# USPAS Course on
# 4th Generation Light Sources II
# ERLs and Thomson Scattering

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## Undulator Radiation Overview

# Summary of Undulator Radiation & Assignments

1. Synchrotron Radiation Overview
    1. Radiation From Electron Bunch
    2. Bending Magnet / Undulator / Wiggler
2. Undulator Radiation Properties
    1. Central Cone
    2. Spatial and Temporal Coherence
3. Home Problems
4. Further Reading on Synchrotron Radiation

# Coherent or Incoherent Radiation From Many Electrons?

Radiation field from a single $k^{th}$ electron in a bunch:

$$E_k = E_0 \exp(i\omega t_k)$$

Radiation field from the whole bunch $\propto$ bunching factor (*b.f.*)

$$b.f. = \frac{1}{N_e}\sum_{k=1}^{N_e} \exp(i\omega t_k)$$

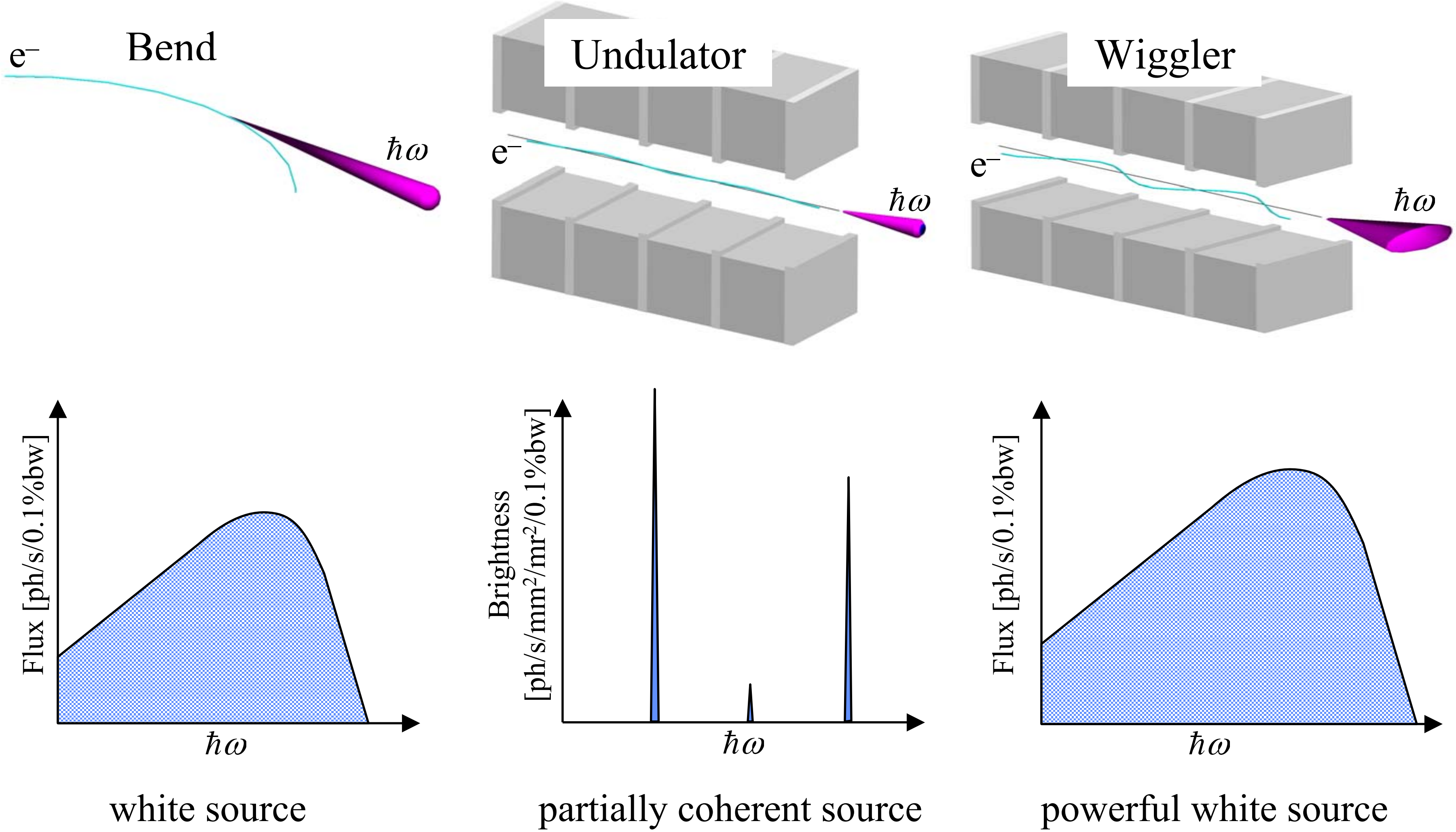
Radiation Intensity: $I = I_0 |b.f.|^2 N_e^2$

single electron

1) "long bunch": $|b.f.|^2 \sim 1/N_e$ => $I = I_0 N_e$ *incoherent (conventional) SR*

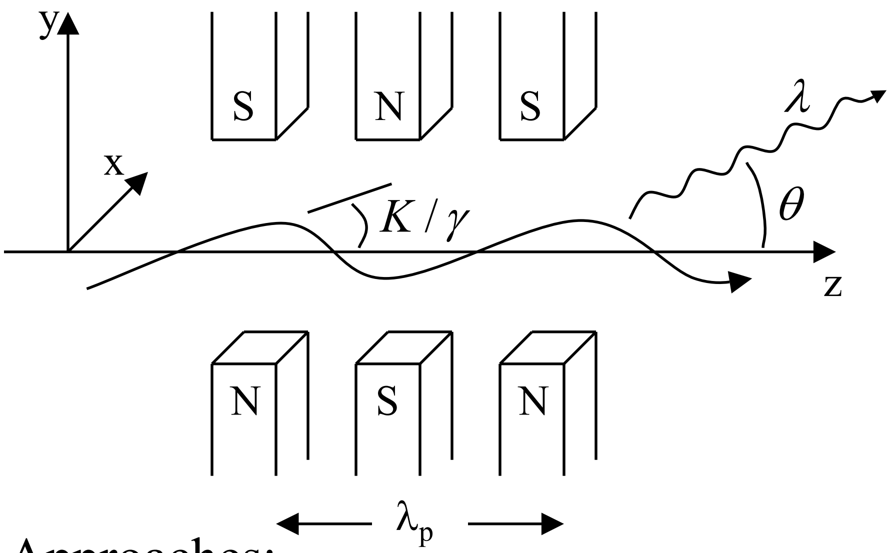
2) "short bunch" or μ-bunching: $|b.f.| \leq 1$ => $I \sim I_0 N_e^2$ *coherent (FELs) SR*

In this course we are dealing mostly with spontaneous (non-FEL) SR



Bend
e⁻
ħω
Undulator
e⁻
ħω
Wiggler
e⁻
ħω
Flux [ph/s/0.1%bw]
ħω
white source
Brightness [ph/s/mm²/mr²/0.1%bw]
ħω
partially coherent source
Flux [ph/s/0.1%bw]
ħω
powerful white source

# Undulator Radiation from Single Electron




$$B_y = B_0 \sin k_p z$$

$$K = 93.4 B_0[\mathrm{T}]\lambda_p[\mathrm{m}]$$

Halbach permanent magnet undulator:

$$B_0[\mathrm{T}] \approx 3.33 \exp[-\kappa(5.47 - 1.8\kappa)]$$

for $SmCo_5$, here $\kappa = gap/\lambda_p$

Approaches:

1. Solve equation of motion* (trivial), grab Jackson and calculate retarded potentials (not so trivial – usually done in the far field approximation). Fourier Transform the field seen by the observer to get the spectrum.

More intuitively in the electron rest frame:

2. Doppler shift to the lab frame (nearly) simple harmonic oscillator radiation.

3. Doppler shift Thomson back-scattered undulator field “photons”. *

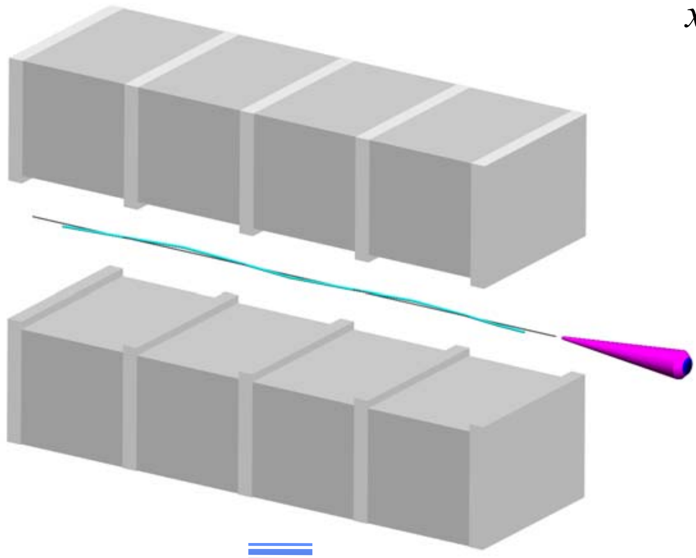
Or simply

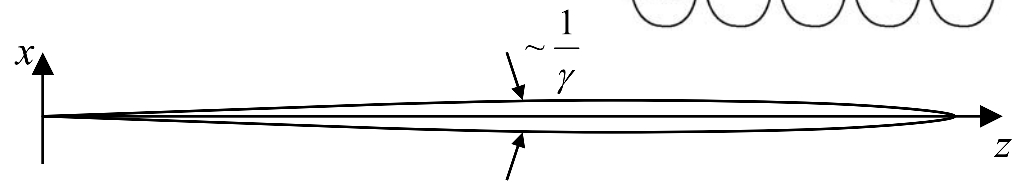
4. Write interference condition of wavefront emitted by the electron.*

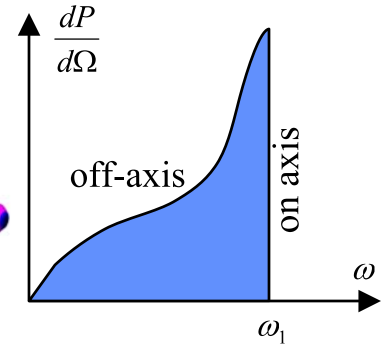
* means home problem

# Steps We Followed in the Previous Lecture

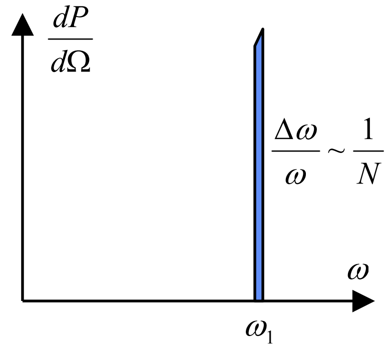
Jefferson Lab





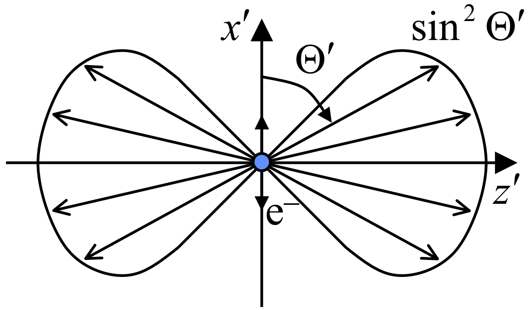




back to lab frame

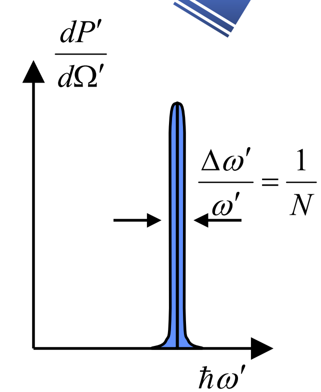



after pin-hole aperture

$$\lambda_p' = \lambda_p / \bar{\gamma}$$




in e⁻ frame




$$\lambda_n = \frac{\lambda_p}{2\gamma^2 n}\left(1 + \tfrac{1}{2}K^2 + \gamma^2\theta^2\right)$$

$$\frac{\Delta\lambda}{\lambda_n} \sim \frac{1}{nN_p}$$ (for fixed $\theta$ only!)

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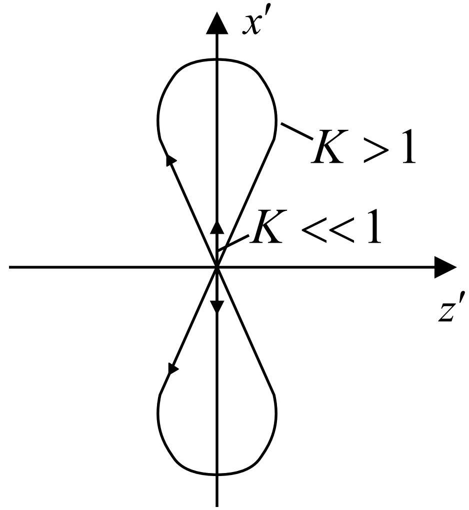
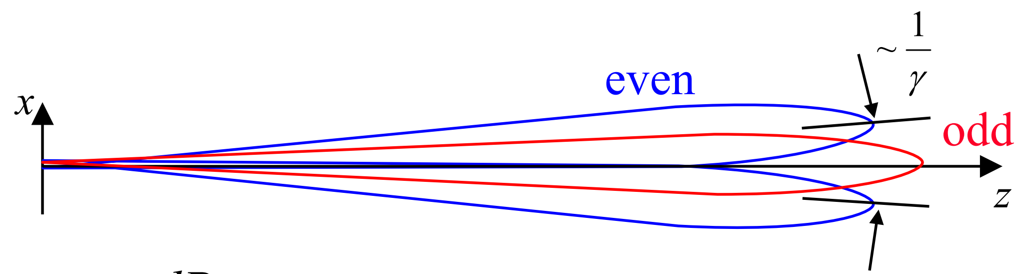
# Higher Harmonics / Wiggler




motion in $e^-$ frame

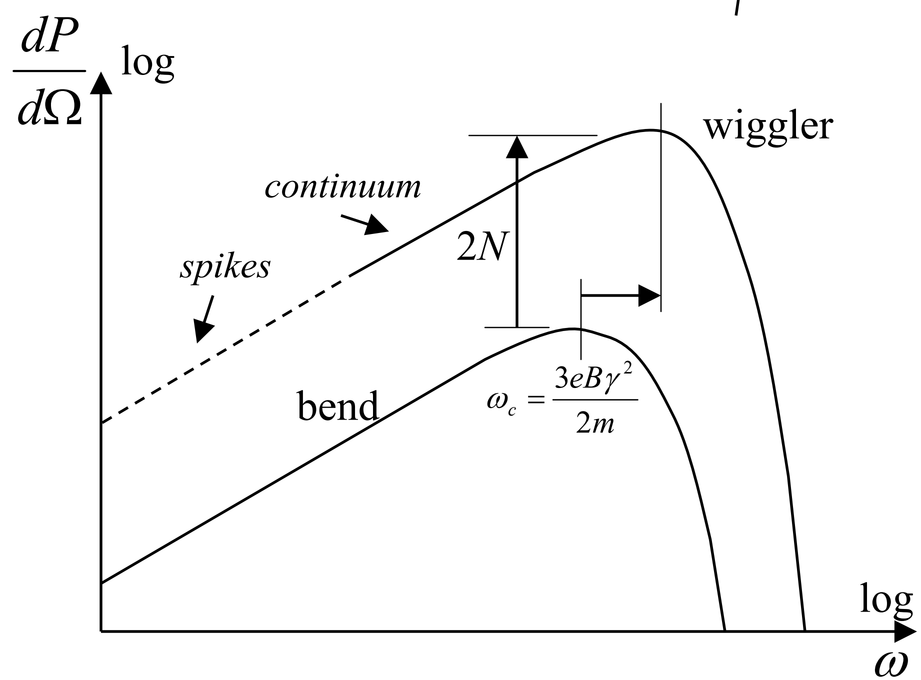



$K \leq 1$ undulator
$K > 1$ wiggler

$$n_c = \frac{3K}{4}\left(1 + \frac{K^2}{2}\right)$$

| $K$ | $n_c$ |
|---|---|
| 1 | 1 |
| 2 | 4 |
| 4 | 27 |
| 8 | 198 |
| 16 | 1548 |

critical harmonic number for wiggler
(in analogy to $\omega_c$ of bending magnet)




wiggler and bend spectra after pin-hole aperture

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# Total Radiation Power

$$P_{tot} = \frac{\pi}{3}\alpha\hbar\omega_1 K^2(1+\tfrac{1}{2}K^2)N\frac{I}{e}$$

or

$$P_{tot}[\mathrm{W}] = 726\frac{E[\mathrm{GeV}]^2 K^2}{\lambda_p[\mathrm{cm}]^2}L[\mathrm{m}]I[\mathrm{A}]$$

e.g. about 1 photon from each electron in a 100-pole undulator, or 1 kW c.w. power from 1 m insertion device for beam current of 100 mA @ 5 GeV, $K = 1.5$, $\lambda_p = 2$ cm

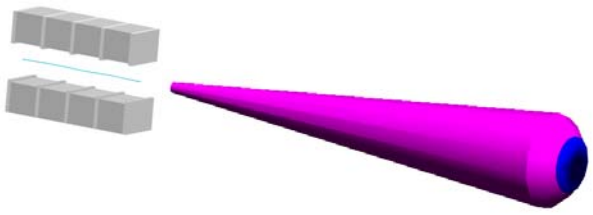
Note: the radiated power is independent from electron beam energy **if** one can keep $B_0\lambda_p \cong$ const, while $\lambda_p \sim \gamma^2$ to provide the same radiation wavelength. (e.g. low energy synchrotron and Thomson scattering light sources)

However, most of this power is discarded (bw ~ 1). Only a small fraction is used.

## Radiation Needed

| | | |
|---|---|---|
| wavelength | 0.1 – 2 Å (if a hard x-ray source) | |
| bw | $10^{-2} - 10^{-4}$ | ← *temporal coherence* |
| small source size & divergence | | ← *spatial coherence* |

# Undulator Central Cone



Select with a pin-hole aperture the cone:

$$\theta_{cen} = \frac{1}{2\gamma}\sqrt{\frac{1+\frac{1}{2}K^2}{nN}} = \sqrt{\frac{\lambda_n}{2L}}$$

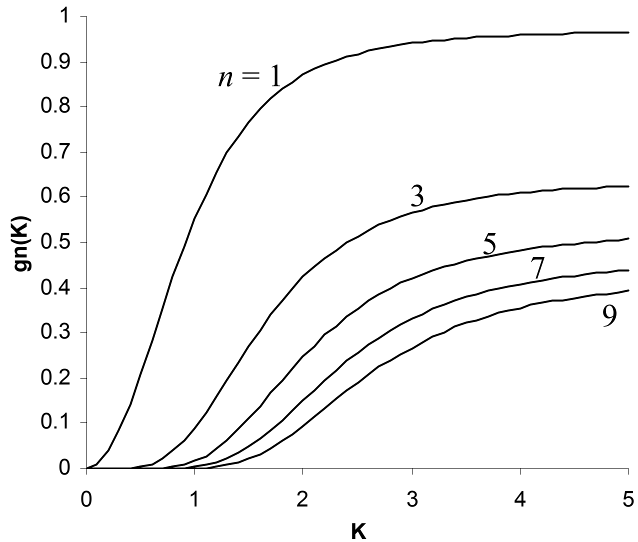
to get bw: $\dfrac{\Delta\omega}{\omega_n} \sim \dfrac{1}{nN}$

Flux in the central cone from $n^{th}$ harmonic in bw $\Delta\omega/\omega_n$:

$$\dot{N}_{ph}\Big|_n = \pi\alpha N \frac{\Delta\omega}{\omega_n}\frac{I}{e} g_n(K) \le \boxed{\pi\alpha \frac{I}{e}\frac{g_n(K)}{n}}$$

Note: the number of photons in bw ~ $1/N$ is about 2 % max of the number of $e^-$ for any-length undulator.

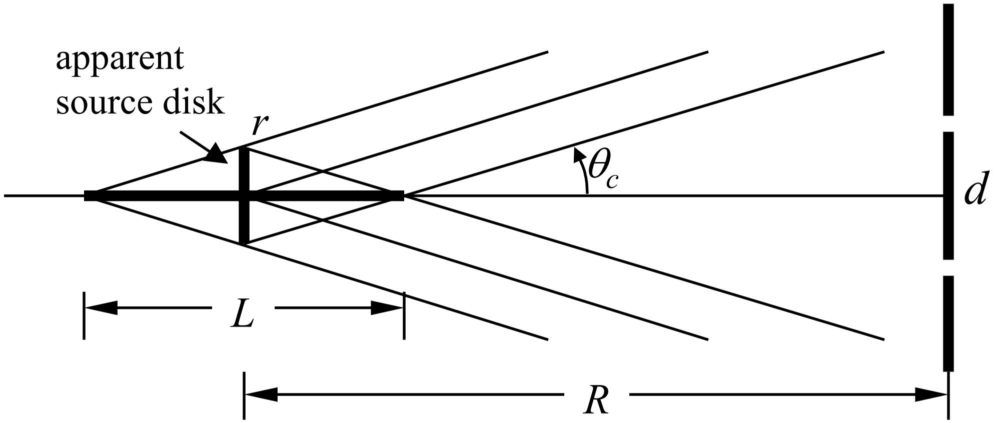
Undulator "efficiency": $\dfrac{P_{cen}}{P_{tot}} \le \dfrac{3g_n(K)}{K^2(1+\frac{1}{2}K^2)}\dfrac{1}{N_p}$




Function $g_n(K) = \dfrac{nK^2[JJ]}{(1+\frac{1}{2}K^2)}$

# A Word on Coherence of Undulator

Radiation contained in the central cone is transversely coherent (no beam emittance!)




Young's double-slit interference condition:

$$\frac{rd}{R} \sim \lambda$$

in Fraunhofer limit:

$$r \sim \theta_c L, \quad \theta_c \sim r/R \quad \Rightarrow \theta_c \sim \sqrt{\lambda / L}$$

same as central cone

Spatial coherence (rms): $r \cdot \theta_c = \lambda / 4\pi$

Temporal coherence: $l_c = \lambda^2 / (2\Delta\lambda)$, $t_c = l_c / c$

Photon degeneracy*: $\Delta_c = \dot{N}_{ph,c} t_c$

| X-ray source | $\Delta_c$ |
|---|---|
| Storage rings | <1 |
| ERLs | >1 |
| XFEL | >>1 |

Next, we will study the effect of finite beam 6D emittance on undulator radiation.

# Home Problems

1) Adopting perturbative approach, solve the equation of motion in planar undulator ($z$ is axis of motion, $y$ is magnetic field axis) and **a)** find the average $z$-velocity of electron. Include higher terms of motion and show that **b)** $x$-velocity exhibits oscillation with odd harmonics present, while $z$-velocity modulation has only even harmonics. How does the amplitude of harmonics of velocity fluctuations in the lab-frame scale with $n$ (harmonic number)? **c)** In helical undulator, magnetic field is such that electrons travel in helical trajectories. Comment on harmonic content if electron trajectory is a pure helix.

2) Write the interference condition of wavefront emitted by the same electron in undulator and derive formulae of radiation wavelength for $n^{th}$ harmonic number and bandwidth at fixed angle (i.e. $1/nN$).

3) Lorentz transform undulator field ($K \ll 1$) into the electron frame. Notice that the field in that frame is similar to that of E&M plane wave. These “virtual” photons can be back-scattered from electron. Lorentz transform Thomson back-scattered photons to the lab frame and show that the radiation frequency is identical to the result obtained earlier.

# Home Problems

4) Consider wiggler with $K = 20$ and pin-hole aperture selecting 1 mrad half angle. At what harmonic number will the radiation spectrum after the aperture appear to be merged? Beam energy is 3 GeV, and beam emittance and energy spread are negligible.

5) Electrons in a bunch always have some inherent divergence spread. Through Doppler shift, this divergence "erodes" monochromatic nature of radiation in the central cone and compromises undulator performance. What is the acceptable level of electron divergence spread for good undulator performance (also known as undulator condition)?

6) Dipole radiation of the fundamental in the electron rest frame has a donut-like appearance with the half radiation being emitted in the direction opposite to $z'$ axis. Discuss the appearance of this part of radiation in the lab frame.

7) Estimate photon degeneracy of undulator radiation from 5 GeV bunch of 100 pC compressed to 100 fs with no energy spread or emittance. Assume $N = 1000$, $K = 1.2$, and $\lambda_p = 1.7$ cm. Compare it with photon degeneracy of a conventional pulsed dye laser ($\lambda = 600$ nm, 60 μJ energy/pulse, 3 ns pulse, 0.3 nm linewidth). Explain the result.

# Further Reading on Synchrotron Radiation


1. K.J. Kim, Characteristics of Synchrotron Radiation, AIP Conference Proceedings **189** (1989) pp.565-632

2. R.P. Walker, Insertion Devices: Undulators and Wigglers, CERN Accelerator School **98-04** (1998) pp.129-190, and references therein. Available on the Internet at http://preprints.cern.ch/cernrep/1998/98-04/98-04.html

3. B. Lengeler, Coherence in X-ray physics, Naturwissenschaften **88** (2001) pp. 249-260, and references therein.

4. D. Attwood, Soft X-rays and Extreme UV Radiation: Principles and Applications, Cambridge University Press, 1999. Chapters 5 (Synchrotron Radiation) and 8 (Coherence at Short Wavelength) and references therein.

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## Brightness, Finite Emittance Effect

# Brightness, Finite Emittance Effect

1. Brightness definition
   1. Geometric Optics
   2. Wave Optics
2. Brightness for Finite Beam Emittance
   1. Electron Distribution Effect
   2. Matching Beam Phase Space
   3. Coherence Properties

# Brightness Definition: Geometric Optics

Brightness is a measure of spatial (transverse) coherence of radiation. Spectral brightness (per 0.1 % BW) is usually quoted as a figure of merit, which also reflects temporal coherence of the beam. The word "spectral" is often omitted. Peak spectral brightness is proportional to photon degeneracy.

For the most parts we will follow K-J Kim's arguments regarding brightness definitions.

A ray coordinate in 4D phase space is defined as $\vec{x} = (x, y), \vec{\varphi} = (\varphi, \psi)$

$$B(\vec{x}, \vec{\varphi}; z) = \frac{d^4 F}{d^2\vec{x}\, d^2\vec{\varphi}}$$

Brightness is invariant in lossless linear optics as well as flux: $F = \int B(\vec{x}, \vec{\varphi}; z)\, d^2\vec{x}\, d^2\vec{\varphi}$

while flux densities are not: $\dfrac{d^2 F}{d^2\vec{\varphi}} = \int B(\vec{x}, \vec{\varphi}; z)\, d^2\vec{x},\ \dfrac{d^2 F}{d^2\vec{x}} = \int B(\vec{x}, \vec{\varphi}; z)\, d^2\vec{\varphi} \quad \neq inv$

# Brightness Definition: Wave Optics

$$B(\vec{x},\vec{\varphi};z) = \frac{d\omega}{\hbar\omega}\frac{2\varepsilon_0 c}{T}\int d^2\vec{\xi}\left\langle E^*_{\omega,\varphi}(\vec{\varphi}+\vec{\xi}/2;z)E_{\omega,\varphi}(\vec{\varphi}-\vec{\xi}/2;z)\right\rangle e^{-ik\vec{\xi}\cdot\vec{x}}$$

$$= \frac{d\omega}{\hbar\omega}\frac{2\varepsilon_0 c}{\lambda^2 T}\int d^2\vec{y}\left\langle E^*_{\omega,x}(\vec{x}+\vec{y}/2;z)E_{\omega,x}(\vec{x}-\vec{y}/2;z)\right\rangle e^{-ik\vec{\varphi}\cdot\vec{y}}$$

here electric field in frequency domain is given in either coordinate or angular representation. Far-field (angular) pattern is equivalent to the Fourier transform of the near-field (coordinate) pattern:

$$E_{\omega,\varphi} = \frac{1}{\lambda^2}\int E_{\omega,x}(\vec{x};z)e^{-ik\vec{\varphi}\cdot\vec{x}}d^2\vec{x} \quad \Leftrightarrow \quad E_{\omega,x} = \int E_{\omega,\varphi}(\vec{x};z)e^{-ik\vec{\varphi}\cdot\vec{x}}d^2\vec{\varphi}$$

A word of caution: brightness as defined in wave optics may have negative values when diffraction becomes important. One way to deal with that is to evaluate brightness when diffraction is not important (e.g. $z = 0$) and use optics transform thereafter.

# Diffraction Limit

Gaussian laser beam equation:

$$E(\vec{x},z) = E_0 \frac{w_0}{w(z)} \exp\left\{ i\left[ kz - \cot\left(\frac{z}{z_R}\right)\right] - \vec{x}^2 \left[\frac{1}{w^2(z)} - \frac{ik}{2R(z)}\right]\right\}$$

$$w^2(z) = w_0^2 (1 + z^2 / z_R^2)$$

$$z_R = \pi w_0^2 / \lambda$$

$$R(z) = z(1 + z_R^2 / z^2)$$

With corresponding brightness:

$$B(\vec{x},\vec{\varphi};z) = B_0 \exp\left\{ -\frac{1}{2}\left[\frac{(\vec{x} - z\vec{\varphi})^2}{\sigma_r^2} + \frac{\vec{\varphi}^2}{\sigma_{r'}^2}\right]\right\}$$

$$\sigma_r = w_0 / 2, \;\; \sigma_{r'} = 1 / k w_0$$

$$\sigma_r \sigma_{r'} = \lambda / 4\pi$$

$$\sigma_r / \sigma_{r'} = z_R$$

$$B_0 = \frac{F}{(2\pi \sigma_r \sigma_{r'})^2}$$

$$F_{coh} = \frac{B_0}{(\lambda / 2)^2}$$

# Effect of Electron Distribution

Previous result from undulator treatment:

$$E_{\omega,\varphi}(\vec{\varphi};0) = \frac{e}{4\pi\varepsilon_0 c}\frac{\omega}{\lambda\sqrt{2\pi}}\int dt' \mathrm{e}^{i\omega t(t')}\vec{n}\times(\vec{n}\times\vec{\beta}(t')), \quad \text{here } \vec{n} = (\vec{\varphi}, 1-\vec{\varphi}^2/2)$$

The field in terms of reference electron trajectory for i[th]-electron is given by:

$$E^{\mathrm{i}}_{\omega,\varphi}(\vec{\varphi};0) = E^{0}_{\omega,\varphi}(\vec{\varphi}-\vec{\varphi}^{\mathrm{i}}_e;0)\mathrm{e}^{\underbrace{i\omega(t-\vec{\varphi}\cdot\vec{x}^{\mathrm{i}}_e/c)}_{\text{phase of i}^{\text{th}}\text{-electron}}}$$

For brightness we need to evaluate the following ensemble average for all electrons:

$$\left\langle E^{*}_{\omega,\varphi}(\vec{\varphi}_1;0)E_{\omega,\varphi}(\vec{\varphi}_2;0)\right\rangle = \sum_{\mathrm{i}=1}^{N_e}\left\langle E^{\mathrm{i}*}_{\omega,\varphi}(\vec{\varphi}_1;0)E^{\mathrm{i}}_{\omega,\varphi}(\vec{\varphi}_2;0)\right\rangle \qquad \propto N_e$$

$$+\sum_{\mathrm{i}\neq\mathrm{j}}\left\langle E^{\mathrm{i}*}_{\omega,\varphi}(\vec{\varphi}_1;0)E^{\mathrm{j}}_{\omega,\varphi}(\vec{\varphi}_2;0)\right\rangle \qquad \propto N_e(N_e-1)\,\mathrm{e}^{-k^2\sigma_z^2}$$

2[nd] term is the "FEL" term. Typically $N_e \mathrm{e}^{-k^2\sigma_z^2} << 1$, so only the 1[st] term is important.

# Effect of Electron Distribution (contd.)

$$\left\langle E^{*}_{\omega,\varphi}(\vec{\varphi}_1;0)E_{\omega,\varphi}(\vec{\varphi}_2;0)\right\rangle \approx N_e\left\langle \mathrm{e}^{ik\vec{x}_e^{\mathrm{i}}\cdot(\vec{\varphi}_1-\vec{\varphi}_2)}E^{0*}_{\omega,\varphi}(\vec{\varphi}_1-\vec{\varphi}_e^{\mathrm{i}};0)E^{0}_{\omega,\varphi}(\vec{\varphi}_2-\vec{\varphi}_e^{\mathrm{i}};0)\right\rangle$$

$$B(\vec{x},\vec{\varphi};0) = N_e\left\langle B^0(\vec{x}-\vec{x}_e^{\mathrm{i}},\vec{\varphi}-\vec{\varphi}_e^{\mathrm{i}};0)\right\rangle$$

$$= N_e\int B^0(\vec{x}-\vec{x}_e,\vec{\varphi}-\vec{\varphi}_e;0)f(\vec{x}_e,\vec{\varphi}_e;0)d^2\vec{x}_e d^2\vec{\varphi}_e$$

electron distribution

Brightness due to single electron has been already introduced. Total brightness becomes a convolution of single electron brightness with electron distribution function.

Brightness on axis due to single electron:

flux in the central cone

$$B^0(0,0;0) = \frac{F^0}{(\lambda/2)^2}$$

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# Finite Beam Emittance Effect

Oftentimes brightness from a single electron is approximated by Gaussian:

$$B^0(\vec{x},\vec{\varphi};0) = \frac{F^0}{(\lambda/2)^2}\exp\left\{-\frac{1}{2}\left[\frac{\vec{x}^2}{\sigma_r^2}+\frac{\vec{\varphi}^2}{\sigma_{r'}^2}\right]\right\}$$

$$\sigma_r = \sqrt{2\lambda L}/4\pi, \quad \sigma_{r'} = \sqrt{\lambda/2L}$$

Including the electron beam effects, amplitude and sigma's of brightness become:

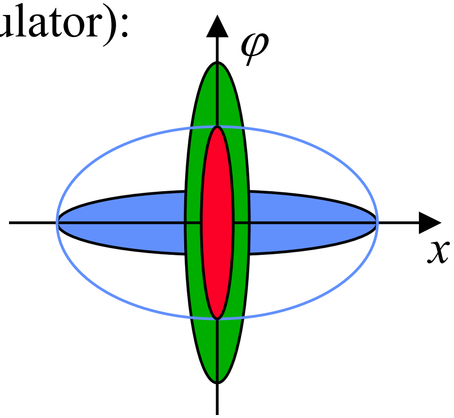
$$B(0,0;0) = \frac{F}{(2\pi)^2\sigma_{Tx}\sigma_{Tx'}\sigma_{Ty}\sigma_{Ty}}$$

$$\sigma_{Tx}^2 = \sigma_r^2 + \sigma_x^2 + a^2 + \tfrac{1}{12}\sigma_{x'}^2 L^2 + \tfrac{1}{36}\varphi^2 L^2 \qquad \sigma_{Tx'}^2 = \sigma_{r'}^2 + \sigma_{x'}^2$$

$$\sigma_{Ty}^2 = \sigma_r^2 + \sigma_y^2 + \tfrac{1}{12}\sigma_{y'}^2 L^2 + \tfrac{1}{36}\psi^2 L^2 \qquad \sigma_{Ty'}^2 = \sigma_{r'}^2 + \sigma_{y'}^2$$

# Matching Electron Beam

Matched β-function is given by (beam waist at the center of undulator):




$$\beta_{x,y}^{opt} = \sigma_r / \sigma_{r'} = L / 2\pi$$

Brightness on axis becomes:

$$B(0,0;0) = \frac{F}{(\lambda/2)^2} \frac{1}{\left(1 + \frac{\varepsilon_x}{\lambda/4\pi}\right)\left(1 + \frac{\varepsilon_y}{\lambda/4\pi}\right)}$$

← transversely coherent fraction of the central cone flux

Matched β-function has a broad minimum (for $\varepsilon/(\lambda/4\pi) << 1$ or $\varepsilon/(\lambda/4\pi) >> 1$)

$$\sigma_T \sigma_{T'} = \begin{cases} \sqrt{2}\,\min & \text{for} \quad \beta \approx 2L\varepsilon/\lambda \\ \min & \text{for} \quad \beta = L/2\pi \\ \sqrt{2}\,\min & \text{for} \quad \beta \approx \lambda L/(8\pi^2 \varepsilon) \end{cases}$$

also if $\varepsilon \sim \lambda / 4\pi \Rightarrow$

$\beta \approx 6\beta^{opt} \approx L$ is still acceptable

# Energy Spread of the Beam

Energy spread of the beam can degrade brightness of undulators with many periods.

If the number of undulator periods is much greater than $N_\delta \approx 0.2/\sigma_\delta$ , brightness will not grow with the number of periods.

Maximal spectral brightness on axis becomes

$$B(0,0;0) = \frac{F}{(\lambda/2)^2} \frac{1}{\left(1+\frac{\varepsilon_x}{\lambda/4\pi}\right)\left(1+\frac{\varepsilon_y}{\lambda/4\pi}\right)} \frac{1}{\sqrt{1+\left(\frac{N}{N_\delta}\right)^2}}$$

# Photon Degeneracy

Number of photons in a single quantum mode:

$$\hbar k \sigma_x \sigma_\varphi \approx \frac{\hbar}{2} \qquad \hbar k \sigma_y \sigma_\psi \approx \frac{\hbar}{2} \qquad \sigma_E \sigma_t \approx \frac{\hbar}{2}$$

Peak brightness is a measure of photon degeneracy

$$\Delta_c = B_{peak} \left( \frac{\lambda}{2} \right)^3 \frac{\Delta\lambda}{\lambda} \frac{1}{c}$$

E.g. maximum photon degeneracy that is available from undulator (non-FEL)

$$\Delta_c^{\max} \approx \alpha \frac{\lambda_n}{\sigma_z} N_e N \cdot g_n(K) \quad \text{more typically, however: } \Delta_c \approx 10^{-3} \alpha \frac{\lambda_n^3}{\varepsilon_x \varepsilon_y \varepsilon_z} N_e \frac{g_n(K)}{n}$$

diffraction-limited    emittance dominated