USPAS Course on

4^{th} Generation Light Sources II
ERLs and Thomson Scattering

G. A. Krafft and I. V. Bazarov

Jefferson Lab and Cornell University

Overview of Transverse Dynamics in Transport Lines & RF Structures

CHESS / LEPP

Optics Overview

1. First-Order Optics
   1. Matrix Formalism for Changing Energy
   2. Higher-Order Terms
2. Quantum Excitation in Transport Lines
   1. Transverse Emittance
   2. Energy Spread

Summary and Scope

Purpose of this lecture is to present a brief review of beam transport matrix formalism and concepts, paying attention to special cases of particular importance to linear accelerators.

Effects of spontaneous synchrotron radiation on beam phase-space density will be briefly reviewed.

This is not meant to be a compete tutorial of beam optics formalism. For thorough treatment refer to

1) K.L. Brown, R.V. Servranckx, "1st- and 2nd-order charged particle optics", SLAC-PUB-3381, July 1984

2) R.H. Helm, R. Miller, "Particle Dynamics", in Linear Accelerators, ed. P.M. Lapostolle and A.L. Septier, 1970

3) M. Sands, SLAC-121, November 1970

First-Order Optics

The deviation of an arbitrary trajectory from the central trajectory is given by 6-D vector:

$$\mathbf{X}^t(s) = [x(s), x'(s), y(s), y'(s), l(s), \delta(s)]$$

This, along with knowing the central trajectory and its momentum $P_0(s)$, fully describes the system.

Vector $\mathbf{X}(a)$ at position *a* is transformed to $\mathbf{X}(b)$ at position *b* by

$$X_i(b) = \sum_{j=1}^{6} R_{ij} X_j(a) + \sum_{j,k=1}^{6} T_{ijk} X_j(a) X_k(a) + \sum_{j,k,l=1}^{6} U_{ijkl} X_j(a) X_k(a) X_l(a) + \dots$$

The above is essentially Taylor expansion. First-order optics formalism is derived by neglecting higher-order terms in equation of motion and retaining only the first term in the expansion. Higher-order matrix elements are obtained by including other significant terms, and can be represented as a combination of cosine, sine and off-momentum (dispersion) trajectories of the first-order Taylor expansion, their derivatives and multipole strengths of external magnetic fields

Transfer Matrix for Decoupled Motion

If all magnets have mid-plane symmetry about $y = 0$, then the first-order transfer matrix simplifies to

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_x(s) & s_x(s) & 0 & 0 & 0 & d_x(s) \\ c'_x(s) & s'_x(s) & 0 & 0 & 0 & d'_x(s) \\ 0 & 0 & c_y(s) & s_y(s) & 0 & 0 \\ 0 & 0 & c'_y(s) & s'_y(s) & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Beam can be represented by Σ-matrix, which is 6×6 matrix in most general case, but for decoupled motion it could be represented by three 2×2 matrices, e.g.:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

, beam ellipse equation at location a: $\mathbf{x}_a^t \boldsymbol{\sigma}_a^{-1} \mathbf{x}_a = 1$

with the corresponding area: $A_a = \pi \sqrt{|\boldsymbol{\sigma}_a|}$

Changing Energy Case

Transformation rule for σ-matrix:

$$\boldsymbol{\sigma}_b = \mathbf{R}\boldsymbol{\sigma}_a\mathbf{R}^t$$

Liouville's theorem requires that the phase space area be conserved

$$A_b = \pi\sqrt{|\boldsymbol{\sigma}_b|} = \pi\sqrt{|\mathbf{R}\boldsymbol{\sigma}_a\mathbf{R}^t|} = const = \pi\varepsilon$$

$$\Rightarrow \quad |\mathbf{R}|^2 = 1, \quad \text{i.e.} \quad |\mathbf{R}| = 1$$

For changing energy case, one has to redefine transformation in terms of conjugate pairs of variables $\{x, p_z x'\}$ and $\{y, p_z y'\}$

$$\begin{bmatrix} x_b \\ x'_b p_b \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12}/p_a \\ R_{21}p_b & R_{22}p_b/p_a \end{bmatrix}\begin{bmatrix} x_a \\ x'_a p_a \end{bmatrix} \quad \text{or} \quad \sqrt{p_b}\begin{bmatrix} x_b \\ x'_b \end{bmatrix} = \begin{bmatrix} \rho R_{11} & \rho R_{12} \\ \rho R_{21} & \rho R_{22} \end{bmatrix}\sqrt{p_a}\begin{bmatrix} x_a \\ x'_a \end{bmatrix}$$

$$\rho = \sqrt{p_b/p_a}$$

Now invariant of motion is normalized phase space area

$A_n = \pi p\sqrt{|\boldsymbol{\sigma}|}$, normalized emittance defined as $\varepsilon_n = \beta\gamma \cdot \varepsilon$

CORNELL UNIVERSITY
CHESS / LEPP

Courant-Snyder Notation

Initially introduced for a periodic case of a synchrotron or storage ring. E.g. consider stability condition for periodic lattice:

$$\begin{bmatrix} x_{n+1} \\ x'_{n+1} \end{bmatrix} = \mathbf{M} \begin{bmatrix} x_n \\ x'_n \end{bmatrix}$$

Transfer matrix, **M**, is the same for all orbits and |**M**| = 1, thus,

$$x_{n+1} - \mathrm{Tr}(\mathbf{M}) x_n + x_{n-1} = 0$$

General solution is of the form:

$$x_n = a_1 \lambda_1^n + a_2 \lambda_2^n$$

here $a_{1,2}$ are constants and $\lambda_{1,2}$ are the roots of characteristic equation

$$\lambda^2 - \mathrm{Tr}(\mathbf{M})\lambda + 1 = 0$$

Motion is stable when eigenvalues are of the form $\lambda_{1,2} = \mathrm{e}^{\pm i \mu_C}$, which is the case when

$|\mathrm{Tr}(\mathbf{M})| < 2,$ $\quad \mu$ is known as orbital betatron phase advance : $\cos \mu_C = \frac{1}{2} \mathrm{Tr}(\mathbf{M})$

Courant-Snyder Notation (contd.)

The most general form for matrix **M** is given by

$$\mathbf{M} = \begin{bmatrix} \cos\mu_C + \alpha\sin\mu_C & \beta\sin\mu_C \\ -\gamma\sin\mu_C & \cos\mu_C - \alpha\sin\mu_C \end{bmatrix} = \mathbf{I}\cos\mu_C + \mathbf{J}\sin\mu_C$$

where α, β and γ are Courant-Snyder parameters (also called Twiss parameters along with dispersion function), **I** is a unit matrix, and

$$\mathbf{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}, \text{ with } \mathrm{Tr}(\mathbf{J}) = 0,\ \mathbf{J}^2 = -\mathbf{I} \text{ or } \beta\gamma = 1 + \alpha^2$$

Alternatively, one can arrive at Courant-Snyder notation by considering solution to the (Hill's) equations of motion in phase-amplitude form:

$$x(s) = A\sqrt{\beta(s)}\cos(\mu(s) + \phi)$$

$$x'(s) = -\frac{A}{\sqrt{\beta(s)}}[\alpha(s)\cos(\mu(s) + \phi) + \sin(\mu(s) + \phi)]$$

β-function satisfies $2\beta\beta'' - \beta'^2 + 4\beta^2 K = 0;\ \mu' = 1/\beta,\ \alpha \equiv -\beta'/2,\ \gamma \equiv \dfrac{1+\alpha^2}{\beta}$

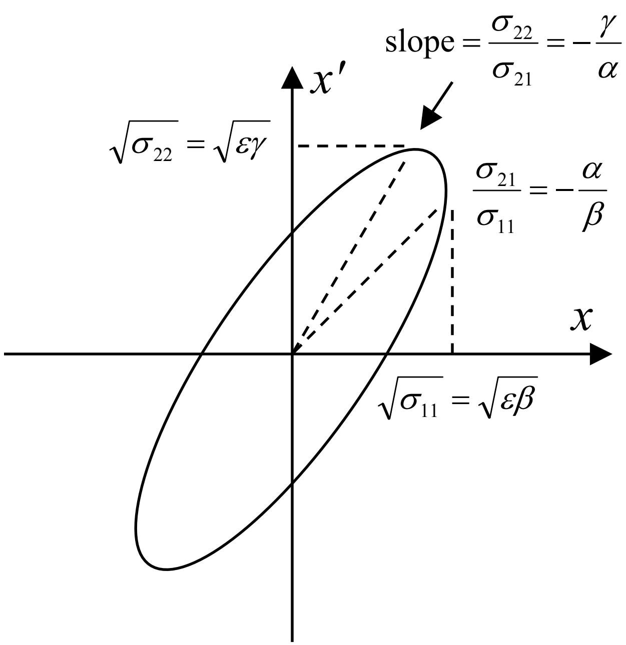
Courant-Snyder Representation of σ-matrix

Require that phase space ellipse is unchanged for a successive orbit

$$\mathbf{M}\,\boldsymbol{\sigma}\,\mathbf{M}^t = \boldsymbol{\sigma}$$

then σ-matrix has a form

$$\boldsymbol{\sigma} = \varepsilon\begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix}$$



One can write transformation rule for vector $[\beta, \alpha, \gamma]^t$

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_a = \frac{P_b}{P_a}\begin{bmatrix} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ -R_{11}R_{21} & R_{11}R_{22}+R_{12}R_{21} & -R_{12}R_{22} \\ R_{21}^2 & -2R_{21}R_{22} & R_{22}^2 \end{bmatrix}\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_a$$

Transfer Matrix for RF structure

At high enough energies, transverse momentum is conserved in the RF

$$(\beta\gamma)x' = const$$

Integrating the above one finds

$$x_b - x_a \approx \frac{x'_a \gamma_a}{\gamma'} \ln \frac{\gamma_b}{\gamma_a}$$

Corresponding transfer matrix

$$\mathbf{R} = \begin{bmatrix} 1 & \frac{\gamma_a}{\gamma'} \ln \frac{\gamma_b}{\gamma_a} \\ 0 & \frac{\gamma_a}{\gamma_b} \end{bmatrix},$$

as expected $|\mathbf{R}| = \gamma_a / \gamma_b$. This is a good approximation at higher energies, while at lower energies, azimuthal magnetic and radial electric fields of RF cause focusing:

$$\mathbf{R} = \begin{bmatrix} \cos A - \sqrt{2} \sin A & \sqrt{8} \frac{\gamma_a}{\gamma'} \sin A \\ -\frac{3\gamma'}{\sqrt{8}\gamma_b} \sin A & \frac{\gamma_a}{\gamma_b} (\cos A - \sqrt{2} \sin A) \end{bmatrix}, \quad A = (1/\sqrt{8}) \ln(\gamma_b / \gamma_a)$$

First-Order Path Length and Dispersion

To utilize small emittance beams, insertion devices have to be placed in places where dispersion and its derivative is zero. Thus, one ought to use achromatic transport systems. Furthermore, to utilize short bunches, one needs to control path length to be independent of particles' momentum, or linearly changing with momentum for bunch compression

Path length is given by $L = \int \sqrt{(1 + x/\rho)^2 + x'^2 + y'^2}\, ds$

$$l = \int_0^s x(\tau) h(\tau) d\tau + \text{higher order terms}$$

$$= l_0 + x_0 \int_0^s c_x(\tau) h(\tau) d\tau + x_0' \int_0^s s_x(\tau) h(\tau) d\tau + \delta \int_0^s d_x(\tau) h(\tau) d\tau = R_{51} x_0 + R_{52} x_0' + l_0 + R_{56} \delta$$

First-order dispersion

$$d_x = R_{16} = s_x \int_0^s c_x(\tau) h(\tau) d\tau - c_x \int_0^s s_x(\tau) h(\tau) d\tau = R_{12} R_{51} - R_{11} R_{52}$$

$$d_x' = R_{26} = s_x' \int_0^s c_x(\tau) h(\tau) d\tau - c_x' \int_0^s s_x(\tau) h(\tau) d\tau = R_{22} R_{51} - R_{21} R_{52}$$

Isochronous transport line has to be an achromat as well: $R_{51} = R_{52} = R_{56} = 0$

Second-Order Path Length

Generally, higher-order optics effects are harmful as they tend to distort phase space, making effective emittance larger. Second-order path length, however, can be used to correct longitudinal phase space after the linac.

Second-order "momentum compaction", similar to first-order matrix element R_{56} is defined as:

$$T_{566} = \frac{1}{2}\frac{\partial^2 l}{\partial \delta^2} \qquad \text{or} \qquad T_{566} = \int \left[\frac{d_{(2)}}{\rho} + \frac{d^2}{2\rho} + \frac{d'^2}{2} \right] ds$$

Usually, the most dominant term is the one with 2nd-order dispersion, $d_{(2)} = \frac{1}{2}\frac{\partial^2 \langle x \rangle}{\partial \delta^2}$

Differential equation for second-order dispersion can be found to be

$$d''_{(2)} + K(s) d_{(2)} = -h + k_1 d - \tfrac{1}{2} k_2 d^2 + (h^3 + 2k_1 h) d^2 + \tfrac{1}{2} h d'^2 + h' d' d + 2h^2 d$$

Quantum Excitation: Energy Spread

Electron beam emits synchrotron radiation as it is being bent in dipole magnets. Probability distribution of the number of photons emitted by a single electron is described by Poisson distribution, and in the approximation of large number of photons by Gaussian distribution.

E.g. N_{ph} photons are emitted with energy E_{ph}. Random walk growth of energy spread from the average is simply:

$$\sigma_E^2 = N_{ph} \cdot E_{ph}^2$$

If photons are emitted with spectral distribution of $N(E_{ph})$, then one has to integrate

$$\sigma_E^2 = \int E_{ph}^2 N_{ph}(E_{ph}) dE_{ph}$$

A. Energy spread growth in bends

$$\sigma_E^{\ 2} = \frac{55}{32\sqrt{3}\pi} C_\gamma \hbar c \left(mc^2\right)^4 \gamma^7 \int \frac{ds}{\rho^3} , \qquad (7)$$

Sand's radiation constant for e$^-$: $C_\gamma = \frac{4\pi r_c}{3\left(mc^2\right)^3} = 8.86 \cdot 10^{-5} \frac{\mathrm{m}}{\mathrm{GeV}^3}$.

For constant bending radius ρ and total angle Θ ($\Theta = 2\pi$ for a ring) energy spread becomes:

$$\frac{\sigma_E^{\ 2}}{E^2} = 2.6 \cdot 10^{-10} E^5 \left(\mathrm{GeV}^5\right) \frac{1}{\rho^2 \left(\mathrm{m}^2\right)} \frac{\Theta}{2\pi} . \qquad (8)$$

B. Energy spread growth in undulators

$$\sigma_E{}^2 = \int \varepsilon^2 N_{\mathrm{ph}}(\varepsilon) d\varepsilon \approx N_{\mathrm{ph}} \varepsilon_\gamma{}^2 , \qquad (9)$$

with $N_{\mathrm{ph}} = \dfrac{E_\gamma}{\varepsilon_\gamma} \approx 0.763 \dfrac{K^2\left(1+\frac{1}{2}K^2\right)}{\lambda_p(\mathrm{cm})} L_{\mathrm{u}}(\mathrm{m})$, where

$$\varepsilon_\gamma = \frac{2hc}{\lambda_{\mathrm{p}}} \frac{2\gamma^2}{\left(1+\frac{1}{2}K^2\right)}, \qquad \varepsilon_\gamma(\mathrm{eV}) = 950 \frac{E^2\left(\mathrm{GeV}^2\right)}{\lambda_{\mathrm{p}}(\mathrm{cm})\left(1+\frac{1}{2}K^2\right)}, \qquad (10)$$

$$E_\gamma = \frac{4\pi^2}{3} r_{\mathrm{c}} mc^2 \frac{1}{\lambda_{\mathrm{p}}{}^2} \gamma^2 K^2 L_{\mathrm{u}}, \qquad E_\gamma(\mathrm{eV}) = 725 \frac{E^2\left(\mathrm{GeV}^2\right) K^2}{\lambda_{\mathrm{p}}{}^2\left(\mathrm{cm}^2\right)} L_{\mathrm{u}}(\mathrm{m}). \quad (11)$$

$$\frac{\sigma_E{}^2}{E^2} = 7 \cdot 10^{-13} \frac{E^2\left(\mathrm{GeV}^2\right) K^2}{\lambda_{\mathrm{p}}{}^3\left(\mathrm{cm}^3\right)\left(1+\frac{1}{2}K^2\right)} L_{\mathrm{u}}(\mathrm{m}). \qquad (12)$$

Note: The radiation regime in ERL undulators should be far from SASE to keep energy spread from IDs as estimated above.

II. Transverse Emittance Growth

Consider transverse motion:

$$u = u_\beta + \eta \frac{\Delta E}{E}, \qquad u' = u'_\beta + \eta' \frac{\Delta E}{E}, \tag{14}$$

where $u_\beta(s) = a\sqrt{\beta(s)}e^{i\psi(s)}$.

Emission of a photon leads to:

$$\begin{aligned} \delta u &= 0 = \delta u_\beta + \eta \frac{\varepsilon}{E}, & \delta u_\beta &= -\eta \frac{\varepsilon}{E}, \\ \delta u' &= 0 = \delta u'_\beta + \eta' \frac{\varepsilon}{E}, & \delta u'_\beta &= -\eta' \frac{\varepsilon}{E}, \end{aligned} \tag{15}$$

with respective change of the phase ellipse $a^2 = \gamma u^2 + 2\alpha u u' + \beta u'^2$:

$$\delta\left(a^2\right) = \gamma\delta\left(u_\beta{}^2\right) + 2\alpha\delta\left(u_\beta u'_\beta\right) + \beta\delta\left(u'_\beta{}^2\right). \tag{16}$$

$$\left\langle \delta a^2 \right\rangle = \frac{\varepsilon^2}{E^2} H(s), \tag{17}$$

Emission of a photon leads to:

$$\delta u = 0 = \delta u_\beta + \eta \frac{\varepsilon}{E}, \qquad \delta u_\beta = -\eta \frac{\varepsilon}{E},$$
$$\delta u' = 0 = \delta u'_\beta + \eta' \frac{\varepsilon}{E}, \qquad \delta u'_\beta = -\eta' \frac{\varepsilon}{E}, \tag{15}$$

with respective change of the phase ellipse $a^2 = \gamma u^2 + 2\alpha u u' + \beta u'^2$:

$$\delta\left(a^2\right) = \gamma\delta\left(u_\beta{}^2\right) + 2\alpha\delta\left(u_\beta u'_\beta\right) + \beta\delta\left(u'_\beta{}^2\right). \tag{16}$$

$$\left\langle \delta a^2 \right\rangle_\psi = \frac{\varepsilon^2}{E^2} H(s), \tag{17}$$

here $H(s) = \beta\eta'^2 + 2\alpha\eta\eta' + \gamma\eta^2$. (18)

Emittance growth due to quantum excitation becomes:

$$\epsilon_u = \frac{\sigma_u{}^2}{\beta_u} = \frac{1}{2}\Delta\left\langle a^2 \right\rangle = \frac{1}{2cE^2}\int ds \int \varepsilon^2 \dot{N}_{\mathrm{ph}}(\varepsilon) H(s) d\varepsilon\,. \tag{19}$$

____

A. Emittance growth in bends

$$\epsilon_x = \frac{55 C_\gamma \hbar c \left(mc^2\right)^2}{32\pi\sqrt{3}} \gamma^5 \int \frac{H ds}{\rho}, \tag{20}$$

For isomagnetic ring:

$$\epsilon_x = \frac{55 C_\gamma \hbar c \left(mc^2\right)^2}{32\pi\sqrt{3}} \gamma^5 \Theta \frac{\langle H \rangle}{\rho^2}, \tag{21}$$

where $\langle H \rangle$ and ρ are the average value of H- function in a dipole and dipole bend radius respectfully.

B. Emittance growth in undulators

$$\epsilon \approx \frac{1}{2}\frac{\varepsilon_\gamma^{\ 2}}{E^2}\langle H\rangle N_{\mathrm{ph}}, \qquad (22)$$

$$\langle H\rangle = \frac{1}{L_\mathrm{u}}\int_0^{L_\mathrm{u}}(\beta\eta'^2 + 2\alpha\eta\eta' + \alpha\eta^2)ds. \qquad (23)$$

For sinusoidal undulator field $B(s) = B_0 \cos k_\mathrm{p} s$ differential equation for η:

$$\eta'' = \frac{1}{\rho} = \frac{1}{\rho_\mathrm{u}}\cos k_\mathrm{p} s, \qquad \text{here } k_\mathrm{p} = \frac{2\pi}{\lambda_\mathrm{p}}. \qquad (24)$$

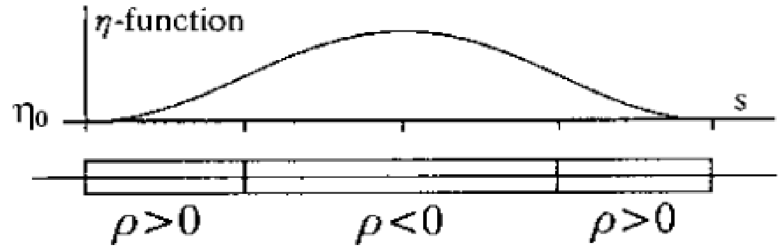


Fig. 2. Dispersion function in one period of a undulator magnet.

$$\eta(s) = \frac{1}{k_p{}^2 \rho_u}\left(1 - \cos k_p s\right) + \eta_0, \qquad \eta'(s) = \frac{1}{k_p \rho_u} \sin k_p s. \tag{25}$$

For an undulator located with the beam waist at its center (β^*) we have:

$$\begin{aligned} \langle H \rangle &= \frac{\beta^*}{2k_p{}^2 \rho_u{}^2}\left(1 + \frac{L_u{}^2}{12\beta^{*2}} + \frac{2\eta_0{}^2 k_p{}^2 \rho_u{}^2}{\beta^{*2}} + \frac{8\eta_0 \rho_u}{\beta^{*2}} + \frac{11}{2\beta^{*2} k_p{}^2}\right) \\ &\approx \frac{\beta^*}{2k_p{}^2 \rho_u{}^2}\left(1 + \frac{L_u{}^2}{12\beta^{*2}} + \frac{2\eta_0{}^2 k_p{}^2 \rho_u{}^2}{\beta^{*2}}\right). \end{aligned} \tag{26}$$