Physics 854
Accelerator Physics
Homework 6 Solutions

1.

$$r_c = 3, r_b = 1, \omega_0 = 100kHz$$

$$E_0 = 100GeV \rightarrow \gamma = 106.6$$

$$E_{tr} = 50GeV \rightarrow \gamma_{tr} = 53.3$$

$$\eta_c = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2} = -2.64 \times 10^{-4}$$

$$(\Omega - n\omega_0)^2 = -\frac{n^2\omega_0^2}{\beta^2\gamma^2}\left(\frac{q\eta_c I_0}{4\pi\epsilon_0 c\beta E_0}\left(1 + 2ln\left(\frac{r_c}{r_b}\right)\right)\right)$$

$$(\Omega - n\omega_0)^2 = -n^2\, 87.7435 \times 10^{-7}$$

$$\Omega - n\omega_0 = \pm in2.962 \times 10^{-3}$$

$$n = 1 \rightarrow \Omega - n\omega_0 = \pm i2.962 \times 10^{-3}$$

$$Growth\ time = \frac{1}{2.962 \times 10^{-3}} = 337.61s$$

$$\omega_n = n\omega_0 - ni\hat{\omega} \pm i2.962 \times 10^{-3}$$

$$n = 1 \rightarrow 2.962 \times 10^{-3} = \delta_0\eta_c\omega_0$$

$$\delta_0 = \frac{2.962 \times 10^{-3}}{2.64 \times 10^{-4} \times 2\pi \times 100 \times 10^3}$$

$$\delta_0 = 1.78 \times 10^{-5}$$

2.

$$\lambda = \frac{2.3\text{ cm}}{2\left(7000/0.511\right)^2}\left(1+K^2/2\right)$$
$$= 0.6159\text{ Angstrom} \quad \text{for } K = 0.1$$
$$= 1.839\text{ Angstrom} \quad \text{for } K = 2$$
$$\lambda = 1.499\times10^{-4}\text{ m}$$
$$\gamma^2 = \frac{2.3\text{ cm}\cdot1.5}{2\cdot1.499\times10^{-2}\text{ cm}} = 115.08$$
$$E_{beam} = 5.48\text{ MeV}$$

3. From $d\tau = dt/\gamma$, and the expression for the Lorentz-invariant power

$$P = -\frac{q^2}{6\pi\varepsilon_0 c}\frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau} = -\frac{q^2\gamma^2}{6\pi\varepsilon_0 c}\left[\begin{matrix}\frac{d\gamma}{dt}\frac{d\gamma}{dt} - \frac{d(\gamma\beta_x)}{dt}\frac{d(\gamma\beta_x)}{dt} \\ -\frac{d(\gamma\beta_y)}{dt}\frac{d(\gamma\beta_y)}{dt} - \frac{d(\gamma\beta_z)}{dt}\frac{d(\gamma\beta_z)}{dt}\end{matrix}\right]$$

$$P = -\frac{q^2\gamma^2}{6\pi\varepsilon_0 c}\left[\begin{matrix}\left(1-\beta^2\right)\frac{d\gamma}{dt}\frac{d\gamma}{dt} \\ -2\gamma\frac{d\gamma}{dt}\vec{\beta}\cdot\dot{\vec{\beta}} - \gamma^2\dot{\vec{\beta}}\cdot\dot{\vec{\beta}}\end{matrix}\right]$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left[\frac{1}{\sqrt{1-\beta_x^2-\beta_y^2-\beta_z^2}}\right] = \gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}$$

$$\therefore P = -\frac{q^2\gamma^2}{6\pi\varepsilon_0 c}\left[\begin{matrix}\gamma^4\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 \\ -2\gamma^4\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \gamma^2\dot{\vec{\beta}}\cdot\dot{\vec{\beta}}\end{matrix}\right]$$

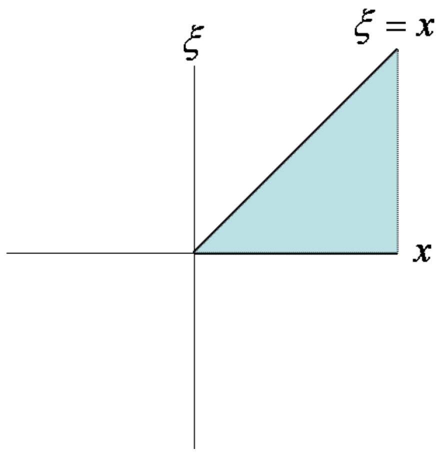
$$= \frac{q^2\gamma^6}{6\pi\varepsilon_0 c}\left[\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 + \left(1-\beta^2\right)\dot{\vec{\beta}}\cdot\dot{\vec{\beta}}\right]$$

$$\left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2 = \beta^2\dot{\vec{\beta}}\cdot\dot{\vec{\beta}} - \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2$$

$$= \frac{q^2\gamma^6}{6\pi\varepsilon_0 c}\left[\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 + \dot{\vec{\beta}}\cdot\dot{\vec{\beta}} - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2 - \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2\right]$$

$$= \frac{q^2\gamma^6}{6\pi\varepsilon_0 c}\left[\dot{\vec{\beta}}\cdot\dot{\vec{\beta}} - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right]$$

4. The key to this problem is to realize that when you interchange the order of integration, one must re-express the new limits of integration properly. Clearly, the range of integration in the $\xi - x$ plane is the blue wedge extending to positive infinity,



so

$$P = \frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\omega_c\gamma\int_0^\infty \xi\int_\xi^\infty K_{5/3}(x)\,dx\,d\xi = \frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\omega_c\gamma\int_0^\infty K_{5/3}(x)\int_0^x \xi\,d\xi\,dx$$

$$= \frac{\sqrt{3}}{16\pi^2\varepsilon_0}\frac{e^2}{\rho}\omega_c\gamma\int_0^\infty x^2 K_{5/3}(x)\,dx = \frac{e^2 c}{6\pi\varepsilon_0\rho^2}\gamma^4$$

A similar calculation allows evaluation of the average energy of the emitted photons

$$\langle E_\gamma\rangle = \frac{\int \hbar\omega\frac{d\dot{n}_\gamma}{d\omega}d\omega}{\int \frac{d\dot{n}_\gamma}{d\omega}d\omega} = \frac{\hbar\omega_c\int_0^\infty \xi\int_\xi^\infty K_{5/3}(x)\,dx\,d\xi}{\int_0^\infty\int_\xi^\infty K_{5/3}(x)\,dx\,d\xi} = \frac{\frac{16\pi}{2\cdot 9\sqrt{3}}}{\frac{5\pi}{3}}\hbar\omega_c = \frac{8}{15\sqrt{3}}\hbar\omega_c\,.$$