1. The key to this problem is to realize that the βs in the expression for γ have a time derivative in them. So an argument of this type is valid

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\int_{t_1}^{t_2}\sqrt{dt^2-\left(1/c^2\right)\left(dx^2+dy^2+dz^2\right)}.$$

But the invariance of the space-time interval shows that under a general Lorentz transformation

$$\int_{t_1}^{t_2}\sqrt{dt^2-\left(1/c^2\right)\left(dx^2+dy^2+dz^2\right)}=\int_{t'_1}^{t'_2}\sqrt{dt'^2-\left(1/c^2\right)\left(dx'^2+dy'^2+dz'^2\right)}$$

Therefore

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\int_{t'_1}^{t'_2}\sqrt{1-\beta'^2}\,dt'.$$

The square root differential expression is well defined because the four-velocity of any massive particle is always time-like.

A more formal and precise argument is the following. Divide the closed interval $\left[t_1,t_2\right]$ into N equal sub-intervals of duration $\Delta t=\left(t_2-t_1\right)/N$ labeled by the index i: $I_i=\left[t_1+\left(i-1\right)\Delta t,t_1+i\Delta t\right]$. By the general Lorentz transformation between frames we may establish the coordinates of the space-time events $c\left(t_1\right),\vec{x}\left(t_1\right)$ and $c\left(t_1+i\Delta t\right),\vec{x}\left(t_1+i\Delta t\right)$ in the prime frame. Call the coordinates in the K' frame $c\left(t'_0\right),\vec{x}'_0$ and $ct'_i,\vec{x}'_i$ respectively. Recall the mean value theorem from calculus

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\sum_{i=1}^{N}\int_{I_i}\sqrt{1-\beta^2}\,dt=\sum_{i=1}^{N}\sqrt{1-\beta^2\left(T_i\right)}\Delta t$$

for some $T_i\in I_i$. In the limit $N\to\infty$, the intervals become infinitesimals and

$$\vec{\beta}\left(T_i\right)\to\frac{\Delta\vec{x}_i}{\Delta t}\equiv\frac{\vec{x}_i-\vec{x}_{i-1}}{\Delta t}.$$

This means

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\lim_{N\to\infty}\sum_{i=1}^{N}\sqrt{\Delta t^2-\left(1/c^2\right)\left|\Delta\vec{x}_i\right|^2}$$

The Lorentz transformations are linear, and so the differentials $c\Delta t$ and $\Delta\vec{x}$ transform in the same way as ct and $\vec{x}$. Therefore, by the invariance of the space-time interval under Lorentz transformations,

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt=\lim_{N\to\infty}\sum_{i=1}^{N}\sqrt{\Delta t_i'^2-\left(1/c^2\right)\left|\Delta\vec{x}'_i\right|^2}=\int_{t'_1}^{t'_2}\sqrt{1-\beta'^2}\,dt',$$

because the durations of all the intervals in the prime frame approach zero as N increases without bound. It should be noted that $\Delta t_i'$ is NOT necessarily constant as i changes when there is acceleration in the orbit.

2. By the relativistic Lorentz Force equation

$$\vec{\mathrm{v}} \cdot \frac{d\left(\gamma m \vec{\mathrm{v}}\right)}{d\tau} = \vec{\mathrm{v}} \cdot q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right) = q\vec{\mathrm{v}} \cdot \vec{E}$$

Now

$$\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1-\left|\vec{\mathrm{v}}\right|^2 / c^2}} = \frac{\vec{\mathrm{v}} \cdot \left(d\vec{\mathrm{v}} / dt\right)}{c^2\left(1-\beta^2\right)^{3/2}} = \frac{\gamma^3}{c^2}\vec{\mathrm{v}} \cdot \left(d\vec{\mathrm{v}} / dt\right)$$

and

$$q\vec{\mathrm{v}} \cdot \vec{E} = \gamma\vec{\mathrm{v}} \cdot \frac{d\left(m\vec{\mathrm{v}}\right)}{dt} + \frac{m\left|\vec{\mathrm{v}}\right|^2}{c^2}\gamma^3\vec{\mathrm{v}} \cdot \left(d\vec{\mathrm{v}} / dt\right)$$
$$= \gamma m\left(1+\beta^2\gamma^2\right)\vec{\mathrm{v}} \cdot \left(d\vec{\mathrm{v}} / dt\right) = \gamma^3 m\vec{\mathrm{v}} \cdot \left(d\vec{\mathrm{v}} / dt\right).$$

So

$$\frac{d\gamma}{dt} = \frac{q\vec{\mathrm{v}} \cdot \vec{E}}{mc^2}.$$

Finally

$$\frac{d\left(\gamma m \mathrm{v}_x\right)}{d\tau} = qE_x u^0 + 0 + qcB_z u^2 - qcB_y u^3$$
$$\gamma\frac{d\left(\gamma m \mathrm{v}_x\right)}{dt} = qE_x\gamma + q\mathrm{v}_y B_z\gamma - q\mathrm{v}_z B_y\gamma$$
$$\frac{d\left(\gamma m \mathrm{v}_x\right)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)_x$$
$$\frac{d\left(\gamma m \mathrm{v}_y\right)}{d\tau} = qE_y u^0 - qcB_z u^1 + 0 + qcB_x u^3$$
$$\gamma\frac{d\left(\gamma m \mathrm{v}_y\right)}{dt} = qE_y\gamma - q\mathrm{v}_x B_z\gamma + q\mathrm{v}_z B_x\gamma$$
$$\frac{d\left(\gamma m \mathrm{v}_y\right)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)_y$$
$$\frac{d\left(\gamma m \mathrm{v}_z\right)}{d\tau} = qE_z u^0 + qcB_y u^1 - qcB_x u^2 + 0$$
$$\gamma\frac{d\left(\gamma m \mathrm{v}_z\right)}{dt} = qE_z\gamma + q\mathrm{v}_x B_y\gamma - q\mathrm{v}_y B_x\gamma$$
$$\frac{d\left(\gamma m \mathrm{v}_z\right)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)_z$$

3. My solution

$$\frac{d\gamma}{dt}=0\rightarrow\frac{d|\vec{\mathrm{v}}|}{dt}=0\quad |\vec{\mathrm{v}}|=const=\mathrm{v}_0$$

$$\frac{d\mathrm{v}_x}{dt}=\frac{qB}{\gamma m}\mathrm{v}_y\qquad\frac{d\mathrm{v}_y}{dt}=-\frac{qB}{\gamma m}\mathrm{v}_x$$

$$\frac{d^2\mathrm{v}_x}{dt^2}+\Omega_c^2\mathrm{v}_x=0\qquad\frac{d^2\mathrm{v}_y}{dt^2}+\Omega_c^2\mathrm{v}_y=0$$

$$\mathrm{v}_x(t)=A\cos(\Omega_c t+\delta)\qquad\mathrm{v}_y(t)=-A\sin(\Omega_c t+\delta)$$

$$|\vec{\mathrm{v}}|=A\rightarrow A=\mathrm{v}_0$$

$$x(t)=x_c+\frac{\mathrm{v}_0}{\Omega_c}\sin(\Omega_c t+\delta)\qquad y(t)=y_c+\frac{\mathrm{v}_0}{\Omega_c}\cos(\Omega_c t+\delta)$$

$$r=\frac{\mathrm{v}_0}{\Omega_c}=\frac{\beta c}{qB/\gamma m}$$

It turns out that Serkan Golge doesn't like me to invoke "unphysical" things like time derivatives of acceleration. For his benefit one can also get it like this

$$\mathrm{v}_x(t)=C+\Omega_c y(t)$$

$$\frac{d\mathrm{v}_y}{dt}=-\frac{qB}{\gamma m}\left(C+\Omega_c y(t)\right)$$

$$\frac{d^2y}{dt^2}+\Omega_c^2 y=-\frac{qB}{\gamma m}C$$

$$y(t)=-\frac{C}{\Omega_c}+A'\cos(\Omega_c t+\delta)$$

$$\mathrm{v}_x(t)=\Omega_c A'\cos(\Omega_c t+\delta)\qquad\mathrm{v}_y(t)=-\Omega_c A'\sin(\Omega_c t+\delta)\qquad\therefore\Omega_c A'=\mathrm{v}_0$$

$$x(t)=C'+A'\sin(\Omega_c t+\delta)$$

$$A'=r=\frac{\mathrm{v}_0}{\Omega_c}$$

4. This problem is a perfect example of the use of the magnetic rigidity. The electron relativistic momentum is $p=\sqrt{\gamma^2-1}m_0c=\sqrt{\gamma^2-1}\left(0.511\ \mathrm{MeV}/c\right)$, the magnetic rigidity is $\sqrt{\gamma^2-1}\left(0.511\ \mathrm{MV}/c\right)$, and 1 T = 1 (V sec)/($\mathrm{m}^2$). Now

$$B=\frac{(B\rho)}{L}2\sin(\theta/2)$$

for magnets in the normal configuration. The bend angles are $\pi/16 = 0.19635$ rad for the first arc and $\pi/32 = 0.098175$ rad for the rest of the arcs.

$$B_1 = \frac{\sqrt{(605/0.511)^2 - 1} \times 0.511 \times 10^6\,\text{V}}{2.998 \times 10^8\ \text{m/sec}(1\ \text{m})} 2\sin 0.098175 = 0.3956\ \text{T} = 3.956\ \text{kG}$$

$$B_2 = \frac{\sqrt{(1693/0.511)^2 - 1} \times 0.511 \times 10^6\,\text{V}}{2.998 \times 10^8\ \text{m/sec}(1\ \text{m})} 2\sin 0.04909 = 0.5542\ \text{T} = 5.542\ \text{kG}$$

$$B_3 = \frac{\sqrt{(2781/0.511)^2 - 1} \times 0.511 \times 10^6\,\text{V}}{2.998 \times 10^8\ \text{m/sec}(2\ \text{m})} 2\sin 0.04909 = 0.4552\ \text{T} = 4.552\ \text{kG}$$

$$B_4 = \frac{\sqrt{(3868/0.511)^2 - 1} \times 0.511 \times 10^6\,\text{V}}{2.998 \times 10^8\ \text{m/sec}(3\ \text{m})} 2\sin 0.04909 = 0.4220\ \text{T} = 4.220\ \text{kG}$$

$$B_5 = \frac{\sqrt{(4956/0.511)^2 - 1} \times 0.511 \times 10^6\,\text{V}}{2.998 \times 10^8\ \text{m/sec}(3\ \text{m})} 2\sin 0.04909 = 0.5408\ \text{T} = 5.408\ \text{kG}$$

Arc	Electron Energy (MeV)	Number of Dipoles	Dipole Length (m)	Bend Angle (rad)	Magnetic Field (T)
1	605	16	1	0.19635	0.3956
2	1693	32	1	0.098175	0.5542
3	2781	32	2	0.098175	0.4552
4	3868	32	3	0.098175	0.4220
5	4956	32	3	0.098175	0.5408

5. If $\phi = 180^\circ$, then the transfer matrix is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

In practice this condition may be achieve with a single transfer matrix, 2 transfer matrices of 90 degrees phase advance, 3 of 60 degrees of phase advance, etc.