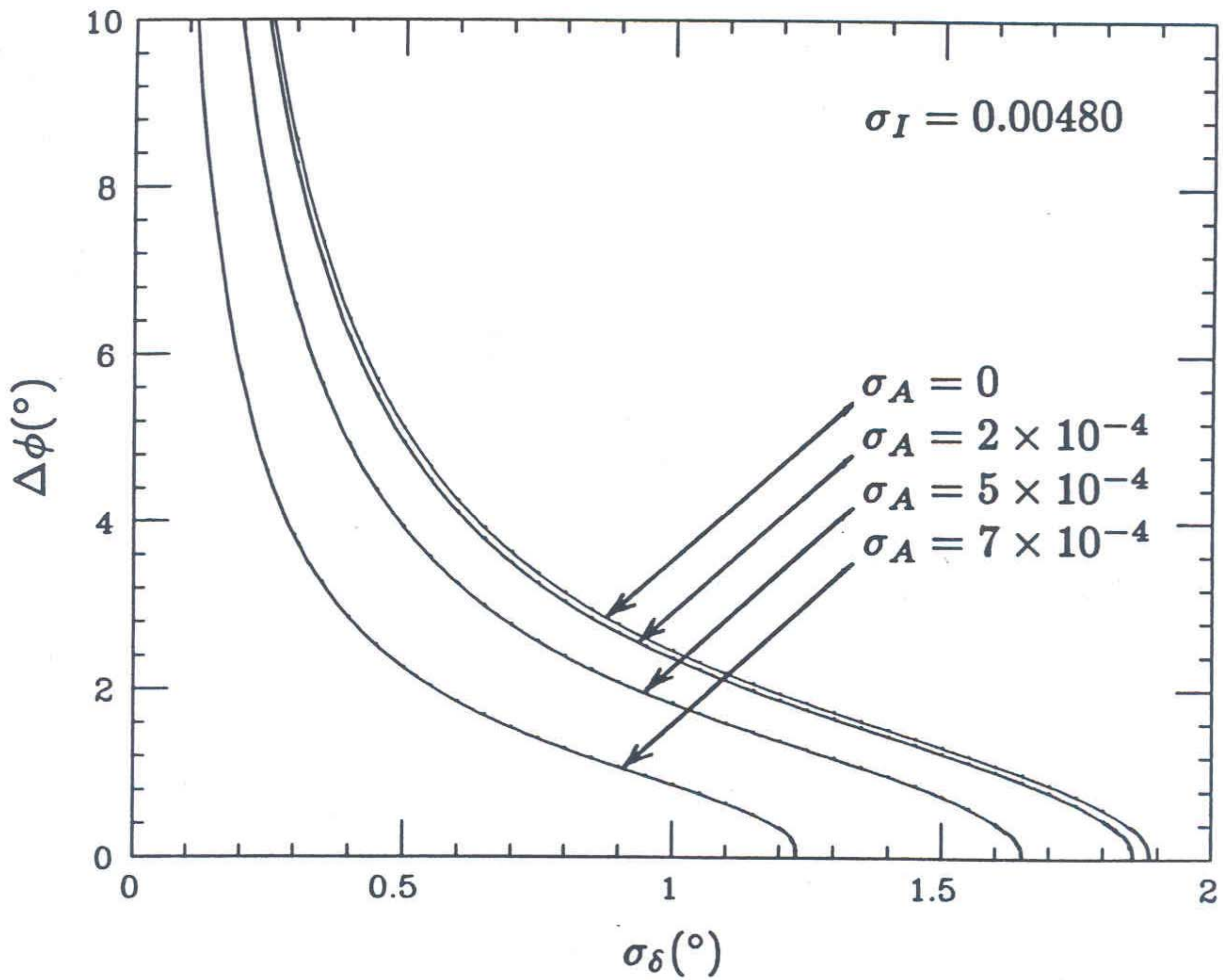
$$+ \sigma_A^2 / N$$

$$D_2 \approx \left(\sigma_I^2 + \sigma_\delta^2\right)^2 / 2 + \sigma_A^2$$

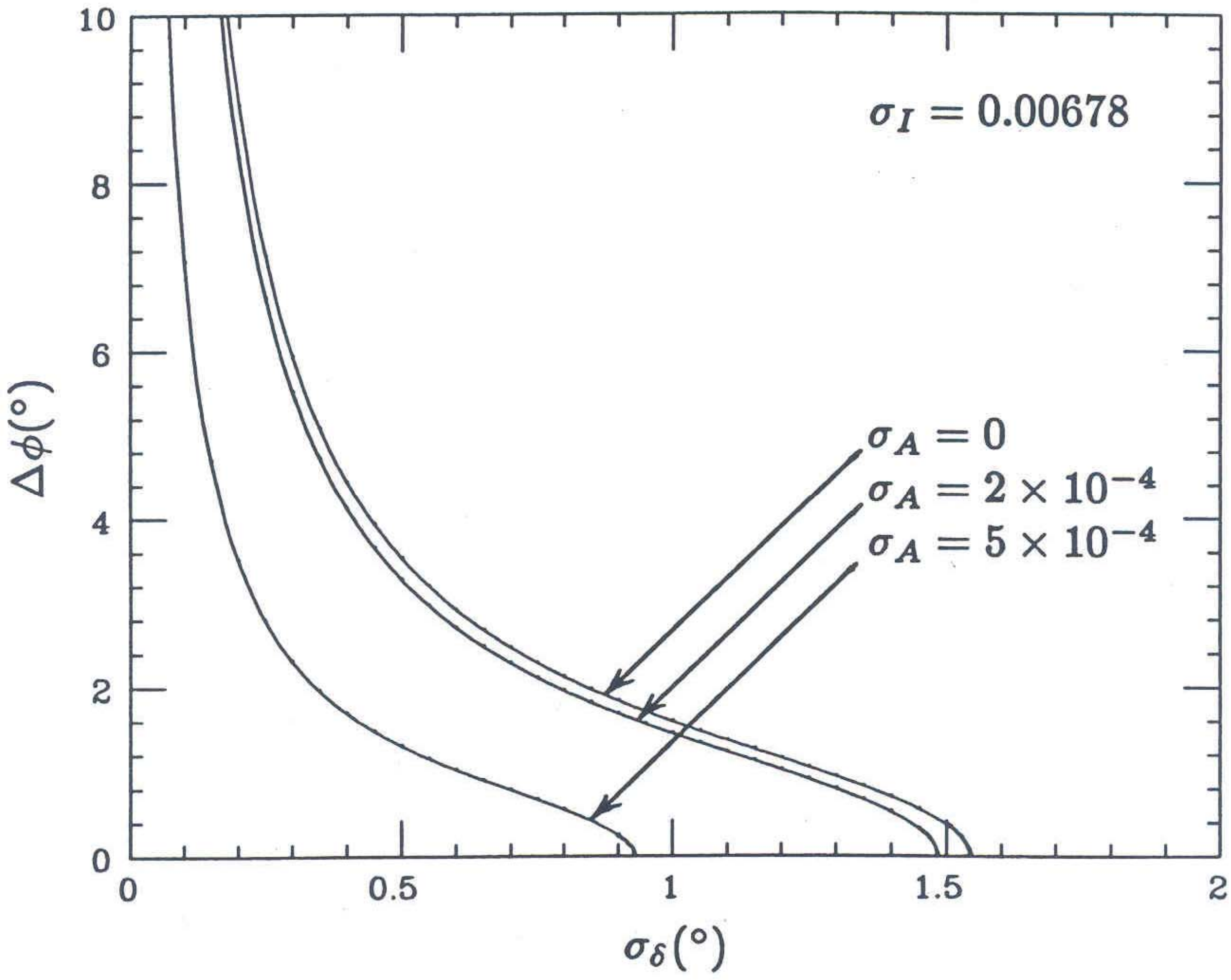
To fourth order in the small quantities. Note the central limit theorem gives that the overall energy distribution is Gaussian for large N

Combinations leading to 2.5×10^{-5} energy spread

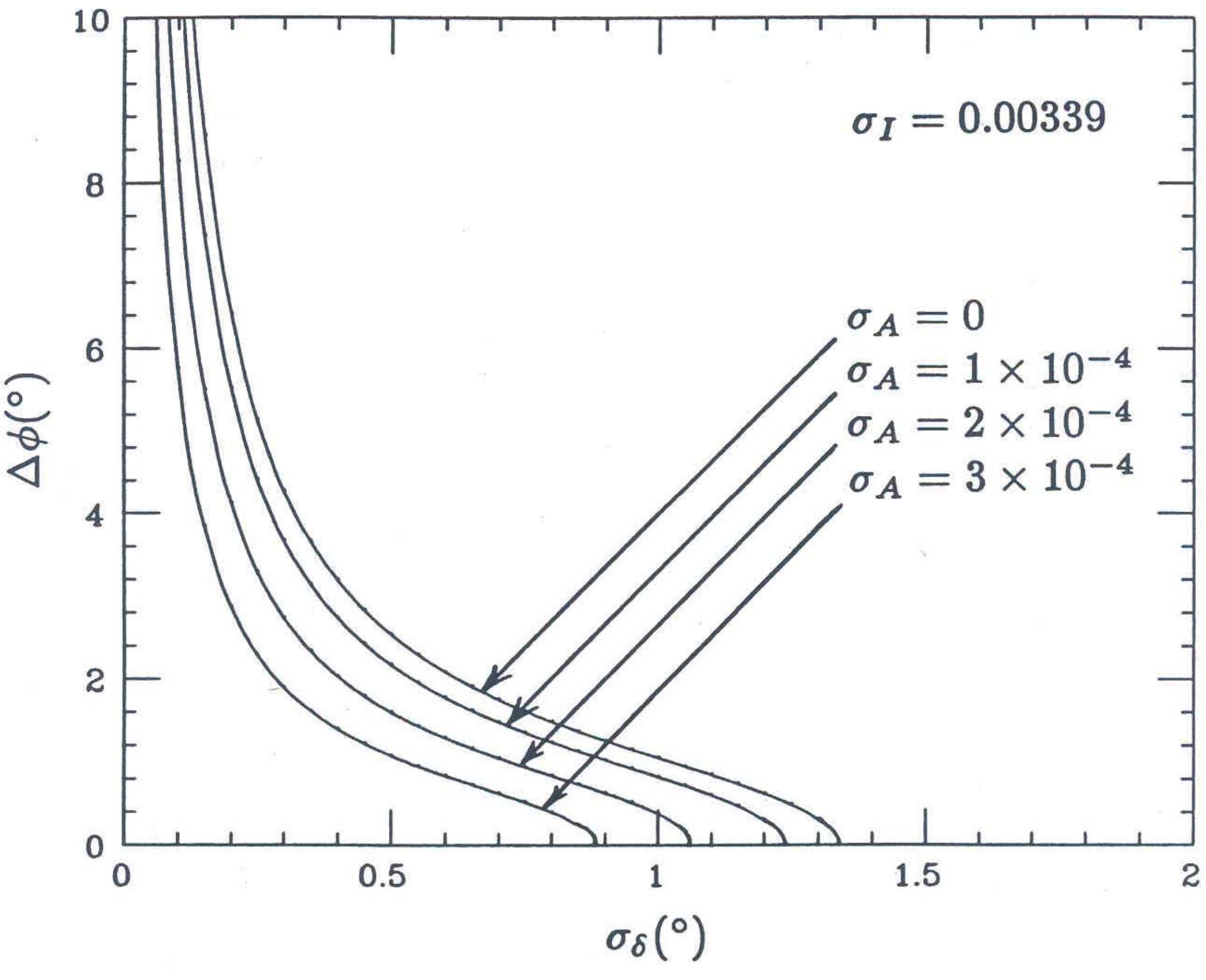




$\sigma_I = 0.00480$
$\sigma_A = 0$
$\sigma_A = 2 \times 10^{-4}$
$\sigma_A = 5 \times 10^{-4}$
$\sigma_A = 7 \times 10^{-4}$
$\Delta\phi(°)$
$\sigma_\delta(°)$
0
2
4
6
8
10
0
0.5
1
1.5
2



$\sigma_I = 0.00678$
$\sigma_A = 0$
$\sigma_A = 2 \times 10^{-4}$
$\sigma_A = 5 \times 10^{-4}$
$\Delta\phi(°)$
$\sigma_\delta(°)$
0
2
4
6
8
10
0
0.5
1
1.5
2


$\sigma_I = 0.00339$
$\sigma_A = 0$
$\sigma_A = 1 \times 10^{-4}$
$\sigma_A = 2 \times 10^{-4}$
$\sigma_A = 3 \times 10^{-4}$
$\Delta\phi(°)$
$\sigma_\delta(°)$
0
2
4
6
8
10
0
0.5
1
1.5
2