**2.12** Consider a periodic focusing function $K(s+L) = K(s)$ with $|\mathrm{Tr} M_{s+l,s}| < 2$, and two distinct points within the period $s$ and $s'$. Note that the transfer matrix for the motion through a period starting at $s$ followed by a transfer from $s$ to $s'$ may be computed in two different ways depending on whether the period matrix at $s$ or the period matrix at $s'$ is used.

(a) Show, for example by direct computation, that $\mathrm{Tr} M_{s+L,s} = \mathrm{Tr} M_{s'+L,s'}$, and hence the phase advance $\mu_L$ is independent of the point $s$ chosen.

(b) Recall that the transfer matrix for a period is thus

$$M_{s+L,s} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\mu_L + \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \sin\mu_L$$

Show that

$$\begin{pmatrix} \alpha(s') & \beta(s') \\ -\gamma(s') & -\alpha(s') \end{pmatrix} = M_{s',s} \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} M_{s',s}^{-1}.$$

(c) By multiplication of the matrices in (b), reproduce the ellipse transformation formulas. Reproduce the differential equations for $\alpha$, $\beta$, and $\gamma$ by considering an infinitesimal displacement from $s$ to $s + ds$. This problem demonstrates that $\alpha$, $\beta$, and $\gamma$ as defined by the period transfer matrices are the same as those defined by ellipse transformation and indeed, as those in the pseudoharmonic solution to Hill's equation.

**2.13** (a) Show explicitly that the matrix representation

$$M_{s',s} = \begin{pmatrix} \sqrt{\frac{\beta(s')}{\beta(s)}}\left(\cos\Delta\mu_{s',s} + \alpha(s)\sin\Delta\mu_{s',s}\right) & \sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s} \\ -\frac{1}{\sqrt{\beta(s')\beta(s)}}\begin{bmatrix} (1+\alpha(s')\alpha(s))\sin\Delta\mu_{s',s} \\ +(\alpha(s')-\alpha(s))\cos\Delta\mu_{s',s} \end{bmatrix} & \sqrt{\frac{\beta(s)}{\beta(s')}}\left(\cos\Delta\mu_{s',s} - \alpha(s')\sin\Delta\mu_{s',s}\right) \end{pmatrix}$$

satisfies $M_{s'',s} = M_{s'',s'} M_{s',s}$.

(b) Demonstrate the following important formula for the phase advance

$$\tan\Delta\mu_{s',s} = \frac{(M_{s',s})_{12}}{\beta(s)(M_{s',s})_{11} + \alpha(s)(M_{s',s})_{12}}$$

(c) Using the addition formula for tan, the ellipse transformation formula, and the result from part (b), show that the phase advance adds properly

$$\Delta\mu_{s'',s} = \Delta\mu_{s'',s'} + \Delta\mu_{s',s}$$

**2.15** Starting with the 3 by 3 transfer matrix for a sector dipole bend magnet in the bending plane,

$$\begin{pmatrix} \cos\Theta & \rho\sin\Theta & \rho(1-\cos\Theta) \\ -\sin\Theta/\rho & \cos\Theta & \sin\Theta \\ 0 & 0 & 1 \end{pmatrix},$$

derive the 3 by 3 transfer matrix for a rectangular bend magnet in the bend plane, assuming that the edge focusing on each side of the rectangular bend may be treated as a thin lens. The total bending angle is denoted by $\Theta$. You may assume $\Theta << 1$.