

USPAS Course on
**Recirculated and Energy
Recovered Linear Accelerators**

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Lecture 6



Lecture Outline

- Invariant Ellipses Generated by a Unimodular Linear Transformations
- Detailed Solution of Hill's Equation
 - General Formula for Phase Advance
 - Transfer Matrix in Terms of β -function
 - Periodic Solutions
- Non-periodic Solutions
 - Formulas for β -function and Phase Advance
- Dispersion Calculations
- Beam Matching



Significance of matrix parameters

Another way to interpret the parameters α , β , and γ , which represent the unimodular matrix M (these parameters are sometimes called the Twiss parameters or Twiss representation for the matrix) is as the “coordinates” of that specific set of ellipses that are mapped onto each other, or are invariant, under the linear action of the matrix. This result is demonstrated in

Thm: For the unimodular linear transformation

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

with $|\text{Tr}(M)| < 2$, the ellipses



$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$

are invariant under the linear action of M , where c is any constant. Furthermore, these are the only invariant ellipses. Note that the theorem does not apply to $\pm I$, because $|\text{Tr}(\pm I)| = 2$.

Pf: The inverse to M is clearly

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) - \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

By the ellipse transformation formulas, for example

$$\begin{aligned} \beta' &= \beta^2 (\sin^2 \mu) \gamma + 2(-\beta \sin \mu)(\cos \mu + \alpha \sin \mu)\alpha + (\cos \mu + \alpha \sin \mu)^2 \beta \\ &= \beta \sin^2 \mu (1 + \alpha^2) - 2\beta \alpha^2 \sin^2 \mu + \beta \cos^2 \mu + \beta \alpha^2 \sin^2 \mu \\ &= (\sin^2 \mu + \cos^2 \mu) \beta = \beta \end{aligned}$$



Similar calculations demonstrate that $\alpha' = \alpha$ and $\gamma' = \gamma$. As $\det(M) = 1$, $c' = c$, and therefore the ellipse is invariant. Conversely, suppose that an ellipse is invariant. By the ellipse transformation formula, the specific ellipse

$$\gamma_i x^2 + 2\alpha_i xy + \beta_i y^2 = \varepsilon$$

is invariant under the transformation by M only if

$$\begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} (\cos \mu - \alpha \sin \mu)^2 & 2(\cos \mu - \alpha \sin \mu)(\gamma \sin \mu) & (\gamma \sin \mu)^2 \\ -(\cos \mu - \alpha \sin \mu)(\beta \sin \mu) & 1 - 2\beta\gamma \sin^2 \mu & (\cos \mu + \alpha \sin \mu)(\gamma \sin \mu) \\ (\beta \sin \mu)^2 & -2(\cos \mu + \alpha \sin \mu)(\beta \sin \mu) & (\cos \mu + \alpha \sin \mu)^2 \end{pmatrix} \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix}$$

$$\equiv T_M \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} \equiv T_M \vec{v},$$



i.e., if the vector \vec{v} is *ANY* eigenvector of T_M with eigenvalue 1. All possible solutions may be obtained by investigating the eigenvalues and eigenvectors of T_M . Now

$$T_M \vec{v}_\lambda = \lambda \vec{v}_\lambda \quad \text{has a solution when } \text{Det}(T_M - \lambda I) = 0$$

i.e.,

$$\left(\lambda^2 + [2 - 4 \cos^2 \mu] \lambda + 1 \right) (1 - \lambda) = 0$$

Therefore, M generates a transformation matrix T_M with at least one eigenvalue equal to 1. For there to be more than one solution with $\lambda = 1$,

$$1 + [2 - 4 \cos^2 \mu] + 1 = 0, \quad \cos^2 \mu = 1, \quad \text{or } M = \pm I$$



and we note that *all* ellipses are invariant when $M = \pm I$. But, these two cases are excluded by hypothesis. Therefore, M generates a transformation matrix T_M which always possesses a single nondegenerate eigenvalue 1; the set of eigenvectors corresponding to the eigenvalue 1, all proportional to each other, are the only vectors whose components $(\gamma_i, \alpha_i, \beta_i)$ yield equations for the invariant ellipses. For concreteness, compute that eigenvector with eigenvalue 1 normalized so $\beta_i \gamma_i - \alpha_i^2 = 1$

$$\vec{v}_{1,i} = \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} = \beta \begin{pmatrix} -M_{21} / M_{12} \\ (M_{11} - M_{22}) / 2M_{12} \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix}$$

All other eigenvectors with eigenvalue 1 have $\vec{v}_1 = \varepsilon \vec{v}_{1,i} / c$, for some value c .



Because $\text{Det}(M) = 1$, the eigenvector $\vec{v}_{1,i}$ clearly yields the invariant ellipse

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon.$$

Likewise, the proportional eigenvector \vec{v}_1 generates the similar ellipse

$$\frac{\varepsilon}{c} (\gamma x^2 + 2\alpha xy + \beta y^2) = \varepsilon$$

Because we have enumerated all possible eigenvectors with eigenvalue 1, all ellipses invariant under the action of M , are of the form

$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$



To summarize, this theorem gives a way to tie the mathematical representation of a unimodular matrix in terms of its α , β , and γ , and its phase advance, to the equations of the ellipses invariant under the matrix transformation. The equations of the invariant ellipses when properly normalized have precisely the same α , β , and γ as in the Twiss representation of the matrix, but varying c .

Finally note that throughout this calculation c acts merely as a scale parameter for the ellipse. All ellipses similar to the starting ellipse, i.e., ellipses whose equations have the same α , β , and γ , but with different c , are also invariant under the action of M .

Later, it will be shown that more generally

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + (\beta x' + \alpha x)^2 \right) / \beta$$

is an invariant of the equations of transverse motion.



Applications to transverse beam optics

When the motion of particles in transverse phase space is considered, linear optics provides a good first approximation of the transverse particle motion. Beams of particles are represented by ellipses in phase space (i.e. in the (x, x') space). To the extent that the transverse forces are linear in the deviation of the particles from some pre-defined central orbit, the motion may be analyzed by applying ellipse transformation techniques.

Transverse Optics Conventions: positions are measured in terms of length and angles are measured by radian measure. The area in phase space divided by π , ε , measured in m-rad, is called the emittance. In such applications, α has no units, β has units m/radian. Codes that calculate β , by widely accepted convention, drop the per radian when reporting results, it is implicit that the units for x' are radians.



Definition of the Linear Transport Matrix

Within a linear optics description of transverse particle motion, the particle transverse coordinates at a location s along the beam line are described by a vector

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

If the differential equation giving the evolution of x is linear, one may define a linear transport matrix $M_{s',s}$ relating the coordinates at s' to those at s by

$$\begin{pmatrix} x(s') \\ \frac{dx}{ds}(s') \end{pmatrix} = M_{s',s} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$



From the definitions, the concatenation rule $M_{s'',s} = M_{s'',s'}M_{s',s}$ must apply for all s' such that $s < s' < s''$ where the multiplication is the usual matrix multiplication.

Pf: The equations of motion, linear in x and dx/ds , generate a motion with

$$M_{s'',s} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} x(s'') \\ \frac{dx}{ds}(s'') \end{pmatrix} = M_{s'',s'} \begin{pmatrix} x(s') \\ \frac{dx}{ds}(s') \end{pmatrix} = M_{s'',s'} M_{s',s} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

for all initial conditions $(x(s), dx/ds(s))$, thus $M_{s'',s} = M_{s'',s'}M_{s',s}$.

Clearly $M_{s,s} = I$. As is shown next, the matrix $M_{s',s}$ is in general a member of the unimodular subgroup of the general linear group.



Ellipse Transformations Generated by Hill's Equation

The equation governing the linear transverse dynamics in a particle accelerator, without acceleration, is *Hill's equation**

$$\frac{d^2 x}{ds^2} + K(s)x = 0 \quad \text{Eqn. (2)}$$

The transformation matrix taking a solution through an infinitesimal distance ds is

$$\begin{pmatrix} x(s+ds) \\ \frac{dx}{ds}(s+ds) \end{pmatrix} = \begin{pmatrix} 1 & ds \\ -K(s)ds \text{ rad} & 1 \end{pmatrix} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} \equiv M_{s+ds,s} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

* Strictly speaking, Hill studied Eqn. (2) with periodic K . It was first applied to circular accelerators which had a periodicity given by the circumference of the machine. It is a now standard in the field of beam optics, to still refer to Eqn. 2 as Hill's equation, even in cases, as in linear accelerators, where there is no periodicity.



Suppose we are given the phase space ellipse

$$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = \varepsilon$$

at location s , and we wish to calculate the ellipse parameters, after the motion generated by Hill's equation, at the location $s + ds$

$$\gamma(s + ds)x^2 + 2\alpha(s + ds)xx' + \beta(s + ds)x'^2 = \varepsilon'$$

Because, to order linear in ds , $\text{Det } M_{s+ds,s} = 1$, at all locations s , $\varepsilon' = \varepsilon$, and thus the phase space area of the ellipse after an infinitesimal displacement must equal the phase space area before the displacement. Because the transformation through a finite interval in s can be written as a series of infinitesimal displacement transformations, all of which preserve the phase space area of the transformed ellipse, we come to two important conclusions:



1. The phase space area is preserved after a finite integration of Hill's equation to obtain $M_{s',s}$, the transport matrix which can be used to take an ellipse at s to an ellipse at s' . This conclusion holds generally for all s' and s .
2. Therefore $\text{Det } M_{s',s} = 1$ for all s' and s , independent of the details of the functional form $K(s)$. (If desired, these two conclusions may be verified more analytically by showing that

$$\frac{d}{ds}(\beta\gamma - \alpha^2) = 0 \quad \rightarrow \quad \beta(s)\gamma(s) - \alpha^2(s) = 1, \quad \forall s$$

may be derived directly from Hill's equation.)



Evolution equations for the α , β functions

The ellipse transformation formulas give, to order linear in ds

$$\beta(s + ds) = -2\alpha \frac{ds}{\text{rad}} + \beta(s)$$

$$\alpha(s + ds) = -\gamma(s) \frac{ds}{\text{rad}} + \alpha(s) + \beta(s)K ds \text{ rad}$$

So

$$\frac{d\beta}{ds}(s) = -\frac{2\alpha(s)}{\text{rad}}$$

$$\frac{d\alpha}{ds}(s) = \beta(s)K \text{ rad} - \frac{\gamma(s)}{\text{rad}}$$



Note that these two formulas are independent of the scale of the starting ellipse ε , and in theory may be integrated directly for $\beta(s)$ and $\alpha(s)$ given the focusing function $K(s)$. A somewhat easier approach to obtain $\beta(s)$ is to recall that the maximum extent of an ellipse, x_{\max} , is $(\varepsilon\beta)^{1/2}(s)$, and to solve the differential equation describing its evolution. The above equations may be combined to give the following non-linear equation for $x_{\max}(s) = w(s) = (\varepsilon\beta)^{1/2}(s)$

$$\frac{d^2 w}{ds^2} + K(s) w = \frac{(\varepsilon / \text{rad})^2}{w^3}.$$

Such a differential equation describing the evolution of the maximum extent of an ellipse being transformed is known as an *envelope equation*.



It should be noted, for consistency, that the same $\beta(s) = w^2(s)/\varepsilon$ is obtained if one starts integrating the ellipse evolution equation from a different, but similar, starting ellipse. That this is so is an exercise.

The envelope equation may be solved with the correct boundary conditions, to obtain the β -function. α may then be obtained from the derivative of β , and γ by the usual normalization formula. Types of boundary conditions: Class I—periodic boundary conditions suitable for circular machines or periodic focusing lattices, Class II—initial condition boundary conditions suitable for linacs or recirculating machines.



Solution to Hill's Equation in Amplitude-Phase form

To get a more general expression for the phase advance, consider in more detail the single particle solutions to Hill's equation

$$\frac{d^2 x}{ds^2} + K(s)x = 0$$

From the theory of linear ODEs, the general solution of Hill's equation can be written as the sum of the two linearly independent pseudo-harmonic functions

$$x(s) = Ax_+(s) + Bx_-(s)$$

where

$$x_{\pm}(s) = w(s)e^{\pm i\mu(s)}$$



are two particular solutions to Hill's equation, provided that

$$\frac{d^2 w}{ds^2} + K(s)w = \frac{c^2}{w^3} \quad \text{and} \quad \frac{d\mu}{ds}(s) = \frac{c}{w^2(s)}, \quad \text{Eqns. (3)}$$

and where A , B , and c are constants (in s)

That specific solution with boundary conditions $x(s_1) = x_1$ and $dx/ds(s_1) = x'_1$ has

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} w(s_1)e^{i\mu(s_1)} & w(s_1)e^{-i\mu(s_1)} \\ \left[w'(s_1) + \frac{ic}{w(s_1)} \right] e^{i\mu(s_1)} & \left[w'(s_1) - \frac{ic}{w(s_1)} \right] e^{-i\mu(s_1)} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$



Therefore, the unimodular transfer matrix taking the solution at $s = s_1$ to its coordinates at $s = s_2$ is

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \frac{w(s_2)}{w(s_1)} \cos \Delta\mu_{s_2, s_1} - \frac{w(s_2)w'(s_1)}{c} \sin \Delta\mu_{s_2, s_1} & \frac{w(s_2)w(s_1)}{c} \sin \Delta\mu_{s_2, s_1} \\ -\frac{c}{w(s_2)w(s_1)} \left[1 + \frac{w(s_2)w'(s_2)w(s_1)w'(s_1)}{c^2} \right] \sin \Delta\mu_{s_2, s_1} & \frac{w(s_1)}{w(s_2)} \cos \Delta\mu_{s_2, s_1} + \frac{w'(s_2)w(s_1)}{c} \sin \Delta\mu_{s_2, s_1} \\ -\left[\frac{w'(s_1)}{w(s_2)} - \frac{w'(s_2)}{w(s_1)} \right] \cos \Delta\mu_{s_2, s_1} & \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

where

$$\Delta\mu_{s_2, s_1} = \mu(s_2) - \mu(s_1) = \int_{s_1}^{s_2} \frac{c}{w^2(s)} ds$$



Case I: $K(s)$ periodic in s

Such boundary conditions, which may be used to describe circular or ring-like accelerators, or periodic focusing lattices, have $K(s + L) = K(s)$. L is either the machine circumference or period length of the focusing lattice.

It is natural to assume that there exists a unique periodic solution $w(s)$ to Eqn. (3a) when $K(s)$ is periodic. Here, we will assume this to be the case. Later, it will be shown how to construct the function explicitly. Clearly for w periodic

$$\phi(s) = \mu(s) - \mu_L s \quad \text{with} \quad \mu_L = \int_s^{s+L} \frac{c}{w^2(s)} ds$$

is also periodic by Eqn. (3b), and μ_L is independent of s .



The transfer matrix for a single period reduces to

$$\begin{pmatrix} \cos \mu_L - \frac{w(s)w'(s)}{c} \sin \mu_L & \frac{w^2(s)}{c} \sin \mu_L \\ -\frac{c}{w^2(s)} \left[1 + \frac{w(s)w'(s)w(s)w'(s)}{c^2} \right] \sin \mu_L & \cos \mu_L + \frac{w'(s)w(s)}{c} \sin \mu_L \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu_L) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu_L)$$

where the (now periodic!) matrix functions are

$$\alpha(s) = -\frac{w(s)w'(s)}{c}, \quad \beta(s) = \frac{w^2(s)}{c}, \quad \gamma(s) = \frac{1 + \alpha^2(s)}{\beta(s)}$$

By Thm. (2), these are the ellipse parameters of the periodically repeating, i.e., *matched* ellipses.



General formula for phase advance

In terms of the β -function, the phase advance for the period is

$$\mu_L = \int_0^L \frac{ds}{\beta(s)}$$

and more generally the phase advance between any two longitudinal locations s and s' is

$$\Delta\mu_{s',s} = \int_s^{s'} \frac{ds}{\beta(s)}$$



Transfer Matrix in terms of α and β

Also, the unimodular transfer matrix taking the solution from s to s' is

$$M_{s',s} = \begin{pmatrix} \sqrt{\frac{\beta(s')}{\beta(s)}} (\cos \Delta\mu_{s',s} + \alpha(s) \sin \Delta\mu_{s',s}) & \sqrt{\beta(s')\beta(s)} \sin \Delta\mu_{s',s} \\ -\frac{1}{\sqrt{\beta(s')\beta(s)}} \begin{bmatrix} (1 + \alpha(s')\alpha(s)) \sin \Delta\mu_{s',s} \\ + (\alpha(s') - \alpha(s)) \cos \Delta\mu_{s',s} \end{bmatrix} & \sqrt{\frac{\beta(s)}{\beta(s')}} (\cos \Delta\mu_{s',s} - \alpha(s') \sin \Delta\mu_{s',s}) \end{pmatrix}$$

Note that this final transfer matrix and the final expression for the phase advance do not depend on the constant c . This conclusion might have been anticipated because different particular solutions to Hill's equation exist for all values of c , but from the theory of linear ordinary differential equations, the final motion is unique once x and dx/ds are specified somewhere.



One method to compute the β -function

Our previous work has indicated a method to compute the β -function (and thus w) directly, i.e., without solving the differential equation Eqn. (3). At a given location s , determine the one-period transfer map $M_{s+L,s}(s)$. From this find μ_L (which is independent of the location chosen!) from $\cos \mu_L = (M_{11} + M_{22}) / 2$, and by choosing the sign of μ_L so that $\beta(s) = M_{12}(s) / \sin \mu_L$ is positive. Likewise, $\alpha(s) = (M_{11} - M_{22}) / 2 \sin \mu_L$. Repeat this exercise at every location the β -function is desired.

By construction, the beta-function and the alpha-function, and hence w , are periodic because the single-period transfer map is periodic. It is straightforward to show $w = (c\beta(s))^{1/2}$ satisfies the envelope equation.



Courant-Snyder Invariant

Consider now a single particular solution of the equations of motion generated by Hill's equation. We've seen that once a particle is on an invariant ellipse for a period, it must stay on that ellipse throughout its motion. Because the phase space area of the single period invariant ellipse is preserved by the motion, the quantity that gives the phase space area of the invariant ellipse in terms of the single particle orbit must also be an invariant. This phase space area/ π ,

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + (\beta x' + \alpha x)^2 \right) / \beta$$

is called the Courant-Snyder invariant. It may be verified to be a constant by showing its derivative with respect to s is zero by Hill's equation, or by explicit substitution of the transfer matrix solution which begins at some initial value $s = 0$.



Pseudoharmonic Solution

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}} (\cos \Delta\mu_{s,0} + \alpha_0 \sin \Delta\mu_{s,0}) & \sqrt{\beta(s)\beta_0} \sin \Delta\mu_{s,0} \\ -\frac{1}{\sqrt{\beta(s)\beta_0}} \left[(1 + \alpha(s)\alpha_0) \sin \Delta\mu_{s,0} + (\alpha(s) - \alpha_0) \cos \Delta\mu_{s,0} \right] & \sqrt{\frac{\beta_0}{\beta(s)}} (\cos \Delta\mu_{s,0} - \alpha(s) \sin \Delta\mu_{s,0}) \end{pmatrix} \begin{pmatrix} x_0 \\ \left. \frac{dx}{ds} \right|_0 \end{pmatrix}$$

gives

$$\left(x^2(s) + (\beta(s)x'(s) + \alpha(s)x(s))^2 \right) / \beta(s) = \left(x_0^2 + (\beta_0 x'_0 + \alpha_0 x_0)^2 \right) / \beta_0 \equiv \varepsilon$$

Using the $x(s)$ equation above and the definition of ε , the solution may be written in the standard “pseudoharmonic” form

$$x(s) = \sqrt{\varepsilon\beta(s)} \cos(\Delta\mu_{s,0} - \delta) \quad \text{where} \quad \delta = \tan^{-1} \left(\frac{\beta_0 x'_0 + \alpha_0 x_0}{x_0} \right)$$

The the origin of the terminology “phase advance” is now obvious.



Case II: $K(s)$ not periodic

In a linac or a recirculating linac there is no closed orbit or natural machine periodicity. Designing the transverse optics consists of arranging a focusing lattice that assures the beam particles coming into the front end of the accelerator are accelerated (and sometimes decelerated!) with as small beam loss as is possible. Therefore, it is imperative to know the initial beam phase space injected into the accelerator, in addition to the transfer matrices of all the elements making up the focusing lattice of the machine. An initial ellipse, or a set of initial conditions that somehow bound the phase space of the injected beam, are tracked through the acceleration system element by element to determine the transmission of the beam through the accelerator. The designs are usually made up of well-understood “modules” that yield known and understood transverse beam optical properties.



Definition of β function

Now the pseudoharmonic solution applies even when $K(s)$ is not periodic. Suppose there is an ellipse, the design injected ellipse, which tightly includes the phase space of the beam at injection to the accelerator. Let the ellipse parameters for this ellipse be α_0 , β_0 , and γ_0 . A function $\beta(s)$ is simply defined by the ellipse transformation rule

$$\begin{aligned}\beta(s) &= (M_{12}(s))^2 \gamma_0 - 2M_{12}(s)M_{11}(s)\alpha_0 + (M_{11}(s))^2 \beta_0 \\ &= \left[(M_{12}(s))^2 + (\beta_0 M_{11}(s) - \alpha_0 M_{12}(s))^2 \right] / \beta_0\end{aligned}$$

where

$$M_{s,0} \equiv \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix}$$



One might think to evaluate the phase advance by integrating the beta-function. Generally, it is far easier to evaluate the phase advance using the general formula,

$$\tan \Delta\mu_{s',s} = \frac{(M_{s',s})_{12}}{\beta(s)(M_{s',s})_{11} - \alpha(s)(M_{s',s})_{12}}$$

where $\beta(s)$ and $\alpha(s)$ are the ellipse functions at the entrance of the region described by transport matrix $M_{s',s}$. Applied to the situation at hand yields

$$\tan \Delta\mu_{s,0} = \frac{M_{12}(s)}{\beta_0 M_{11}(s) - \alpha_0 M_{12}(s)}$$



Dispersion Calculation

Begin with the inhomogeneous Hill's equation for the dispersion.

$$\frac{d^2 D}{ds^2} + K(s)D = -\frac{1}{\rho(s)}$$

Write the general solution to the inhomogeneous equation for the dispersion as before.

$$D(s) = D_p(s) + Ax_1(s) + Bx_2(s)$$

Here D_p can be any particular solution. Suppose that the dispersion and its derivative are known at the location s_1 , and we wish to determine their values at s_2 . x_1 and x_2 , because they are solutions to the homogeneous equations, must be transported by the transfer matrix solution M_{s_2, s_1} already found.



To build up the general solution, choose that particular solution of the inhomogeneous equation with boundary conditions

$$D_{p,0}(s_1) = D'_{p,0}(s_1) = 0$$

Evaluate A and B by the requirement that the dispersion and its derivative have the proper value at s_1 (x_1 and x_2 need to be linearly independent!)

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} x_1(s_1) & x_2(s_1) \\ x'_1(s_1) & x'_2(s_1) \end{pmatrix}^{-1} \begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix}$$

$$D(s_2) = D_{p,0}(s_2) + \left(M_{s_2,s_1}\right)_{11} D(s_1) + \left(M_{s_2,s_1}\right)_{12} D'(s_1)$$

$$D'(s_2) = D'_{p,0}(s_2) + \left(M_{s_2,s_1}\right)_{21} D(s_1) + \left(M_{s_2,s_1}\right)_{22} D'(s_1)$$



3 by 3 Matrices for Dispersion Tracking

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} (M_{s_2,s_1})_{11} & (M_{s_2,s_1})_{12} & D_{p,0}(s_2) \\ (M_{s_2,s_1})_{21} & (M_{s_2,s_1})_{22} & D'_{p,0}(s_2) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}$$

Particular solutions to inhomogeneous equation for constant K and constant ρ and vanishing dispersion and derivative at $s = 0$

	$K < 0$	$K = 0$	$K > 0$
$D_{p,0}(s)$	$\frac{1}{ K \rho} (1 - \cosh(\sqrt{ K s}))$	$-\frac{s^2}{2\rho}$	$\frac{1}{K\rho} (\cos(\sqrt{K}s) - 1)$
$D'_{p,0}(s)$	$-\frac{1}{\sqrt{ K \rho}} \sinh(\sqrt{ K s})$	$-\frac{s}{\rho}$	$-\frac{1}{\sqrt{K\rho}} \sin(\sqrt{K}s)$



Longitudinal Stability in Detail

For the microtron

$$K_i = 1 / \rho_i^2$$

$$D_{x,p,0} = \rho_i (1 - \cos(s / \rho_i)) \quad 0 \leq s \leq 2\pi\rho_i$$

$$M_{56} = \int_0^{2\pi\rho_i} (1 - \cos(s / \rho_i)) ds = 2\pi\rho_i$$



$$M_{long} = \begin{pmatrix} 1 & -\frac{M_{56}}{cE_i} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2\pi f_{rf} G \sin \phi_s & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 - \frac{4\pi^2 \rho_i f_{rf}}{c\gamma_i} \Delta\gamma \tan \phi_s & -\frac{2\pi\rho_i}{cE_i} \\ 2\pi f_{rf} G \sin \phi_s & 1 \end{pmatrix}$$

$$\Delta\gamma mc^2 = G \cos \phi_s$$

$$|\text{Tr}(M_{long})| < 2 \rightarrow 0 < v\pi \tan \phi_s < 2$$

Same result for racetrack microtron if no optics or identity optics in the return straights



General Polytron

$$M_{56} = \int_0^{2\pi\rho_i/n} (1 - \cos(s/\rho_i)) ds = 2\pi\rho_i \left\{ 1 - (2n/2\pi) \sin(2\pi/2n) \right\} / n$$

Full one turn map, assuming synchronous phase same for all linacs is M_{long}^n

$$M_{long} = \begin{pmatrix} 1 - \frac{4\pi^2 \rho_i f_{rf}}{c\gamma_i n^2} \left\{ 1 - S(2n/2\pi) \right\} \Delta\gamma \tan \phi_s & -\frac{2\pi\rho_i}{cE_i} \left\{ 1 - S(2n/2\pi) \right\} \\ 2\pi f_{rf} \left(\frac{G}{n} \right) \sin \phi_s & 1 \end{pmatrix}$$

$$S(x) \equiv \sin x / x$$

If ν denotes the full one turn path length increment

$$0 < \nu\pi \tan \phi_s / n^2 < 2$$

$$0 < l\pi \tan \phi_s / n < 2$$



Beam Matching

Fundamentally, in circular accelerators beam matching is applied in order to guarantee that the beam envelope of the real accelerator beam does not depend on time. This requirement is one part of the definition of having a stable beam. With periodic boundary conditions, this means making beam density contours in phase space align with the invariant ellipses (in particular at the injection location!) given by the ellipse functions. Once the particles are on the invariant ellipses they stay there (in the linear approximation!), and the density is preserved because the single particle motion is around the invariant ellipses. In linacs and recirculating linacs, usually different purposes are to be achieved. If there are regions with periodic focusing lattices within the linacs, matching as above ensures that the beam



envelope does not grow going down the lattice. Sometimes it is advantageous to have specific values of the ellipse functions at specific longitudinal locations. Other times, re/matching is done to preserve the beam envelopes of a good beam solution as changes in the lattice are made to achieve other purposes, e.g. changing the dispersion function or changing the chromaticity of regions where there are bends (see the next chapter for definitions). At a minimum, there is usually a matching done in the first parts of the injector, to take the phase space that is generated by the particle source, and change this phase space in a way towards agreement with the nominal transverse focusing design of the rest of the accelerator. The ellipse transformation formulas, solved by computer, are essential for performing this process.

