Homework Problems II

1. Write down the Hamilton-Jacobi equation for the harmonic oscillator. Assuming an action function of the form

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t$$

solve the Hamilton-Jacobi equation for the motion of the harmonic oscillator

$$q(t) = \sqrt{\frac{2\alpha}{k}} \sin \omega(t+\beta)$$

Note the constant β follows directly from

$$\beta = \frac{\partial S}{\partial \alpha}$$

2. To illustrate the use of the Hamilton-Jacobi method to eliminate cyclic variables, use the Hamilton-Jacobi equation for gravitational motion in the plane with Hamiltonian

$$H(p_r, p_\theta, r) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{G}{r}$$

to show the orbit is

$$\frac{1}{r(\theta)} = \frac{1}{r_0}\left(1 + \varepsilon \cos(\theta - \theta_0)\right)$$

One should assume that

$$S = W(r, \alpha, p_{\theta 0}) - \alpha t + p_{\theta 0}\theta$$

Hint: in this problem $\partial S / \partial p_{\theta 0}$ is a constant of the motion.

3. Solve the Hamilton-Jacobi Equation with the lab-frame boundary conditions

$$\dot{x} = \dot{y} = 0, \qquad \dot{z} = \beta_z c \qquad \text{as} \qquad t \to -\infty$$
$$x = y = 0, \qquad z = \beta_z ct \qquad \text{as} \qquad t \to -\infty$$

Verify the Lorentz Transformation Formulas hold between this solution and the beam-frame solution used in the lectures. With this solution, verify Eqns. 3.9 by a lab-frame computation with the standard formula from Jackson

$$\frac{d^2 E}{d\omega d\Omega} = \frac{e^2\omega^2}{8\pi^2 c}\left|\int_{-\infty}^{\infty} \vec{n} \times \left(\vec{n} \times \vec{\beta}(t)\right) e^{i\omega(t - \vec{n}\cdot\vec{r}(t)/c)} dt\right|^2$$

4. Utilizing the fact that the delta-function in the retarded Green function of the wave equation picks out

$$t - t(\xi) = R / c = \sqrt{(x - x(\xi))^2 + (y - y(\xi))^2 + (z - z(\xi))^2} \,/\, c,$$

the Lorentz transformation formula for the coordinates, and the beam-frame and lab-frame solutions to the Hamilton-Jacobi equation, demonstrate that

$$R' = R\gamma(1 - \beta_z \cos\theta)$$

To Lorentz transform the field, this formula is needed