Homework Problems IV
Physics 854

1. Our definition of the energy gain is in terms of the full Fourier integral of the electric field pattern of the cavity. Sometimes, particularly for acceleration through a gap, the Fourier integral is divided into two terms: the gap voltage and the transit time factor. Assume a time dependent voltage $V(t) = V_0 \cos \omega t$ across an accelerating gap of length L and a particle orbit $z(t) = \beta ct$. Show

$$|V_c| = V_0 \frac{\sin(\omega L / 2\beta c)}{\omega L / 2\beta c}.$$

$\frac{\sin(\omega L / 2\beta c)}{\omega L / 2\beta c}$ is called the transit time factor, and quantifies the effective voltage reduction because as the particle transits the gap, the voltage (and hence the electric field) is not always at the maximum value.

To determine the amplitude of V_c the location of the cavity is irrelevant. One obtains a purely real value from the integral by choosing the origin at the center of the gap. Then

$$|V_c| = \frac{V_0}{L} \int_{-L/2\beta c}^{L/2\beta c} \frac{e^{i\omega t} + e^{-i\omega t}}{2} \beta c dt = \frac{V_0}{L} \int_{-L/2}^{L/2} \frac{e^{i\omega z/\beta c} + e^{-i\omega z/\beta c}}{2} dz$$

$$= \frac{V_0}{L} \frac{\beta c}{i\omega} \frac{e^{i\omega L/2\beta c} - e^{-i\omega L/2\beta c} - \left(e^{-i\omega L/2\beta c} - e^{i\omega L/2\beta c}\right)}{2}$$

$$= V_0 \frac{2\sin(\omega L / 2\beta c)}{\omega L / \beta c} = V_0 \frac{\sin(\omega L / 2\beta c)}{\omega L / 2\beta c}$$

2. Suppose a transmission line of characteristic impedance Z_0 is terminated with a resistive impedance Z.
   a. With the sign conventions in the lectures show the reflected wave has amplitude

$$V^- = \frac{Z - Z_0}{Z + Z_0} V^+.$$

$$\left(I^+ - I^-\right) Z = i_Z Z = V_Z = V^+ + V^-$$

$$\left(V^+ - V^-\right) \frac{Z}{Z_0} = V^+ + V^-$$

$$\left(Z - Z_0\right) V^+ = \left(Z + Z_0\right) V^-$$

$$\therefore V^- = \frac{\left(Z - Z_0\right)}{\left(Z + Z_0\right)} V^+$$

b. Show the voltage standing wave ratio $\frac{|V^+|+|V^-|}{|V^+|-|V^-|}$ is

$$VSWR = \left(\frac{Z}{Z_0}\right)^{\pm 1},$$

the sign chosen so $VSWR > 1$.
https://en.wikipedia.org/wiki/Standing_wave_ratio

$$\frac{|V^+|+|V^-|}{|V^+|-|V^-|} = \frac{(Z+Z_0+Z-Z_0)V^+}{(Z+Z_0-Z+Z_0)V^+} = \frac{Z}{Z_0}$$

for $Z \geq Z_0$. When $Z < Z_0$, V^+ and V^- have the opposite sign. Then

$$\frac{|V^+|+|V^-|}{|V^+|-|V^-|} = \frac{(Z+Z_0-Z+Z_0)V^+}{(Z+Z_0+Z-Z_0)V^+} = \frac{Z_0}{Z}.$$

c. What is the termination impedance, and what are the answers to a. and b. when the impedance is matched to the transmission line.

When the impedance is matched to the transmission line $Z = Z_0$. The reflected wave vanishes and $VSWR = 1$. The impedance of the transmission line is now simply $V^+/I^+ = V_Z/I_Z = Z_0$. The jargon is "the load is matched to the transmission line".

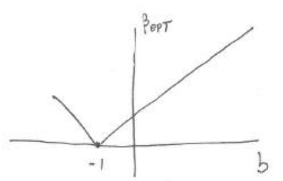
3. Assuming no microphonics, plot β_{opt} and P_g^{opt} as function of b (beam loading) for b = -5 to 5, and explain the results.

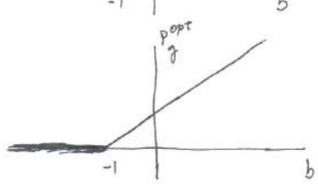
How do the results change if microphonics is present?

From the formula for optimal coupling

$$\beta_{OPT} = |b+1|$$

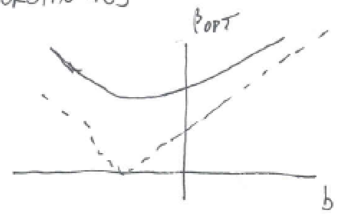
$$P_g^{OPT} = \frac{V_c^2}{2R_a}\left[(b+1) + \sqrt{(b+1)^2}\right] = \frac{V_c^2}{2R_a}\left[b+1 + |b+1|\right]$$





$b=-1$ CORRESPONDS TO GRADIENT IN THE CAVITY BEING PROVIDED DIRECTLY BY THE BEAM

WITH MICROPHONICS



SOME POWER REQUIRED TO CONTROL THE CAVITY