USPAS Course on
Recirculated and Energy
Recovered Linear Accelerators

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Lecture 7
Lecture Outline

. Some General Rules Regarding Recirculated Linac Design
  . Longitudinal Single Particle Dynamics
    – Longitudinal Gymnastics
    – Correcting RF Curvature with RF
    – Correcting RF Curvature with Sextupoles
    – Correction RF Curvature with Linearizers
    – Longitudinal Tune Choices
    – Energy Spread Estimates
  . Transverse Single Particle Dynamics
    – Basic Considerations
    – Betatron Motion Damping and Antidamping
    – RF Focussing
    – Energy Ratio Limits
    – Beam Loss
Some General Rules on Recirculated Linac Design

- Design the nonlinear development of the longitudinal phase space first, adjust the transverse phase space “appropriately” based on the longitudinal design. This is actually a pretty important rule and saves much work if possible because the longitudinal control elements (i.e., RF cavities and places where $M_{56}$ is introduced) can have substantial effects on the transverse dynamics (e.g. through RF focussing or generating dispersion!) BUT the elements controlling the transverse design tend to have somewhat less effect on the longitudinal dynamics, at least insofar as non-linear effects are concerned

- Work to achieve linear (elliptical!) longitudinal phase space densities at specific locations in the design. This may be accomplished by adjusting specific non-linear distortions in the phase space with offsetting distortions introduced as correction elements

- Become familiar with and quantify specific non-linear distortions and the library of “tools” available to correct them
Longitudinal Phase Space

Begin with the action principle [Jackson, Section 12.1]

\[ \text{Action} = \int_{t_1}^{t_2} L[q_i(t), \dot{q}_i(t), t] \, dt \]

Action integral must be Lorentz invariant, as is the proper time, \( d\tau = dt / \gamma \)

\[ \therefore \quad \gamma L \]

must be a Lorentz invariant. So, as we’ve already used previously, the Lagrangian for particle in an EM field must be (MKS units)

\[ L = -mc^2 \sqrt{1 - \beta^2} - e\Phi + eA \cdot v \]

You’ve already seen the E-L Equations yield the correct relativistic equations of motion.
The canonical momenta conjugate to the position coordinates are

\[ P_i = \frac{\partial L}{\partial v_i} = \gamma m v_i + eA_i \]

The Hamiltonian \( H = P \cdot v - L \) is

\[ H(q_i, P_i, t) = \sqrt{(P - eA)^2 + m^2 c^4} + e\Phi \]

and the standard relativistic EM force law may be derived from Hamilton’s canonical equations

\[ \dot{q}_i = \frac{\partial H}{\partial P_i} \quad \quad \quad \dot{P}_i = -\frac{\partial H}{\partial q_i} \]

The coordinate pairs \((q_i, P_i)\), e.g. \((z, P_z)\), form 2-D spaces called phase space.
Usually in accelerators, the longitudinal velocity is much bigger than the transverse velocity, and one can treat the longitudinal dynamics separately from the transverse dynamics.

\[ H_{\text{eff}}(z, P_z) = \sqrt{(P_z - eA_x(z))^2 + m^2 c^4} + e\phi(z) \]

\[ \approx P_z - eA_x(z) + e\phi(z) \quad \text{Extreme Relativistic} \]

\[ \approx mc^2 + \frac{(P_z - eA_x(z))^2}{2m} + e\phi(z) \quad \text{Nonrelativistic} \]

Which is clearly close to the total energy in this case!
Canonical Variables for Longitudinal Motion

In extreme relativistic limit

\[(z, E)\]

or

\[(\phi, \Delta E)\]

Form a canonically conjugate pair.
Liouville’s Theorem

For a 3-dimensional Hamiltonian system, the sum of the projected phase space area is preserved

\[ \frac{d}{dt} g_t^* \left( \sum_i dP_i \wedge dq_i \right) = \sum_i \left( \frac{d}{dt} g_t^* dP_i \wedge dq_i + dP_i \wedge \frac{d}{dt} g_t^* dq_i \right) \]

\[ = \sum_{i=1}^{3} \sum_{j=1}^{3} \left( - \frac{\partial^2 H}{\partial q_j \partial q_i} dq_j \wedge dq_i - \frac{\partial^2 H}{\partial p_j \partial q_i} dP_j \wedge dq_i \right) \]

\[ = \sum_{i=1}^{3} \sum_{j=1}^{3} \left( \frac{\partial^2 H}{\partial q_j \partial p_i} dP_i \wedge dq_j + \frac{\partial^2 H}{\partial p_j \partial p_i} dP_j \wedge dP_i \right) = 0 \]

Here

\[ g_t : P \rightarrow P \]

is the phase flow function.
Liouville’s Theorem

Have also used, e.g.

\[
\frac{d}{dt} g^*_t dP_i = -\sum_{j=1}^{3} \frac{\partial^2 H}{\partial q_j \partial q_i} dq_j - \sum_{j=1}^{3} \frac{\partial^2 H}{\partial P_j \partial q_i} dP_j
\]

This equation follows from the Hamilton equations of motion

Corollary 1 (Liouville’s Theorem): The local full 6-d phase volume is preserved

\[
\frac{d}{dt} g^*_t (dq_1 \Delta dq_2 \Delta dq_3 \Delta dP_1 \Delta dP_2 \Delta dP_3) = 0
\]

Corollary 2 (Longitudinal Liouville’s Theorem): In the case that the longitudinal motion uncoupled from the transverse motion, the longitudinal phase space density is preserved
Longitudinal Emittance

Definition: Utilizing the single particle distribution function to define averages as before, we define the longitudinal emittance to be the following phase space average:

\[ \varepsilon_z = \sqrt{\left\langle \left( z - \left\langle z \right\rangle \right)^2 \right\rangle \left\langle \left( P_z - \left\langle P_z \right\rangle \right)^2 \right\rangle - \left\langle (z - \left\langle z \right\rangle)(P_z - \left\langle P_z \right\rangle) \right\rangle^2} / mc \]

Units in this definition are m. For perfectly linear restoring forces, one can show that this quantity is preserved with acceleration. However, this quantity can go both up and down depending on manipulations done to “straighten out” curvatures in phase space.

In linacs and recirculating linacs especially, this quantity provides a great “metric” for evaluating and comparing different accelerator designs. Smaller longitudinal emittance implies that one is able to compress to smaller bunches.
Cautions

This definition is good enough for our purposes, but there is no real “standard” definition in the field. For example, sometimes emittances are computed with subsets of the total number of particles. When “chirping” is discussed, sometimes the intrinsic phase and energy spread is being talked about, etc.

Ideally, one would like to obtain a final accelerated emittance at the same level as comes out of the gun. Cannot do that exactly, but quantifying various sources of emittance growth in the linacs can provide a path to optimize the end use emittance.

I, personally, like the units keV degrees for my emittance reporting because bunch durations are easily measured in RF degrees and energy spreads are typically 10s of keVs. To convert

\[ \varepsilon_z [\text{keV degrees}] = \varepsilon_z [\text{m}] \frac{511 \text{ keV} \cdot 360 \text{ degrees}}{\lambda_{RF} [\text{m}]} \]
Homework

Suppose for a moment that one could create a distribution with no intrinsic spread but which had a parabolic distortion in the phase space. Compute the longitudinal emittance as a function of the parabolic distortion. Does your result approach the proper limit as $\Delta z_{\text{min}}$ goes to zero?

\[
f(z, \Delta E) = A \delta \left( z + z_{\text{min}} \left( \frac{\Delta E}{\Delta E_{\text{max}}} \right)^2 \right) \left[ \Theta(\Delta E + \Delta E_{\text{max}}) - \Theta(\Delta E - \Delta E_{\text{max}}) \right]
\]
Bunching

Fundamental problem: Take the first phase space distribution to the second one
What Tends to Happen!

Usually, it doesn’t work out the first time because of distortions
Bunching Elements

RF Bunchers

\[ \phi_s \approx 90^\circ \] at the bunching zero crossing

Relatively low gradient
Usually done at low energy
Not too much curvature distortion introduced

Other Linac Cavities

\[ \phi_s \approx 1 - 10^\circ \] offset in the bunching direction

Relatively high gradient
Can be done at high energy
Lots of curvature distortion introduced because at peak of \cos

Bend regions can generate \( M_{56} \) as discussed previously
Some design rules

- Avoid overbunching, especially at non-relativistic energies
- When possible, reduce any phase space non-linearities with available correction elements.
- When possible, choose setups that are easy to set up and rapidly diagnose
  - Favorite positions are zero crossings and crests of RF elements
- Try to reduce sensitivities to RF drifts by making beta functions small at locations where RF cavities do substantial acceleration. This saves on a lot of operational setup and troubleshooting grief.
- Design bunching program first, and then the transverse optics because generally the transverse optics depend quite a bit on the RF cavity settings and phases, whereas the longitudinal dynamics is by contrast insensitive to the details of the transverse settings
Schematic of CEBAF Injector

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31 March 2005

USPAS Recirculated and Energy Recovered Linacs

Thomas Jefferson National Accelerator Facility
Phase Space from CEBAF Bunching

\[ \Delta E \]

-3 cm \hspace{1cm} 3 cm \hspace{1cm} Z

\[ \Delta E \]

5 keV \hspace{1cm} Z

-5 keV
Calculated Longitudinal Phase Space
## Injector Phasing Procedure

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Setup Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buncher Phase</td>
<td>-19 deg from zero</td>
</tr>
<tr>
<td>Buncher Gradient</td>
<td>40 kV/m</td>
</tr>
<tr>
<td>Phase Transfer</td>
<td></td>
</tr>
<tr>
<td>Capture Section Phase</td>
<td>+16.5 deg from crest</td>
</tr>
<tr>
<td>Capture Section Gradient</td>
<td>1.3 MV/m</td>
</tr>
<tr>
<td>Energy @ 500 keV</td>
<td></td>
</tr>
<tr>
<td>Unit Phases</td>
<td>-7.5 deg crested</td>
</tr>
<tr>
<td>40 MeV Accelerator Phases</td>
<td>crested</td>
</tr>
<tr>
<td>Max. 4 MeV energy</td>
<td></td>
</tr>
<tr>
<td>Max. 45 MeV energy</td>
<td></td>
</tr>
</tbody>
</table>

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Effects of Nonlinearities

\[ \Delta E \]

\[ z \]

\[ M_{56} \neq 0 \]

\[ T_{566} \neq 0 \]

\[ \Delta E \]

\[ z \]
Homework

Utilizing the non-relativistic velocity-energy relation, compute the $T_{566}$ for a drift space of length $L$ as a function of $\gamma$. 
Phase Transfer Technique

Simultaneously, digitize phase modulation and arrival time determined by a phase detector
Some Early Results
Phase Space Correction Scheme

- \( \phi_{in} \) to \( \phi_{out} \)
- \( \phi_{in} \) to \( \phi_{out} \)
- \( \phi_{in} \) to \( \phi_{out} \)
Triangle Wave Modulation

Non-harmonic sampling

$\phi_{mod}^{(1)}$

Simultaneous detection and digitization of the arrival phase from the mixer output
Phase Transfer Function, more recently
Tschebyshev Analysis of Nonlinear Transfer Maps

• Concentrate on problem: how can one easily acquire and intelligently analyze and organize, information about the optics of (i.e., the transport maps of) the accelerator including the nonlinearities?

• Basic philosophy: perturb the beam around the operating point, varying one or more variables in a systematic way.

• Triangle wave modulation as in the phase transfer device is very good for producing pictures to compare to the phase space plots, but the Fourier transform of a triangle wave has harmonics of the fundamental frequency mixed in. For example, a tilt in the phase space distribution would produce signals at odd harmonics of the fundamental in the mixer output.

• Question: is there a better way? Yes

• Question: Is there a function set and modulation pattern, such that the function set is cleanly distinguished by Fourier analysis of the modulation pattern applied to the function set? Yes, Tschebyshev polynomials and sinusoidal modulation do the trick!
Definition of Tschebyshev Polynomials

Defining relation

\[ T_n (\cos \theta) = \cos(n \theta) \]

\[ n = 1 \quad T_1 (x) = x \]
\[ n = 2 \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 \]
\[ T_2 (x) = 2x^2 - 1 \]
\[ n = 3 \quad \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \]
\[ T_3 (x) = 4x^3 - 3x \]
\[ n = 4 \quad \cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta - 1 \]
\[ T_4 (x) = 8x^4 - 8x^2 + 1 \]
\[ \vdots \]
General Measurement

\[ x = x_{op} + \Delta x \cos(\omega_m t) \]
Orthogonality and Tschebyshev Expansions

A general continuous function, \( f(x) \), defined on the domain \([-1,1]\) may be expanded in a uniformly convergent series

\[
f(x) = \sum_{n=0}^{\infty} a_n T_n(x)
\]

The expansion coefficients may be obtained by the overlap integral

\[
a_n = \int_{-1}^{1} \frac{f(x)T_n(x)}{(1-x^2)^{1/2}} \, dx
\]
Alternatively, and this is the main idea of the analysis, the expansion coefficients may be obtained by performing a very simple measurement, namely, modulate the input with a sinusoidal oscillation throughout [-1,1] and Fourier transform the resulting data.

\[ a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(t) \cos(n\omega t) \, dt \]

The amplitudes of the expansion coefficients appear directly as the size of the peaks in the Fourier analysis of the output data, because

\[ T_n (\cos \theta) = \cos(n \theta) \]
$n = 1$
$n = 2$
\[ n = 3 \]
$n = 4$
Spectrum from sinusoidal phase modulation
Analysis Results

Expansion Coefficients for phase-phase correlation

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n (°)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.458</td>
</tr>
<tr>
<td>2</td>
<td>0.599</td>
</tr>
<tr>
<td>3</td>
<td>0.072</td>
</tr>
<tr>
<td>4</td>
<td>-0.182</td>
</tr>
<tr>
<td>5</td>
<td>-0.031</td>
</tr>
</tbody>
</table>
Comparison Analysis Results and Original Data
The notion of longitudinal emittance and longitudinal phase space have been introduced

We have given an indication how longitudinal phase space may be manipulated in order to achieve design goals

We have indicated some of the reasoning pertinent to the design of the CEBAF injector at Jefferson Lab

We have introduced a means of quantifying nonlinear distortions in phase space and a means of quantifying the nonlinear transfer characteristics of nonlinear maps
Energy Spread, Intrinsic

\[ \varepsilon_L \approx \Delta E_b \Delta \phi_b / 4 \]

Intrinsic Phase Spread
\[ \sigma_\phi \approx 2 \varepsilon_L / \Delta E_a \]

Set in injector/initial bunching
Longitudinal Optimization

Energy spread from perfectly phased linac

$$\frac{\sigma_E}{E} = \sqrt{\frac{\sigma_{E,\text{inj}}^2}{E^2} + \frac{\sigma_\phi^4}{2}}$$

Using definition of longitudinal emittance, derive an optimum

$$\sigma_{\phi,\text{opt}} = \left(\frac{\varepsilon_L}{E}\right)^{1/3}$$

And minimum energy spread out the end

$$\frac{\sigma_E}{E} = \sqrt{\frac{3}{2}} \left(\frac{\varepsilon_L}{E}\right)^{2/3}$$
Longitudinal Optimization

Measurements of longitudinal emittance

<table>
<thead>
<tr>
<th>Source</th>
<th>Energy (MeV)</th>
<th>Emittance (keV deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Injector</td>
<td>45</td>
<td>&lt;6.7</td>
</tr>
<tr>
<td>Nlinac exit</td>
<td>500</td>
<td>&lt;12.5</td>
</tr>
</tbody>
</table>

Optimal spread using larger number $2 \times 10^{-5}$ @ 1 GeV ($5 \times 10^{-6}$ @ 4 GeV)
Short Bunch Configuration

Beam current ($\mu$A) vs. bunch length (fs)
High Charge Accelerator Design

- Must include effects of space charge (interparticle interactions!) in the design.
- These effects are calculated by self-consistently solving Poisson’s equations, or more generally the full Maxwell equations, with the particle motion, which gives the charge densities and currents for the source terms of the self fields.
- Usually, some kind of “leap-frogging” is done in the calculations.
- At Jefferson Lab we’ve been happy with PARMELA simulations, in contrast to people that exist in the nC regime.
- One advantage of the regime we operate in, around 100 pC, seems to have much less predictive trouble than higher charge-per-bunch. My inclination, (see our ERL talks in the future!), is to avoid going to higher bunch charges as much as possible because there seems to be very few clean results once heavy space-charge gets involved (perhaps this is just a statement about how complicated space-charge is, no one wants to take the time to do it right!)
Jefferson Lab FEL

Jefferson Lab FEL

Thomas Jefferson National Accelerator Facility

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## FEL Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Designed</th>
<th>Measured</th>
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</thead>
<tbody>
<tr>
<td>Kinetic Energy</td>
<td>48 MeV</td>
<td>48.0 MeV</td>
</tr>
<tr>
<td>Average current</td>
<td>5 mA</td>
<td>4.8 mA</td>
</tr>
<tr>
<td>Bunch charge</td>
<td>60 pC</td>
<td>Up to 60 pC</td>
</tr>
<tr>
<td>Bunch length (rms)</td>
<td>&lt;1 ps</td>
<td>0.4±0.1 ps</td>
</tr>
<tr>
<td>Peak current</td>
<td>22 A</td>
<td>Up to 60 A</td>
</tr>
<tr>
<td>Trans. Emittance (rms)</td>
<td>&lt;8.7 mm-mr</td>
<td>7.5±1.5 mm-mr</td>
</tr>
<tr>
<td>Long. Emittance (rms)</td>
<td>33 keV-deg</td>
<td>26±7 keV-deg</td>
</tr>
<tr>
<td>Pulse repetition frequency (PRF)</td>
<td>18.7 MHz, x2</td>
<td>18.7 MHz, x0.25, x0.5, x2, and x4</td>
</tr>
</tbody>
</table>
Correction of Nonlinearities by Sextupoles

Basic Idea: Use sextupoles to get $T_{566}$ in a bending arc to compensate any curvature induced terms.

X-ray Set-up

IR wiggler
Optical system
Dump

Linac

e^- recirculation beam line

sextupoles

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Longitudinal Phase Space Manipulations

Simulation calculations of longitudinal dynamics of JLAB FEL
Transfer Function Measurements

![Graphs showing drive laser phase vs. time of flight for different pickups.]

### Table: Transfer Function Measurements

<table>
<thead>
<tr>
<th>Experiment</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>Simulation</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
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<tr>
<td></td>
<td>0.1172</td>
<td>-0.0801</td>
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<td>0.1070</td>
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<td>0.0256</td>
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<tr>
<td></td>
<td>0.0008</td>
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<td>0.0006</td>
<td></td>
<td>0.0007</td>
<td>0.0003</td>
<td>0.0004</td>
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</table>

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Correction of Nonlinearities by Sextupoles

Basic Idea is to use sextupoles to get $T_{566}$ in the bending arc to compensate any curvature induced terms.
Correction of Nonlinearities by “Linearizers”

\[ V_c = V_0 \cos \theta \approx V_0 \left(1 - \frac{\theta^2}{2} + \cdots\right) \]

\[ V_{\text{lin}} = \frac{V_0}{9} \cos 3\theta \approx \frac{V_0}{9} \left(1 - \frac{9\theta^2}{2} + \cdots\right) \]

\[ V_c - V_{\text{lin}} = \frac{8V_0}{9} + o\left(\theta^4\right), \quad \text{independent of phase!} \]

Boeing High Average Power FEL
Phase Space Evolution Without Linearizer

![Phase Space Diagram](image)

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Correction of Nonlinearities by “Linearizers”

- Graph showing energy vs. time with a line indicating the Chicane Exit.
- Two graphs showing current vs. voltage for different values, labeled (a) and (b).
Summary

We have shown how proper manipulation of the longitudinal phase space can lead to accelerators with superior beam characteristics.

We have shown how phase space tends to be degraded by generation of “curvatures” in longitudinal phase space, and a means to quantify such effects.

In this lecture and the preceding one, we’ve discussed some of the ways that people have combated this effect through (1) proper RF phase choices, (2) adding sextupoles in recirculation optics, and (3) RF linearization cavities.