Mid Term Examination
Physics 854
Thursday, October 19, 2017

1. Consider a relativistic electron circulating in a betatron at radius R with field index n = 0.6.
    a. If the electron makes 10 revolutions, how many radial and vertical oscillations does it make?

Ten revolutions takes a time $10\times 2\pi R/c$. The oscillation frequency of radial oscillations is $\sqrt{1-n}\Omega_c/2\pi$. This means there are $10\sqrt{1-n}\Omega_c R/c$ radial oscillations. But the betatron equilibrium has $\Omega_c R/c = 1$. So there are $10\sqrt{0.4} = 6.32$ radial oscillations. Likewise, there are $10\sqrt{0.6} = 7.75$ vertical oscillations.

    b. What is the dispersion function at all locations in the betatron?

$$D = R/(1-n)$$

    c. Suppose a particle has a relative momentum error of 10^{-3}. What is the additional path length per turn?

By the expression for the dispersion, the radius of the particle shifts out $D(\Delta p/p)$. The additional path length is $(\Delta p/p)2\pi R/(1-n) = 10^{-3}2\pi R/(1-n)$.

    d. Is this result consistent with the formula

$$M_{56} = \int \frac{D(s)}{\rho(s)} ds ?$$

The path length increase from the definition of M_{56} is $M_{56}(\Delta p/p)$. The integral is

$$M_{56} = \int \frac{R}{(1-n)R} ds = \frac{2\pi R}{1-n},$$

which checks!

2. Suppose a periodic focusing channel has focusing as in Figure 1, where $\psi = \sqrt{K}L = \pi/4$.
    a. What is the (thick lens!) phase advance per full cell?

By the matrix multiplication formulas

$$\cos\mu = \cos\psi \cosh\psi = 1.324609/\sqrt{2} = 0.93664.$$

So

$$\mu = 0.35788 \text{ rad} = 20.51^\circ,$$

and $\sin\mu = 0.35029$.

    b. What are $\beta(s=0)$, $\beta(s=L/2)$, $\beta(s=L)$, and $\beta(s=3L/2)$, expressed in units of L (e.g. 2.34 L)?

The first answer is

$$\beta(0)=\frac{M_{12}(0)}{\sin\mu}=\frac{4L}{\pi(0.35029)}\left(\sin(\pi/4)\cosh(\pi/4)+\cos(\pi/4)\sinh(\pi/4)\right)$$

$$=5.64L$$

Similarly

$$\beta(L/2)=\frac{M_{12}(L/2)}{\sin\mu}=\frac{4L}{\pi(0.35029)}\left(\sin(\pi/4)\cosh(\pi/4)+\sinh(\pi/4)\right)$$

$$=6.56L$$

$$\beta(L)=\frac{M_{12}(L)}{\sin\mu}=\frac{4L}{\pi(0.35029)}\left(\sin(\pi/4)\cosh(\pi/4)+\cos(\pi/4)\sinh(\pi/4)\right)$$

$$=5.64L$$

$$\beta(3L/2)=\frac{M_{12}(3L/2)}{\sin\mu}=\frac{4L}{\pi(0.35029)}\left(\sin(\pi/4)+\cos(\pi/4)\sinh(\pi/4)\right)$$

$$=4.80L$$

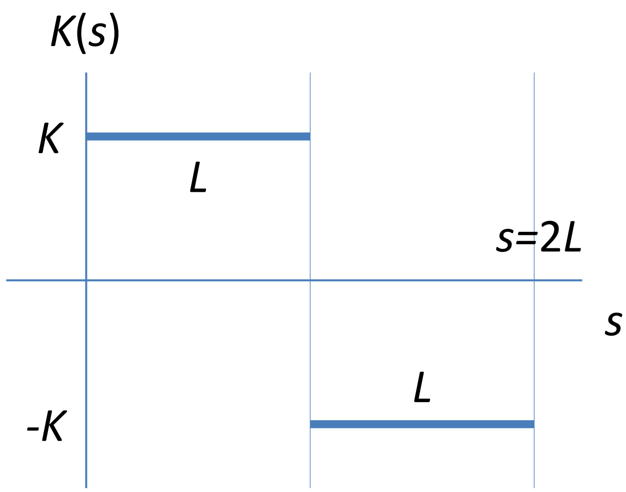

Figure 1: Periodic Focusing Function for Problem 2

3. Consider the following matrix

$$M=\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

   a. Is it unimodular?

Yes

   b. What is the phase advance of this matrix?

$$\cos\mu=\frac{1}{\sqrt{2}}\rightarrow\mu=\frac{\pi}{4}=+45^{\circ}$$

The sign is chosen so β is positive.

   c. What are the $\alpha,\beta,$ and γ (Twiss parameters) for this matrix?

$$M_{11} = M_{22} \to \alpha = 0$$

$$\beta = \frac{M_{12}}{\sin\mu}\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{1} = 1 \to \gamma = 1$$

d. What are M^4, M^{100}, M^{103}, M^{104}, and M^{105}, i. e., what are the individual matrix elements of these powers of *M*?

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\pi/4) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(\pi/4)$$

$$M^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\pi) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(25\pi) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(25\pi)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\pi) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M^{103} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(103\pi/4) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(103\pi/4)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(104\pi/4 - \pi/4) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(104\pi/4 - \pi/4)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(-\pi/4) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(-\pi/4)$$

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = M^{-1}$$

$$M^{104} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(26\pi) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(26\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M^{105} = M^{104}M = M$$

4. Recall our expressions from the homework for the transfer matrix of a FODO system. The one-period transfer matrix starting with the middle of the focusing magnet was

$$M_f = \begin{pmatrix} 1 & 0 \\ -1/(2f) & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/(2f) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - L^2/(2f^2) & 2L + L^2/f \\ -L/(2f^2) + L^2/(4f^3) & 1 - L^2/(2f^2) \end{pmatrix}.$$

and M_d is obtained from M_f by replacing f with $-f$

a. How should one choose *f*, in terms of *L*, so that the phase advance through one period of the FODO system is 90 degrees (corresponding to 1/4 of a transverse oscillation per period)?

$$1 - L^2/2f^2 = 0 \rightarrow f = L/\sqrt{2}$$

b. For the matched phase space (x, x') ellipse in the 90 degree phase advance system, what is the beta-function in the middle of the focusing lens?

$$\sin 90^\circ = 1$$

$$\beta_f = \left(2 + \sqrt{2}\right) L$$

c. For the matched phase space ellipse in the 90 degree phase advance system, what is the beta-function in the middle of the defocusing lens?

$$\beta_d = \left(2 - \sqrt{2}\right) L$$

d. What are the alpha-functions for the matched ellipses at these same two locations?

$$\alpha_f = \alpha_d = 0$$

e. Suppose L is 3 m, and the area of a matched phase space ellipse is $\pi\varepsilon = 2\pi \times 10^{-6}$ m radian, what are the maximum extents (x_{max}) of the matched ellipses in the focusing and defocusing lenses of the 90 degrees phase advance system?

$$x_{\max,f} = \sqrt{\left(2 + \sqrt{2}\right) 3 \cdot 2} \times 10^{-3}\,\text{m} = 4.53 \text{ mm}$$

$$x_{\max,d} = \sqrt{\left(2 - \sqrt{2}\right) 3 \cdot 2} \times 10^{-3}\,\text{m} = 1.87 \text{ mm}$$