Homework Problems II
Accelerator Physics

1. Either $\cosh\sqrt{-k}s$ and $\sinh\sqrt{-k}s$ or $\exp\sqrt{-k}s$ and $\exp-\sqrt{-k}s$ could be used as the fundamental solutions of the homogeneous equation to build up the transfer matrix. Several students used each pair. In this particular solution let's use the pair $\cosh\sqrt{-k}s$ and $\sinh\sqrt{-k}s$.

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}s\right) & \sinh\left(\sqrt{-k}s\right) \\ \sqrt{-k}\sinh\left(\sqrt{-k}s\right) & \sqrt{-k}\cosh\left(\sqrt{-k}s\right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}s_0\right) & \sinh\left(\sqrt{-k}s_0\right) \\ \sqrt{-k}\sinh\left(\sqrt{-k}s_0\right) & \sqrt{-k}\cosh\left(\sqrt{-k}s_0\right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{\sqrt{-k}} \begin{pmatrix} \sqrt{-k}\cosh\left(\sqrt{-k}s_0\right) & -\sinh\left(\sqrt{-k}s_0\right) \\ -\sqrt{-k}\sinh\left(\sqrt{-k}s_0\right) & \cosh\left(\sqrt{-k}s_0\right) \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}s\right) & \sinh\left(\sqrt{-k}s\right) \\ \sqrt{-k}\sinh\left(\sqrt{-k}s\right) & \sqrt{-k}\cosh\left(\sqrt{-k}s\right) \end{pmatrix}$$

$$\times \frac{1}{\sqrt{-k}} \begin{pmatrix} \sqrt{-k}\cosh\left(\sqrt{-k}s_0\right) & -\sinh\left(\sqrt{-k}s_0\right) \\ -\sqrt{-k}\sinh\left(\sqrt{-k}s_0\right) & \cosh\left(\sqrt{-k}s_0\right) \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}(s-s_0)\right) & \sinh\left(\sqrt{-k}(s-s_0)\right)/\sqrt{-k} \\ \sqrt{-k}\sinh\left(\sqrt{-k}(s-s_0)\right) & \cosh\left(\sqrt{-k}(s-s_0)\right) \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}.$$

Using $s-s_0=L$ for a defocusing quad of length *L*, the transfer matrix is

$$\begin{pmatrix} \cosh\left(\sqrt{-k}L\right) & \sinh\left(\sqrt{-k}L\right)/\sqrt{-k} \\ \sqrt{-k}\sinh\left(\sqrt{-k}L\right) & \cosh\left(\sqrt{-k}L\right) \end{pmatrix}.$$

The determinate of this matrix is clearly one.

2. The total transfer matrix is

$$M_{tot} = \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - L_2/f & L_2 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - L_2/f & L_1 + L_2 - L_1 L_2/f \\ -1/f & 1 - L_1/f \end{pmatrix}$$

When the image is focused, all of the rays originating from the same point on the object need to go to the same location on the image. So the image location of all such rays needs to be independent of the initial (emitted) angle. This means for focus $M_{tot,12} = 0$. Setting the matrix element to 0, and dividing by $L_1 L_2$ yields

$$\frac{1}{L_1} + \frac{1}{L_2} = \frac{1}{f}.$$

3.

$$M = \begin{pmatrix} 1 & 0 \\ -1/(2f) & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/(2f) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & L \\ -1/(2f) & 1 - L/(2f) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 - L/(2f) & L \\ -1/(2f) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 + L/f & L \\ 1/(2f) - L/(2f^2) & 1 - L/(2f) \end{pmatrix} \begin{pmatrix} 1 - L/(2f) & L \\ -1/(2f) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 + L/(2f) - L^2/(2f^2) - L/(2f) & 2L + L^2/f \\ \begin{pmatrix} 1/(2f) - L/(2f^2) - L/(4f^2) + L^2/(4f^3) \\ -1/(2f) + L/(4f^2) \end{pmatrix} & L/(2f) - L^2/(2f^2) + 1 - L/(2f) \end{pmatrix}$$
$$= \begin{pmatrix} 1 - L^2/(2f^2) & 2L + L^2/f \\ -L/(2f^2) + L^2/(4f^3) & 1 - L^2/(2f^2) \end{pmatrix}.$$

To get the matrix starting with the middle of the defocusing magnet replace f with $-f$. The resulting transfer matrix is

$$M = \begin{pmatrix} 1 & 0 \\ 1/(2f) & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/(2f) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - L^2/(2f^2) & 2L - L^2/f \\ -L/(2f^2) - L^2/(4f^3) & 1 - L^2/(2f^2) \end{pmatrix}$$

Clearly the matrix traces of the two resulting matrices are the same. Now $\cos 60^\circ = 1/2$, so $L/f = 1$. Because the diagonal elements are equal, $\alpha = 0$. The matched beta in the middle of the focusing magnet is $\beta = 3L/\sin 60^\circ = 6L/\sqrt{3}$. The matched beta in the middle of the defocusing magneti is one-third as much, $\beta = 2L/\sqrt{3}$.

4. This is several pages of algebra and somewhat tedious. However, as a result of this calculation you should be convinced that the formula for the transfer matrix in terms of $\alpha, \beta,$ and $\Delta\mu$ is accurate. If it weren't accurate, there's no way that all the matrix elements would conspire nicely to satisfy the composition formula!

$$M_{s'',s} = M_{s'',s'} M_{s',s} =$$

$$\begin{pmatrix} \sqrt{\dfrac{\beta(s'')}{\beta(s')}}\left(\cos\Delta\mu_{s'',s'} + \alpha(s')\sin\Delta\mu_{s'',s'}\right) & \sqrt{\beta(s'')\beta(s')}\sin\Delta\mu_{s'',s'} \\ -\dfrac{1}{\sqrt{\beta(s'')\beta(s')}}\begin{bmatrix}\left(1+\alpha(s'')\alpha(s')\right)\sin\Delta\mu_{s'',s'} \\ +\left(\alpha(s'')-\alpha(s')\right)\cos\Delta\mu_{s'',s'}\end{bmatrix} & \sqrt{\dfrac{\beta(s')}{\beta(s'')}}\left(\cos\Delta\mu_{s'',s'} - \alpha(s'')\sin\Delta\mu_{s'',s'}\right) \end{pmatrix}$$

$$\times\begin{pmatrix} \sqrt{\dfrac{\beta(s')}{\beta(s)}}\left(\cos\Delta\mu_{s',s} + \alpha(s)\sin\Delta\mu_{s',s}\right) & \sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s} \\ -\dfrac{1}{\sqrt{\beta(s')\beta(s)}}\begin{bmatrix}\left(1+\alpha(s')\alpha(s)\right)\sin\Delta\mu_{s',s} \\ +\left(\alpha(s')-\alpha(s)\right)\cos\Delta\mu_{s',s}\end{bmatrix} & \sqrt{\dfrac{\beta(s)}{\beta(s')}}\left(\cos\Delta\mu_{s',s} - \alpha(s')\sin\Delta\mu_{s',s}\right) \end{pmatrix}.$$

$$\begin{pmatrix} \sqrt{\dfrac{\beta(s")}{\beta(s)}}\begin{bmatrix} \cos\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ +\alpha(s')\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ +\alpha(s)\cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\alpha(s')\alpha(s)\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ -\left[1+\alpha(s')\alpha(s)\right]\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ -\left[\alpha(s')-\alpha(s)\right]\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \end{bmatrix} & \sqrt{\beta(s")\beta(s)}\begin{bmatrix} \cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\alpha(s')\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ -\alpha(s')\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \end{bmatrix} \\ -\dfrac{1}{\sqrt{\beta(s")\beta(s)}}\begin{bmatrix} \left[1+\alpha(s")\alpha(s')\right]\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ +\left[\alpha(s")-\alpha(s')\right]\cos\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ +\alpha(s)\left[1+\alpha(s")\alpha(s')\right]\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\alpha(s)\left[\alpha(s")-\alpha(s')\right]\cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\left[1+\alpha(s')\alpha(s)\right]\cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\left[\alpha(s')-\alpha(s)\right]\cos\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ -\alpha(s")\left[1+\alpha(s')\alpha(s)\right]\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ -\alpha(s")\left[\alpha(s')-\alpha(s)\right]\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \end{bmatrix} & \sqrt{\dfrac{\beta(s)}{\beta(s")}}\begin{bmatrix} -\left[1+\alpha(s")\alpha(s')\right]\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ -\left[\alpha(s")-\alpha(s')\right]\cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ \cos\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ -\alpha(s")\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ -\alpha(s')\cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\alpha(s")\alpha(s')\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \end{bmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\frac{\beta(s")}{\beta(s)}}\begin{bmatrix} \cos\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ +\alpha(s)\cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ -\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\alpha(s)\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \end{bmatrix} & \sqrt{\beta(s")\beta(s)}\begin{bmatrix} \cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \end{bmatrix} \\ -\frac{1}{\sqrt{\beta(s")\beta(s)}}\begin{bmatrix} \sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ +\alpha(s")\cos\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ +\alpha(s)\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\alpha(s)\alpha(s")\cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ +\cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ -\alpha(s)\cos\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ -\alpha(s")\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ -\alpha(s")\alpha(s)\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \end{bmatrix} & \sqrt{\frac{\beta(s)}{\beta(s")}}\begin{bmatrix} -\sin\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ -\alpha(s")\cos\Delta\mu_{s",s'}\sin\Delta\mu_{s',s} \\ \cos\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \\ -\alpha(s")\sin\Delta\mu_{s",s'}\cos\Delta\mu_{s',s} \end{bmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\frac{\beta(s")}{\beta(s)}}\begin{bmatrix} \cos(\Delta\mu_{s",s'}+\Delta\mu_{s',s}) \\ +\alpha(s)\sin(\Delta\mu_{s",s'}+\Delta\mu_{s',s}) \end{bmatrix} & \sqrt{\beta(s")\beta(s)}\sin(\Delta\mu_{s",s'}+\Delta\mu_{s',s}) \\ -\frac{1}{\sqrt{\beta(s")\beta(s)}}\begin{Bmatrix} [1+\alpha(s")\alpha(s)]\sin(\Delta\mu_{s",s'}+\Delta\mu_{s',s}) \\ +[\alpha(s")-\alpha(s)]\cos(\Delta\mu_{s",s'}+\Delta\mu_{s',s}) \end{Bmatrix} & \sqrt{\frac{\beta(s)}{\beta(s")}}\begin{bmatrix} \cos(\Delta\mu_{s",s'}+\Delta\mu_{s',s}) \\ -\alpha(s")\sin(\Delta\mu_{s",s'}+\Delta\mu_{s',s}) \end{bmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\frac{\beta(s")}{\beta(s)}}\left[\cos\Delta\mu_{s",s}+\alpha(s)\sin\Delta\mu_{s",s}\right] & \sqrt{\beta(s")\beta(s)}\sin\Delta\mu_{s",s} \\ -\frac{1}{\sqrt{\beta(s")\beta(s)}}\begin{Bmatrix} [1+\alpha(s")\alpha(s)]\sin\Delta\mu_{s",s} \\ +[\alpha(s")-\alpha(s)]\cos\Delta\mu_{s",s} \end{Bmatrix} & \sqrt{\frac{\beta(s)}{\beta(s")}}\left[\cos\Delta\mu_{s",s}-\alpha(s")\sin\Delta\mu_{s",s}\right] \end{pmatrix}$$

Now

$$(M_{s',s})_{11} = \sqrt{\frac{\beta(s')}{\beta(s)}}\left[\cos\Delta\mu_{s',s}+\alpha(s)\sin\Delta\mu_{s',s}\right]$$

$$(M_{s',s})_{12} = \sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s}$$

$$\beta(s)(M_{s',s})_{11} - \alpha(s)(M_{s',s})_{12} = \sqrt{\beta(s')\beta(s)}\cos\Delta\mu_{s',s}$$

$$\therefore \tan\Delta\mu_{s',s} = \frac{(M_{s',s})_{12}}{\beta(s)(M_{s',s})_{11}-\alpha(s)(M_{s',s})_{12}}$$