Accelerator Physics
Coupling Control

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Lecture 9

Time Depedence

For an oscillation starting in *x*-direction

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{x_0}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_g t) + \frac{x_0}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos\left(\sqrt{\omega_g^2 + 2\omega_s^2}\, t\right)$$

$$\omega_g = \frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} + \frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2}$$

$$\sqrt{\omega_g^2 + 2\omega_s^2} = \frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} - \frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2}$$

$$\therefore x(t) = x_0 \cos\left[\frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right] \cos\left[\frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right]$$

$$\therefore y(t) = -x_0 \sin\left[\frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right] \sin\left[\frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right]$$

Qualitatively

- Oscillation energy migrates $x \rightarrow y \rightarrow x$
- Period for a complete cycle is

$$\frac{4\pi}{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}$$

Becomes longer the weaker the coupling (→compensation)

- If un-coupled oscillation periods different

$$\omega^2 = \frac{\omega_1^2 + \omega_2^2 + 2\omega_s^2 \pm \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 - 4\omega_s^4}}{2}$$

Eigenvectors no longer pure symmetric and antisymmetric but "migration" is fairly generic behavior

Energy and Mode Invariants

- In normal mode coordinates

$$q_+ \frac{1}{\sqrt{2}}(x-y) \quad p_+ = \frac{1}{\sqrt{2}}(\dot{x}-\dot{y}) \quad q_- \frac{1}{\sqrt{2}}(x+y) \quad p_- = \frac{1}{\sqrt{2}}(\dot{x}+\dot{y})$$

- (uncoupled) Hamiltonian is

$$H = \frac{p_+^2}{2m} + \frac{m\omega_+^2 q_+^2}{2} + \frac{p_-^2}{2m} + \frac{m\omega_-^2 q_-^2}{2}$$

and is a constant of the motion.

- So are the individual mode excitation amplitudes

$$\varepsilon_+ = \frac{2}{\omega_+}\left(\frac{p_+^2}{2m} + \frac{m\omega_+^2 q_+^2}{2}\right) \quad \varepsilon_- = \frac{2}{\omega_-}\left(\frac{p_-^2}{2m} + \frac{m\omega_-^2 q_-^2}{2}\right)$$

Solutions Again

- In normal mode coordinates

$$q_+ = \sqrt{\varepsilon_+ \beta_+} \cos\left(\omega_+ t + \delta_+\right) \quad q_- = \sqrt{\varepsilon_- \beta_-} \cos\left(\omega_- t + \delta_-\right)$$

- betas and deltas from

$$\beta_\pm = \frac{1}{m\omega_\pm} \qquad \delta_\pm = \tan^{-1} \frac{p_{\pm 0}}{m\omega_\pm q_0}$$

- back in coupled variables excitation looks like

$$x_1(z) = \sqrt{\varepsilon_+ \beta_+} \cos\left(\omega_+ t\right) \qquad y_1(z) = \sqrt{\varepsilon_+ \beta_+} \cos\left(\omega_+ t + \pi\right)$$

$$x_2(z) = \sqrt{\varepsilon_+ \beta_+} \sin\left(\omega_+ t\right) \qquad y_2(z) = \sqrt{\varepsilon_+ \beta_+} \sin\left(\omega_+ t + \pi\right)$$

$$x_3(z) = \sqrt{\varepsilon_- \beta_-} \cos\left(\omega_- t\right) \qquad y_3(z) = \sqrt{\varepsilon_- \beta_-} \cos\left(\omega_- t\right)$$

$$x_4(z) = \sqrt{\varepsilon_- \beta_-} \sin\left(\omega_- t\right) \qquad y_4(z) = \sqrt{\varepsilon_- \beta_-} \cos\left(\omega_- t\right)$$

Wiedemann Discussion

- 1-D know two pseudoharmonic trajectories

$$u_1 = \sqrt{\beta(z)}\cos\psi(z)$$

$$u_2 = \sqrt{\beta(z)}\sin\psi(z)$$

- z-derivative

$$u_1' = \frac{1}{\sqrt{\beta(z)}}\left(-\alpha\cos\psi(z) - \sin\psi(z)\right)$$

$$u_2' = \frac{1}{\sqrt{\beta(z)}}\left(-\alpha\sin\psi(z) + \cos\psi(z)\right)$$

- Express ellipse functions in terms of solutions

$$u_1^2 + u_2^2 = \beta(z)$$

$$u_1 u_1' + u_2 u_2' = -\alpha$$

$$u_1'^2 + u_2'^2 = \frac{\alpha^2 + 1}{\beta} = \gamma$$

- Change in phase (phase advance) is

$$\cos(\Delta\psi) = \cos(\psi - \psi_0) =$$

$$\frac{u_1 u_{10} + u_2 u_{20}}{\sqrt{u_1^2 + u_2^2}\sqrt{u_{10}^2 + u_{20}^2}}$$

Transfer Matrix

- General solution

$$\begin{pmatrix} x(z) \\ x'(z) \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

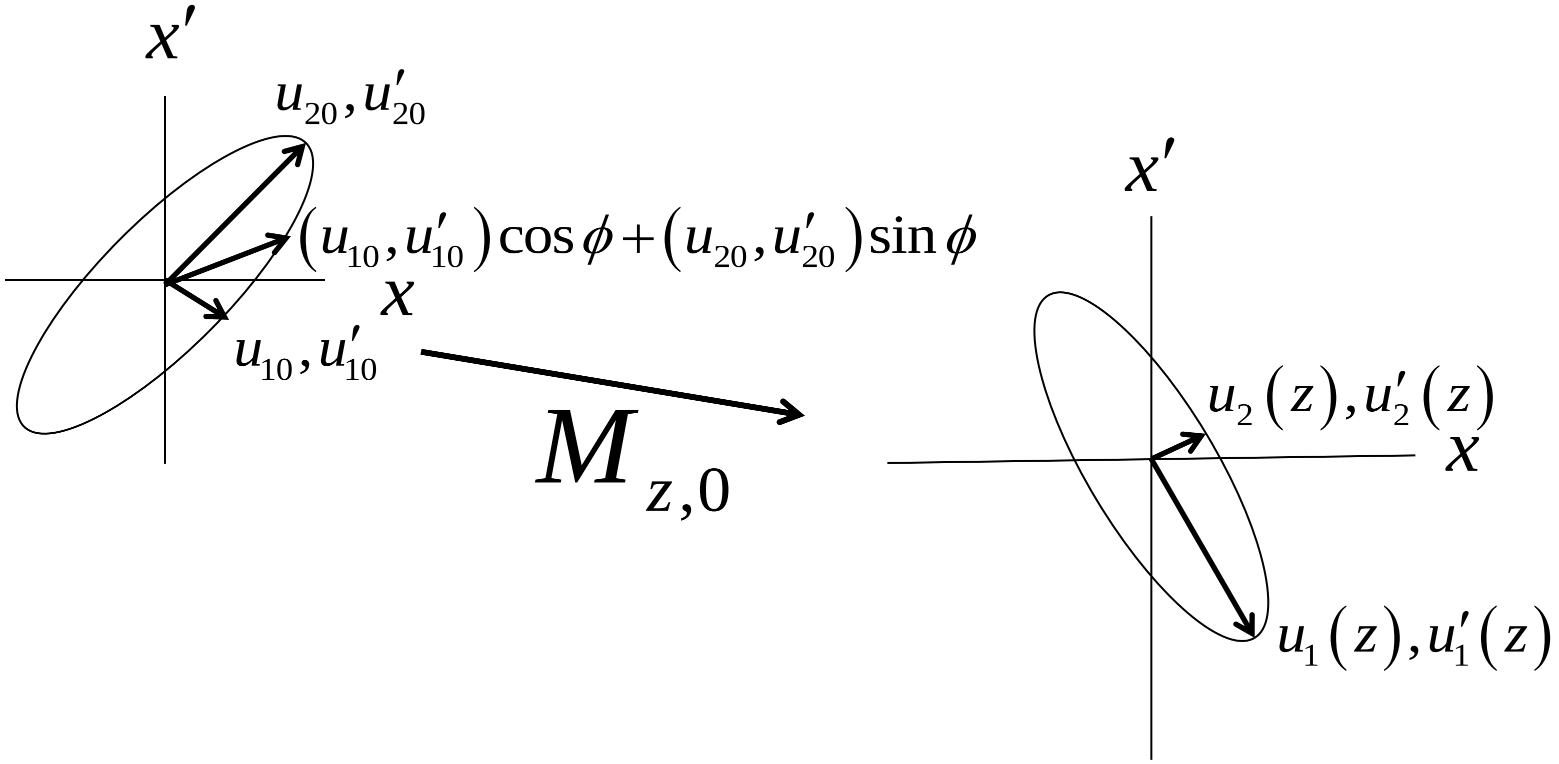
- Initial conditions

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} u_{10} & u_{20} \\ u_{10}' & u_{20}' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{u_{10}u_{20}' - u_{20}u_{10}'} \begin{pmatrix} u_{20}' & -u_{20} \\ -u_{10}' & u_{10} \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

- Transfer matrix

$$M = \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} u_{20}' & -u_{20} \\ -u_{10}' & u_{10} \end{pmatrix} = \begin{pmatrix} u_1 u_{20}' - u_2 u_{10}' & u_2 u_{10} - u_1 u_{20} \\ u_1' u_{20}' - u_2' u_{10}' & u_2' u_{10} - u_1' u_{20} \end{pmatrix}$$

Ellipse Transformation



In 2×2 D

- As in toy problem, invariant is hyperellipsoid in 4 D. Parameterize individual state points as

$$\vec{v}(z) = \sqrt{\varepsilon_I}\left[v_1(z)\cos\vartheta_I - v_2(z)\sin\vartheta_I\right]\cos\chi$$
$$\sqrt{\varepsilon_{II}}\left[v_3(z)\cos\vartheta_{II} - v_4(z)\sin\vartheta_{II}\right]\sin\chi$$

- Compared to uncoupled case have twice the number of matrix elements. Need twice the number of solutions. Wiedemann chooses

$$x_1(z) = \sqrt{\beta_{x_I}(z)}\cos\phi_{x_1}(z) \qquad y_1(z) = \sqrt{\beta_{y_I}(z)}\cos\phi_{y_I}(z)$$
$$x_2(z) = \sqrt{\beta_{x_I}(z)}\sin\phi_{x_1}(z) \qquad y_2(z) = \sqrt{\beta_{y_I}(z)}\sin\phi_{y_I}(z)$$
$$x_3(z) = \sqrt{\beta_{x_{II}}(z)}\cos\phi_{x_{II}}(z) \qquad y_3(z) = \sqrt{\beta_{y_{II}}(z)}\cos\phi_{y_{II}}(z)$$
$$x_4(z) = \sqrt{\beta_{x_{II}}(z)}\sin\phi_{x_{II}}(z) \qquad y_4(z) = \sqrt{\beta_{y_{II}}(z)}\sin\phi_{y_{II}}(z)$$

As Before

$$\beta_{x_I} = x_1^2 + x_2^2 \qquad \beta_{y_I} = y_1^2 + y_2^2$$

$$\beta_{x_{II}} = x_3^2 + x_4^2 \qquad \beta_{y_{II}} = y_3^2 + y_4^2$$

$$\cos\phi_{x_I} = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \qquad \cos\phi_{y_I} = \frac{y_1}{\sqrt{y_1^2 + y_2^2}}$$

$$\cos\phi_{x_{II}} = \frac{x_3}{\sqrt{x_3^2 + x_4^2}} \qquad \cos\phi_{y_I} = \frac{y_3}{\sqrt{y_3^2 + y_4^2}}$$

$$\vdots$$

Convert Fundamental Solutions

- Convert usual phase space variables

$$x_1(z) = \sqrt{\beta_{x_I}} \cos\phi_{x_I} \quad x_1'(z) = \sqrt{\gamma_{x_I}} \cos\psi_{x_I}$$

$$\rightarrow \gamma_{x_I} = \frac{\beta_{x_I}^2 \phi_{x_I}'^2 + \alpha_{x_I}^2}{\beta_{x_I}} \qquad \psi_{x_I} = \phi_{x_I} - \tan^{-1} \frac{\beta_{x_I} \phi_{x_I}'}{\alpha_{x_I}}$$

- Points of hyper-ellipsoid

$$v(z) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \sqrt{\varepsilon_I} \begin{pmatrix} \sqrt{\beta_{x_I}(z)} \cos\left(\phi_{x_I} + \vartheta_I\right) \\ \sqrt{\gamma_{x_I}(z)} \cos\left(\psi_{x_I} + \vartheta_I\right) \\ \sqrt{\beta_{y_I}(z)} \cos\left(\phi_{y_I} + \vartheta_I\right) \\ \sqrt{\gamma_{y_I}(z)} \cos\left(\psi_{y_I} + \vartheta_I\right) \end{pmatrix} \cos\chi + \sqrt{\varepsilon_{II}} \begin{pmatrix} \sqrt{\beta_{x_{II}}(z)} \cos\left(\phi_{x_{II}} + \vartheta_{II}\right) \\ \sqrt{\gamma_{x_{II}}(z)} \cos\left(\psi_{x_{II}} + \vartheta_{II}\right) \\ \sqrt{\beta_{y_{II}}(z)} \cos\left(\phi_{y_{II}} + \vartheta_{II}\right) \\ \sqrt{\gamma_{y_{II}}(z)} \cos\left(\psi_{y_{II}} + \vartheta_{II}\right) \end{pmatrix} \sin\chi$$

Beam size

- Set all derivatives zero

$$\phi_{x,y_{I,II}}(z) = -\vartheta_{I,II}$$

$$\frac{d(x,y)}{d\chi} = 0 \rightarrow \sin^2\chi = \frac{\varepsilon_{II}\beta_{(x,y)}}{E_{(x,y)}} \qquad E_{(x,y)} = \sqrt{\varepsilon_I \beta_{(x,y)_I} + \varepsilon_{II}\beta_{(x,y)_{II}}}$$

$$\psi_{x,y_{I,II}}(z) = -\vartheta_{I,II}$$

$$\frac{d(x',y')}{d\chi} = 0 \rightarrow \sin^2\chi = \frac{\varepsilon_{II}\beta_{(x,y)}}{A_{(x,y)}} \qquad A_{(x,y)} = \sqrt{\varepsilon_I \gamma_{(x,y)_I} + \varepsilon_{II}\gamma_{(x,y)_{II}}}$$

- Projected Ellipse

$$E'_{(x,y)} = -\frac{\varepsilon_I \alpha_{(x,y)_I} + \varepsilon_{II}\alpha_{(x,y)_{II}}}{\sqrt{\varepsilon_I \beta_{(x,y)_I} + \varepsilon_{II}\beta_{(x,y)_{II}}}}$$

$$A^2_{(x,y)}(x,y)^2 - 2E'_{(x,y)}E_{(x,y)}(x,y)(x',y') + E^2_{(x,y)}(x',y')^2 = \varepsilon^2_{(x,y)}$$

$$\varepsilon^2_{(x,y)} = E_{(x,y)}\sqrt{A^2_{(x,y)} - E'^2_{(x,y)}}$$

Beam Tilt

- Project *x*-*y* plane

$$E_{(x,y)} = \sqrt{\varepsilon_I \beta_{(x,y)_I} + \varepsilon_{II} \beta_{(x,y)_{II}}}$$

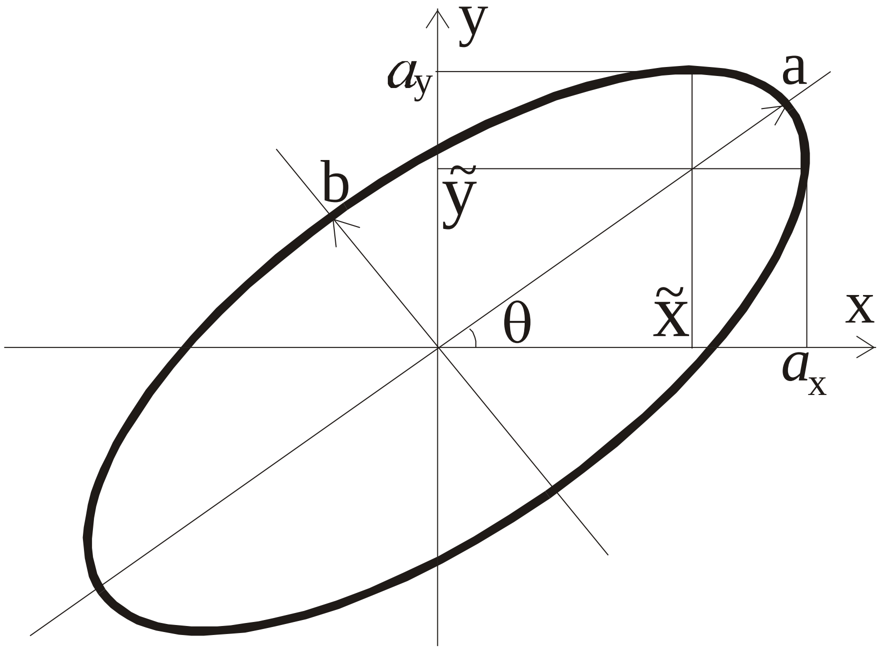
$$E_{xy} = \frac{\varepsilon_I \sqrt{\beta_{x_I} \beta_{y_I}} \cos \Delta\phi_I + \varepsilon_{II} \sqrt{\beta_{x_{II}} \beta_{y_{II}}} \cos \Delta\phi_{II}}{E_x}$$

$$\Delta\phi_{I,II} = \phi_{x_{I,II}} - \phi_{x_{I,II}}$$

$$\tan 2\psi = \frac{2E_x E_y}{E_x^2 - E_y^2} = 2\frac{\varepsilon_I \sqrt{\beta_{x_I} \beta_{y_I}} \cos \Delta\phi_I + \varepsilon_{II} \sqrt{\beta_{x_{II}} \beta_{y_{II}}} \cos \Delta\phi_{II}}{\varepsilon_{x_I} \Delta\beta_I + \varepsilon_{x_{II}} \Delta\beta_{II}}$$

Beam sizes

$$a_x = \sqrt{\varepsilon_1 \beta_{1x} + \varepsilon_2 \beta_{2x}}$$
$$a_y = \sqrt{\varepsilon_1 \beta_{1y} + \varepsilon_2 \beta_{2y}}$$



- Ellipse equation

$$\frac{x^2}{a_x^2} - \frac{2\tilde{\alpha}xy}{a_x a_y} + \frac{y^2}{a_y^2} = 1 - \tilde{\alpha}^2$$

- Ellipse rotation parameter

$$\tilde{\alpha} \equiv \frac{\langle xy \rangle}{\sqrt{\langle x^2 \rangle \langle y^2 \rangle}} = \frac{\tilde{y}}{a_y} = \frac{\tilde{x}}{a_x} = \frac{\sqrt{\beta_{1x}\beta_{1y}}\,\varepsilon_1 \cos \nu_1 + \sqrt{\beta_{2x}\beta_{2y}}\,\varepsilon_2 \cos \nu_2}{\sqrt{\varepsilon_1 \beta_{1x} + \varepsilon_2 \beta_{2x}}\sqrt{\varepsilon_1 \beta_{1y} + \varepsilon_2 \beta_{2y}}}$$

Decoupling

- Counter Wound Solenoids
- Compensating Solenoids or Skew Quads
  - Jefferson Lab
  - MEIC Application
  - Rings
- Flat Beams

Counter-Wound Solenoids

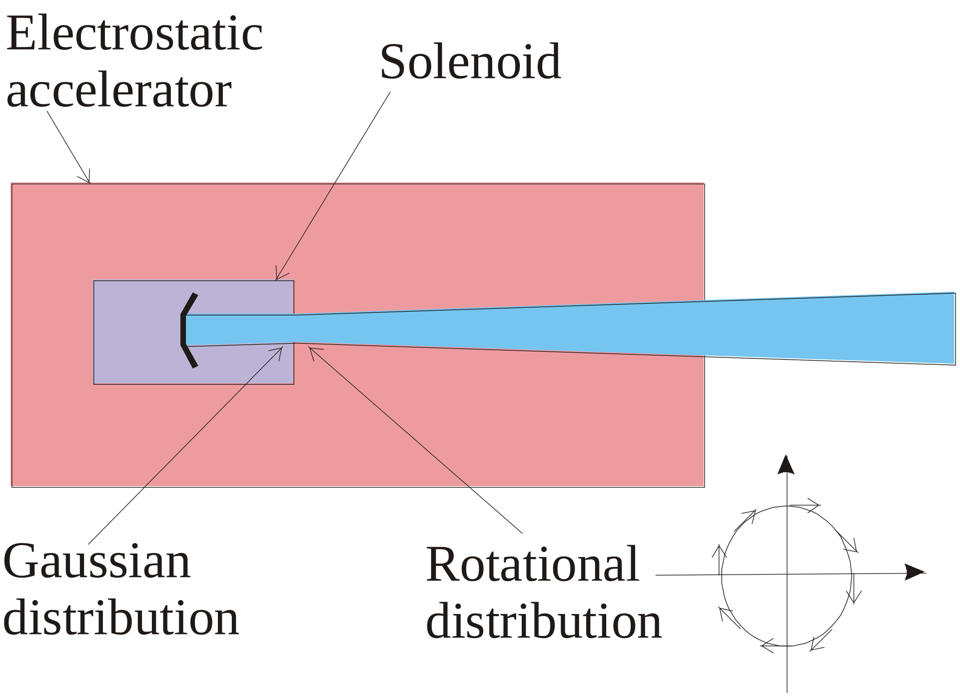
$$
\begin{pmatrix}
\cos^2\Phi & (1/S)\sin 2\Phi & -(1/2)\sin 2\Phi & -(2/S)\sin^2\Phi \\
-(S/4)\sin 2\Phi & \cos^2\Phi & (S/2)\sin^2\Phi & -(1/2)\sin 2\Phi \\
(1/2)\sin 2\Phi & (2/S)\sin^2\Phi & \cos^2\Phi & (1/S)\sin 2\Phi \\
-(S/2)\sin^2\Phi & (1/2)\sin 2\Phi & -(S/4)\sin 2\Phi & \cos^2\Phi
\end{pmatrix}
$$

$$
\times \begin{pmatrix}
\cos^2\Phi & (1/S)\sin 2\Phi & (1/2)\sin 2\Phi & (2/S)\sin^2\Phi \\
-(S/4)\sin 2\Phi & \cos^2\Phi & -(S/2)\sin^2\Phi & (1/2)\sin 2\Phi \\
-(1/2)\sin 2\Phi & -(2/S)\sin^2\Phi & \cos^2\Phi & (1/S)\sin 2\Phi \\
(S/2)\sin^2\Phi & -(1/2)\sin 2\Phi & -(S/4)\sin 2\Phi & \cos^2\Phi
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\cos 2\Phi & (2/S)\sin 2\Phi & 0 & 0 \\
-(S/2)\sin 2\Phi & \cos 2\Phi & 0 & 0 \\
0 & 0 & \cos 2\Phi & (2/S)\sin 2\Phi \\
0 & 0 & -(S/2)\sin 2\Phi & \cos 2\Phi
\end{pmatrix}
$$

Axisymmetric Rotational Distribution

- Fermilab electron cooling



The electron beam distribution is axially symmetric, and uncoupled at the cathode:

$$\mathbf{\Xi}_B = \frac{1}{\varepsilon_T}\begin{bmatrix} \gamma_0 & \alpha_0 & 0 & 0 \\ \alpha_0 & \beta_0 & 0 & 0 \\ 0 & 0 & \gamma_0 & \alpha_0 \\ 0 & 0 & \alpha_0 & \beta_0 \end{bmatrix}$$

where $\varepsilon_T = r_c \sqrt{mkT_c} / P_0$ is the thermal emittance of the beam

Axisymmetric Rotational Distribution

- At the exit of the solenoid the electron beam distribution is still axially symmetric

$$\mathbf{\Xi}_{in} = \mathbf{\Phi}^T\,\mathbf{\Xi}_B\,\mathbf{\Phi} = \frac{1}{\varepsilon_T}\begin{bmatrix} \gamma_0 + \Phi^2\beta_0 & \alpha_0 & 0 & -\Phi\beta_0 \\ \alpha_0 & \beta_0 & \Phi\beta_0 & 0 \\ 0 & \Phi\beta_0 & \gamma_0 + \Phi^2\beta_0 & \alpha_0 \\ -\Phi\beta_0 & 0 & \alpha_0 & \beta_0 \end{bmatrix}$$

where

$$\mathbf{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \Phi & 0 \\ 0 & 0 & 1 & 0 \\ -\Phi & 0 & 0 & 1 \end{bmatrix}$$

  - $\Phi = eB/2P_0c$ is the rotational focusing strength of the solenoid
  - *B* is the solenoid magnetic field.

Axisymmetric Rotational Distribution

- The eigen-vectors of the rotational distribution:

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ i\sqrt{\beta} \\ -i\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix}, \qquad \hat{\mathbf{v}}_2 = \begin{bmatrix} i\sqrt{\beta} \\ -i\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix}$$

  - It corresponds to u = 1/2, $\nu_1 = \nu_2 = \pi/2$

- Then, the matrix $\hat{\mathbf{V}}$ is

$$\hat{\mathbf{V}} = \begin{bmatrix} \sqrt{\beta} & 0 & 0 & -\sqrt{\beta} \\ -\dfrac{\alpha}{\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} & \dfrac{\alpha}{\sqrt{\beta}} \\ 0 & -\sqrt{\beta} & \sqrt{\beta} & 0 \\ \dfrac{1}{2\sqrt{\beta}} & \dfrac{\alpha}{\sqrt{\beta}} & -\dfrac{\alpha}{\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} \end{bmatrix}$$

Axisymmetric Rotational Distribution

- Comparing left and right hand sides of the equation

$$\hat{\mathbf{\Xi}}_{in} = \mathbf{U}\hat{\mathbf{V}} \begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix} \hat{\mathbf{V}}^T \mathbf{U}^T$$

  - One obtains

$$\beta = \frac{\beta_0}{2\sqrt{1+\Phi^2\beta_0^{\ 2}}} \quad ,$$

$$\alpha = \frac{\alpha_0}{2\sqrt{1+\Phi^2\beta_0^{\ 2}}} \quad ,$$

$$\varepsilon_1 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} - \Phi\beta_0} \xrightarrow{\Phi\beta_0 >> 1} 2\Phi\beta_0\varepsilon_T \quad ,$$

$$\varepsilon_2 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} + \Phi\beta_0} \xrightarrow{\Phi\beta_0 >> 1} \frac{\varepsilon_T}{2\Phi\beta_0} \quad .$$

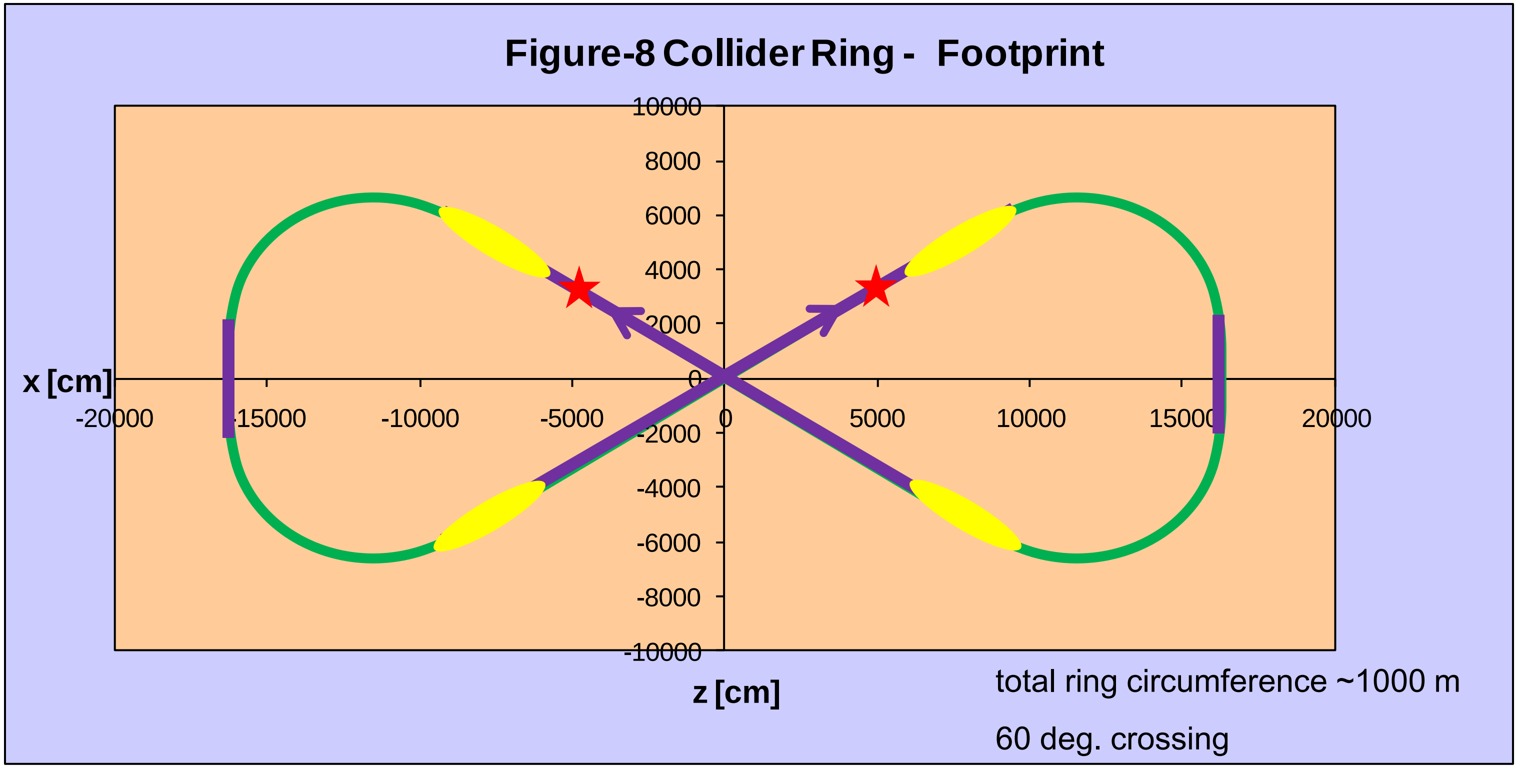
- 4D-emmitance conservation:

$$\varepsilon_1\varepsilon_2 = \varepsilon_T^{\ 2}$$

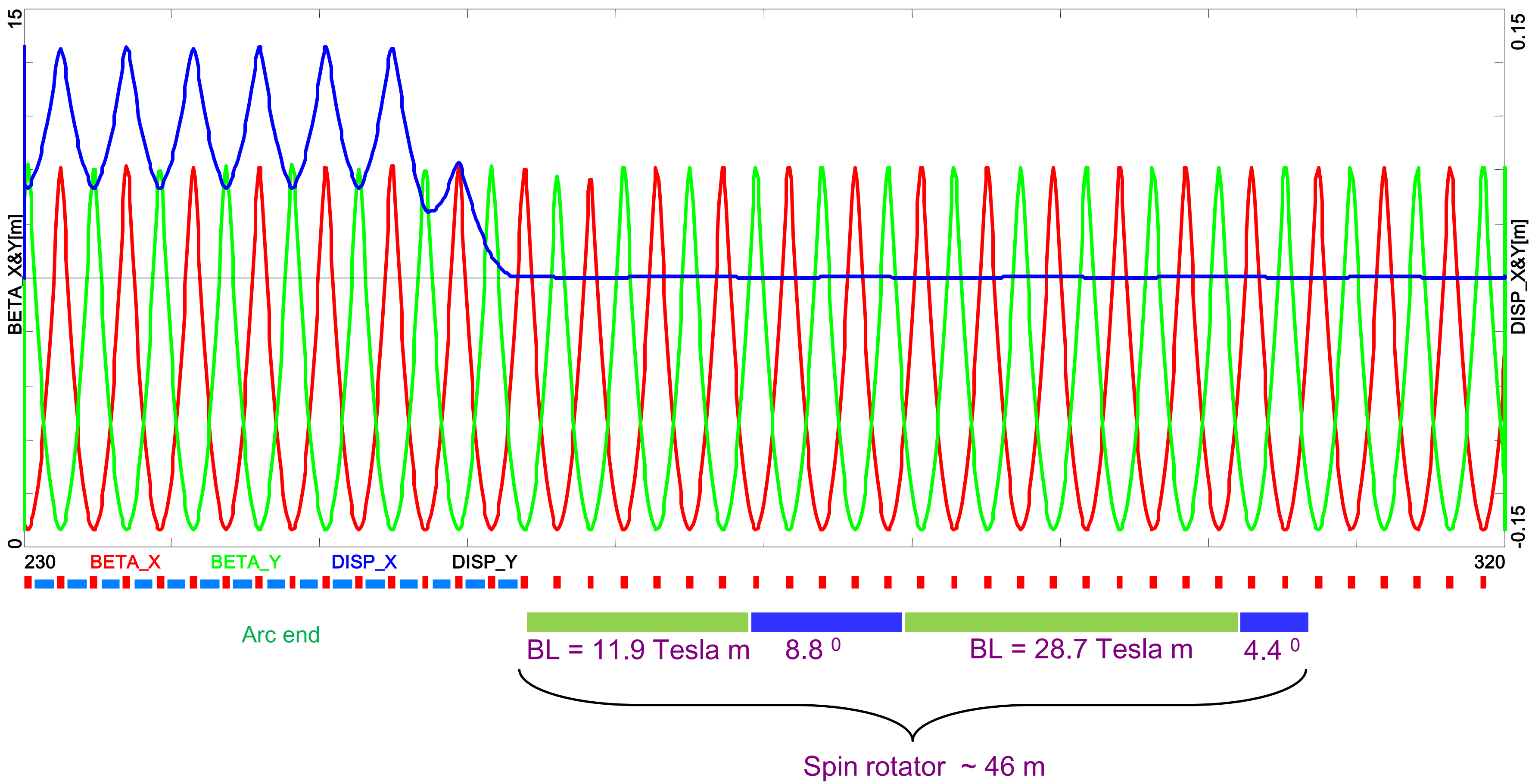
- Rotational emittance estimate

$$\varepsilon_{rot} = r\theta = r(r\Phi) = r^2\Phi = (\varepsilon_T\beta_0)\Phi$$

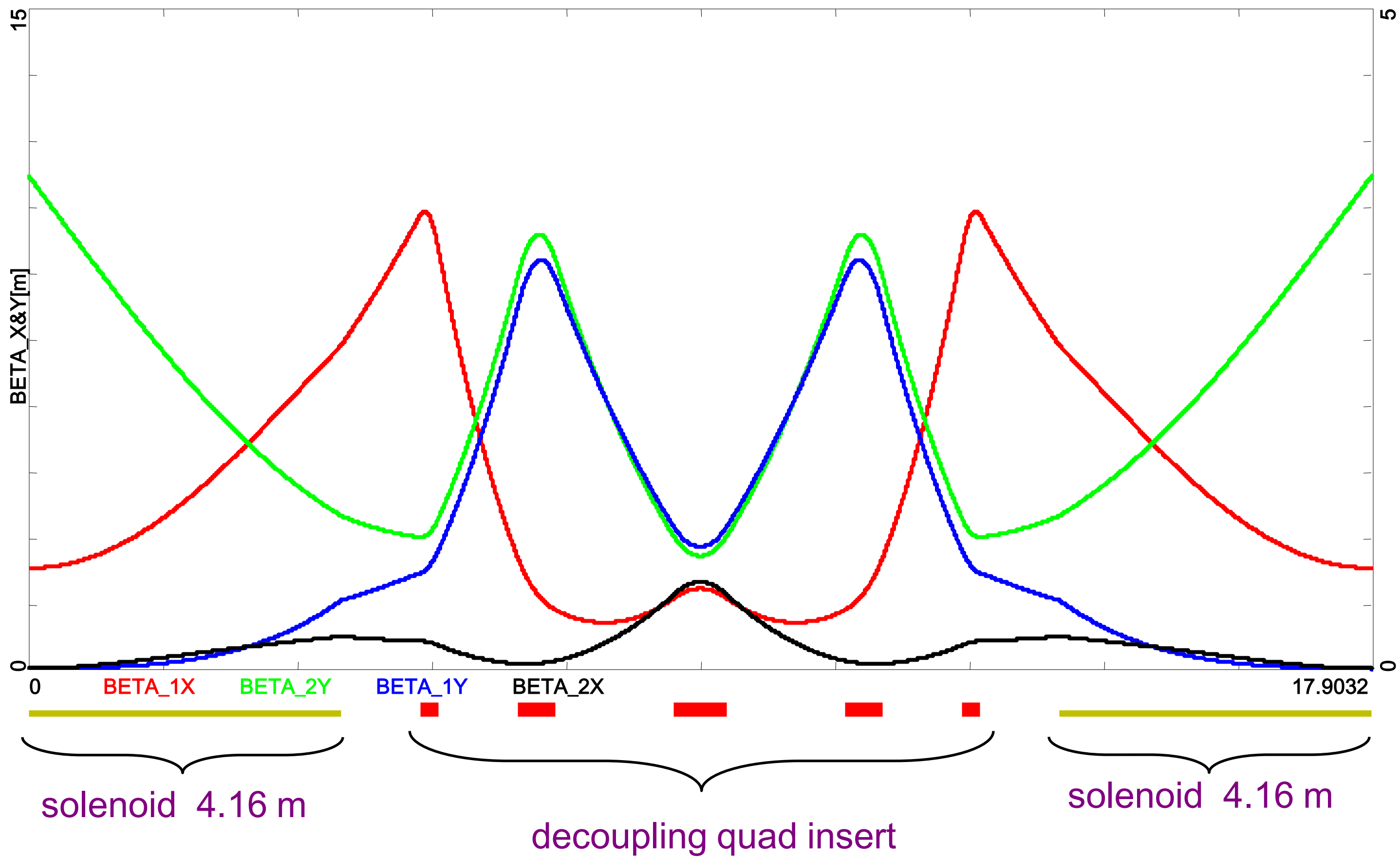
Spin Rotators for Figure-8 Collider Ring



Spin Rotator – Ingredients…



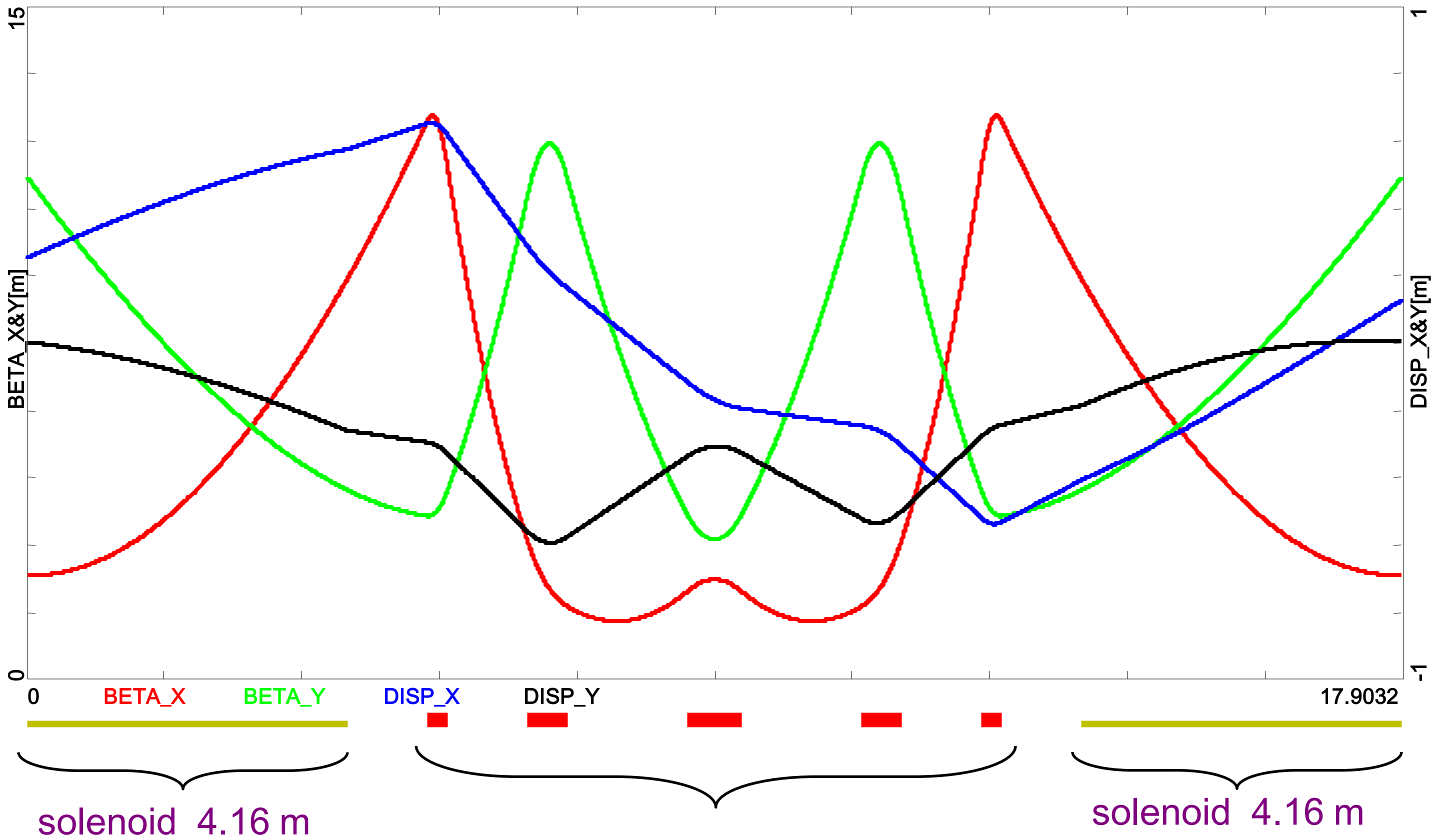
Locally Decoupled Solenoid Pair



$$M = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}$$

Hisham Sayed, PhD Thesis
ODU, 2011

Locally Decoupled Solenoid Pair



$$M = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}$$

Hisham Sayed, PhD Thesis
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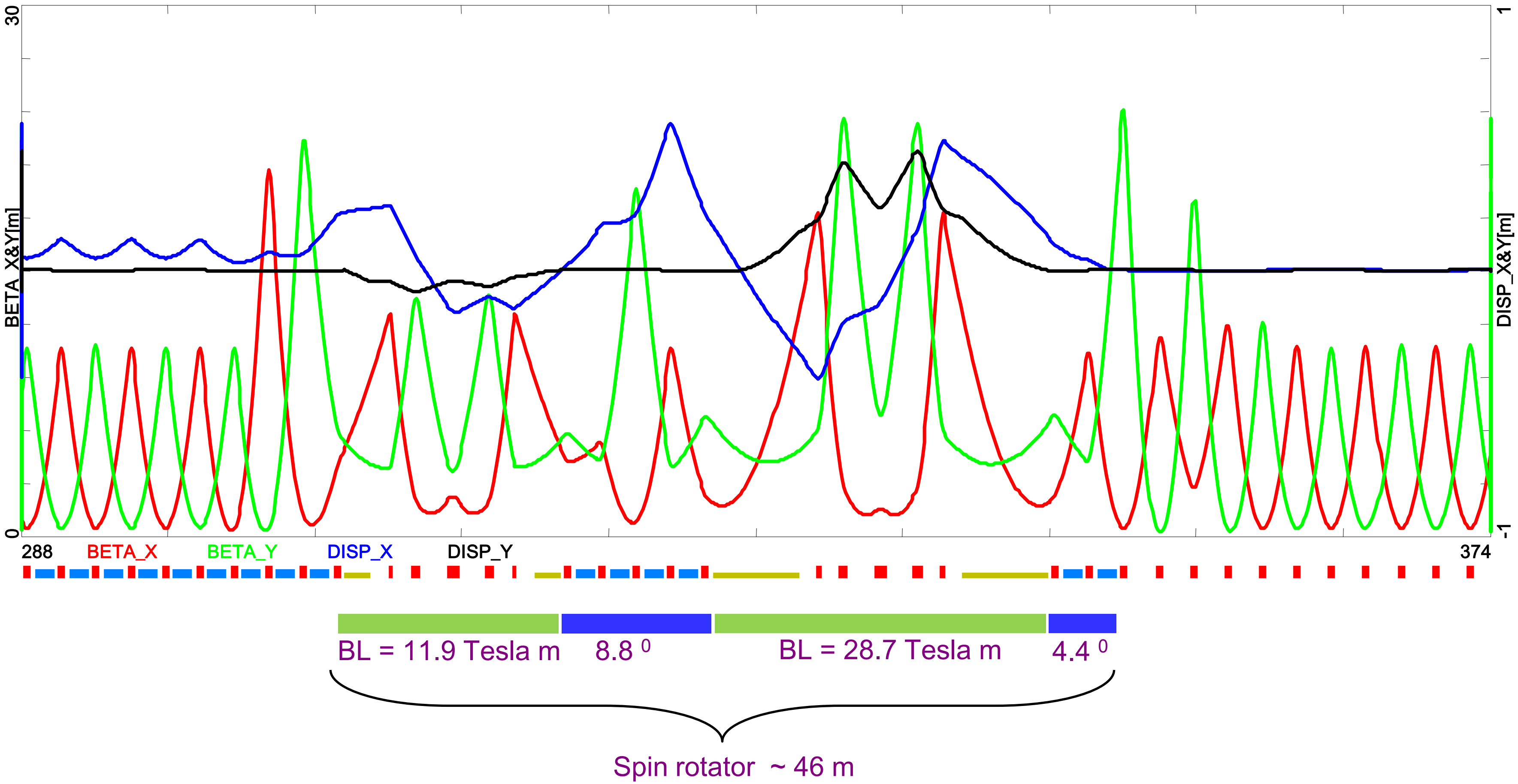
Universal Spin Rotator - Optics

5 GeV



Ionization Cooling in an Axially Symmetric Channel

- A single-particle phase-space trajectory along the beam orbit can be expressed as:

$$\hat{\mathbf{x}}(s) = \mathrm{Re}\left(\sqrt{\varepsilon_1}\hat{\mathbf{v}}_1(s)e^{-i(\psi_1+\mu_1(s))} + \sqrt{\varepsilon_2}\hat{\mathbf{v}}_2(s)e^{-i(\psi_2+\mu_2(s))}\right) \quad ,$$

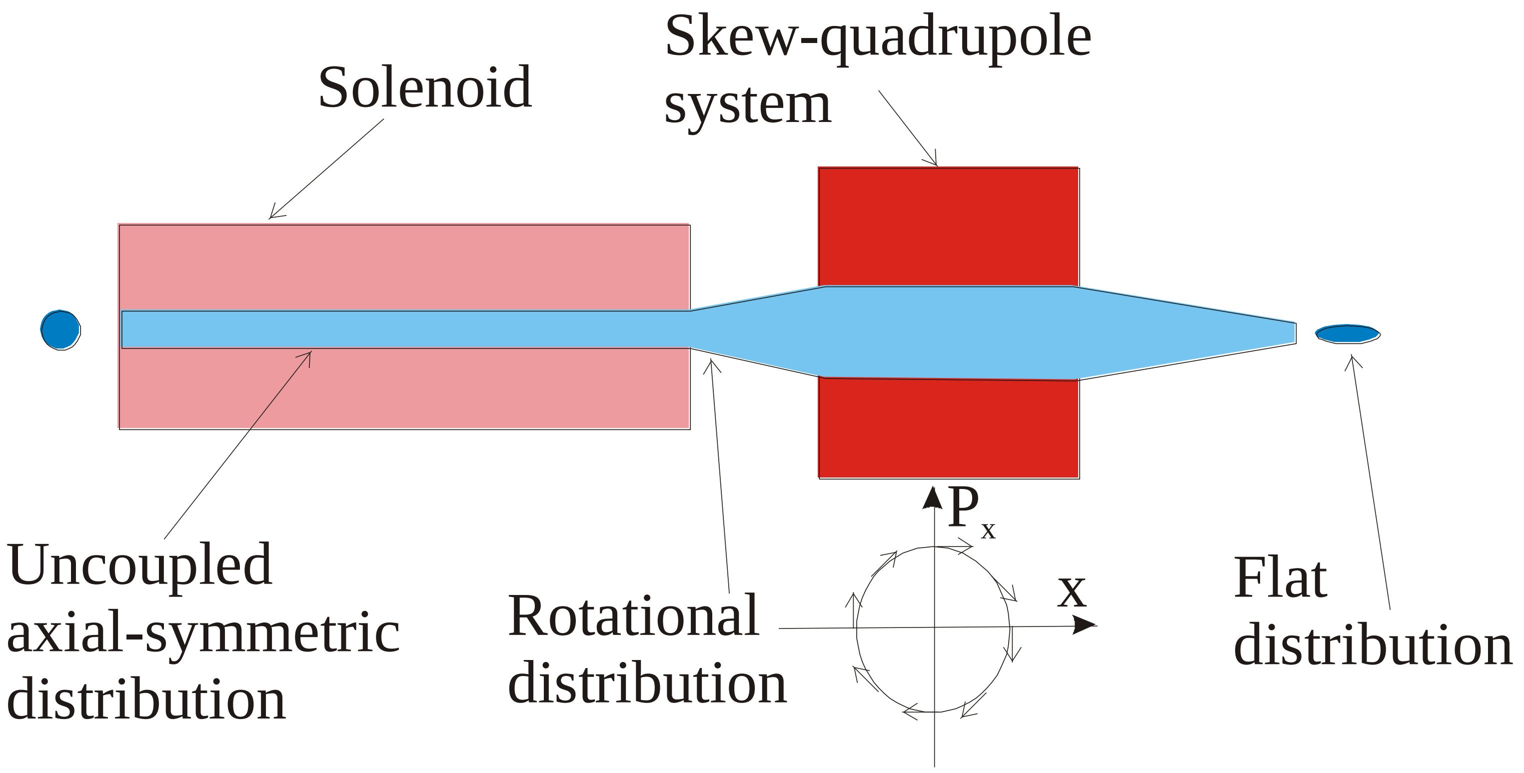
- One can rewrite the above equations in the following compact form

$$\hat{\mathbf{x}}(s) = \hat{\mathbf{V}}(s)\mathbf{a}(s)$$

where

$$\hat{\mathbf{V}}(s) = \left[\hat{\mathbf{v}}_1{}'(s), -\hat{\mathbf{v}}_1{}''(s), \hat{\mathbf{v}}_2{}'(s), -\hat{\mathbf{v}}_2{}''(s)\right] \qquad \mathbf{a}(s) = \begin{bmatrix} \sqrt{\varepsilon_1}\cos(\psi_1+\mu_1(s)) \\ \sqrt{\varepsilon_1}\sin(\psi_1+\mu_1(s)) \\ \sqrt{\varepsilon_2}\cos(\psi_2+\mu_2(s)) \\ \sqrt{\varepsilon_2}\sin(\psi_2+\mu_2(s)) \end{bmatrix}$$

Vertex-to-plane Transformer Insert



Vertex-to-plane Transformer Insert

- Eigen-vectors of the decoupled motion in the coordinate system rotated by 45^0

$$\begin{bmatrix} \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\substack{45\,\mathrm{deg.} \\ rotation}} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \\ \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \end{bmatrix} \xrightarrow{\substack{\beta_1=2\beta \\ \alpha_1=2\alpha}} \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{F}_2 \\ \mathbf{F}_2 \end{bmatrix}$$

  - Rotational eigen-vectors

$$\begin{bmatrix} \mathbf{F}_2 \\ i\mathbf{F}_2 \end{bmatrix} \qquad \begin{bmatrix} i\mathbf{F}_2 \\ \mathbf{F}_2 \end{bmatrix}$$

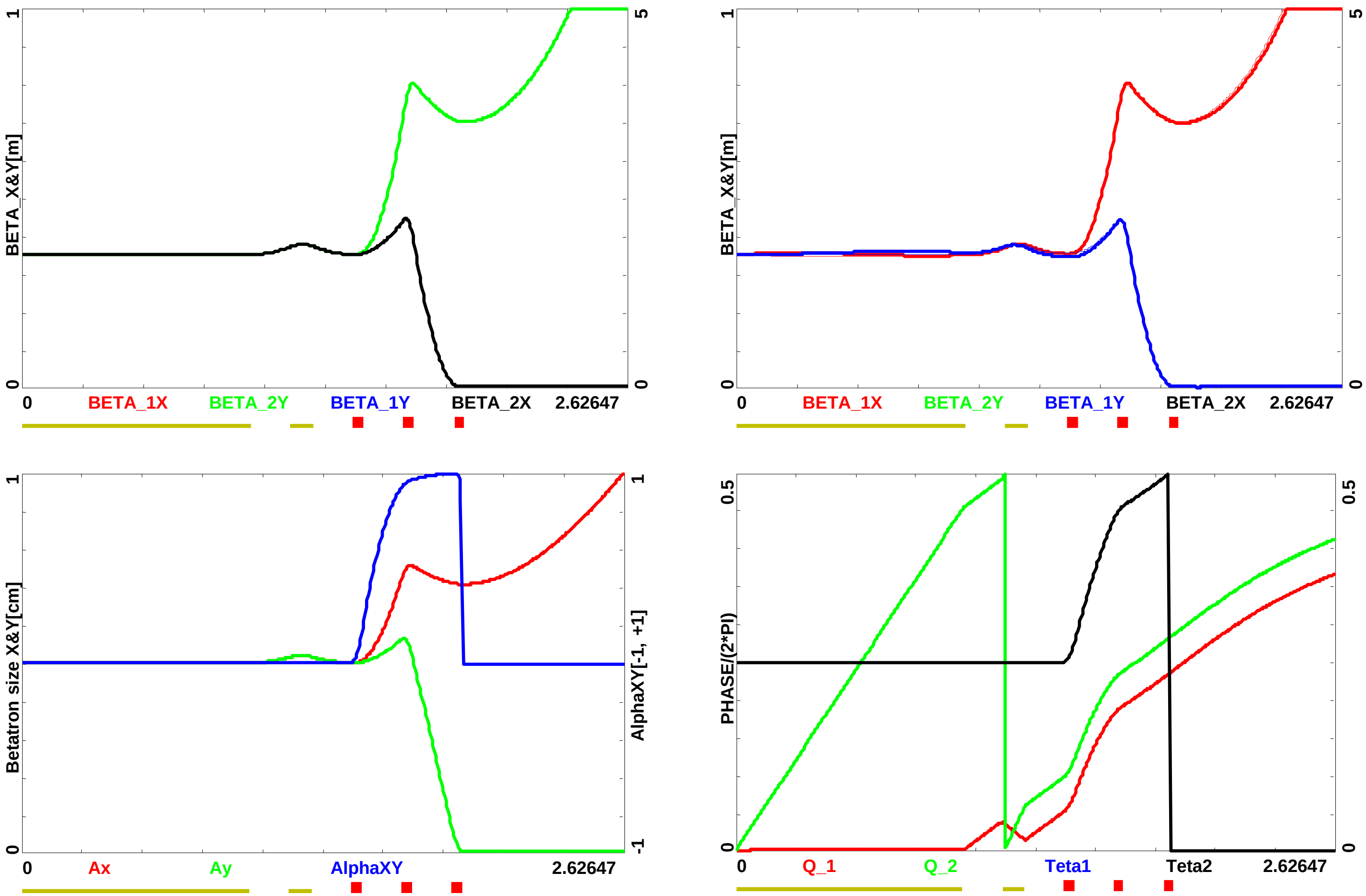
Vertex-to-plane Transformer Insert

- Focusing system with 45^0 difference between the horizontal and vertical betatron phase advances will transform the initial vertex distribution into the flat one
- The resulting 2D emittances are as follows

$$\varepsilon_1 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} - \Phi\beta_0} \quad , \qquad \varepsilon_2 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} + \Phi\beta_0} \quad .$$

- Lattice implementation – Twiss functions, beam sizes etc.

Vertex-to-plane Transformer Insert



E_{kin} = 10 MeV, T_c = 0.2 eV, R_c = 0.5 cm, B_{sol} = 1 kG,

$\Rightarrow \varepsilon_1 = 7.14 \cdot 10^{-3}$ cm, $\varepsilon_2 = 3.24 \cdot 10^{-8}$ cm

Generation and Dynamics of Magnetized Beams for High-Energy Electron Cooling*

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Outline

- Introduction
- Features and parameterization of magnetized beams
- Formation of magnetized bunches:
  - methods and limitations,
  - experiments in rf gun.
- Transport and Manipulation:
  - transverse matching,
  - longitudinal manipulations,
  - decoupling into flat beams.
- Outlook

Required Electron-Beam Parameters

- Cooling interaction time

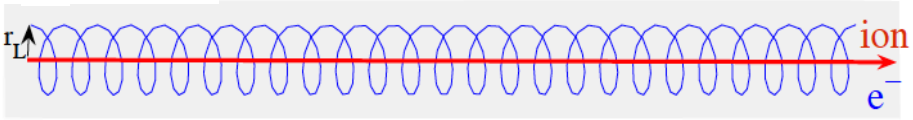
$$\tau \approx \rho / v_{e\perp} \quad \text{(not magnetized)}$$

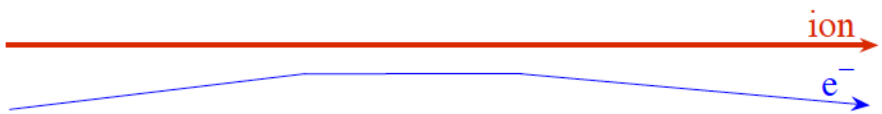
$$\tau \approx \frac{\rho}{v - v_{e\parallel}} \quad \text{(magnetized)}$$

- magnetized cooling less dependent on e- beam transverse emittance (to what extent?)

$$r_L = \frac{v_{e\perp}}{eB_z}$$

(magnetized)





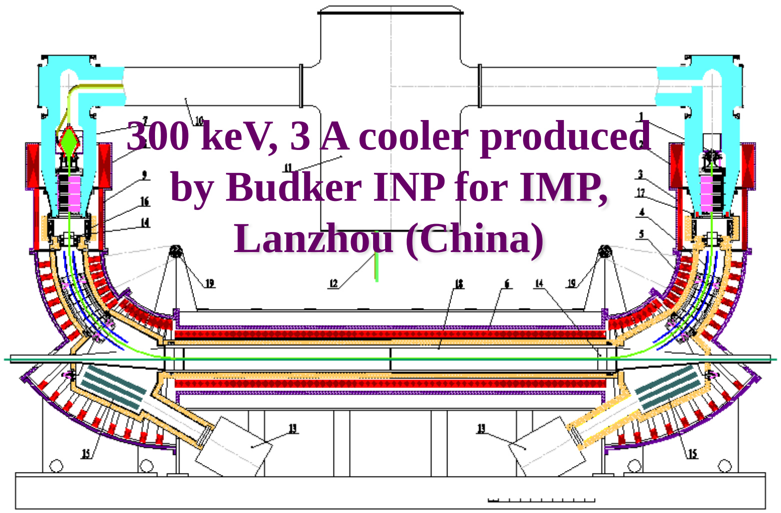
(not magnetized)

- electron-cooling accelerator provides beam eventually matched to cooling-solenoid section

Cooler configurations

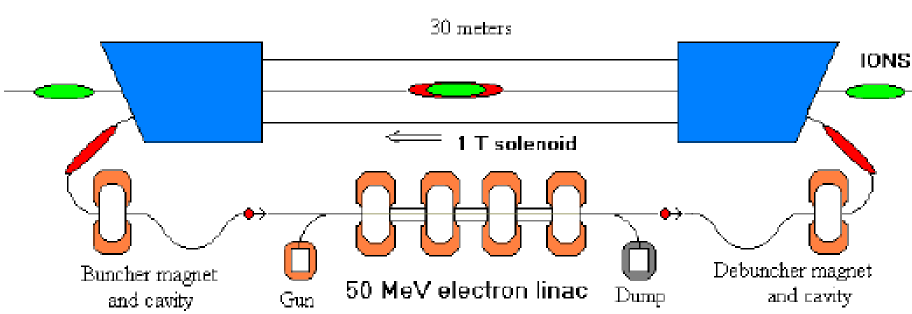
e.g. see S. Nagaitsev, et al, PRL96, 044801 (2006)

- low-energy coolers:
  - lattice (bends) embedded in magnetic fields,
  - based on DC electron sources,
  - no further acceleration or bunching, needed.



300 keV, 3 A cooler produced by Budker INP for IMP, Lanzhou (China)

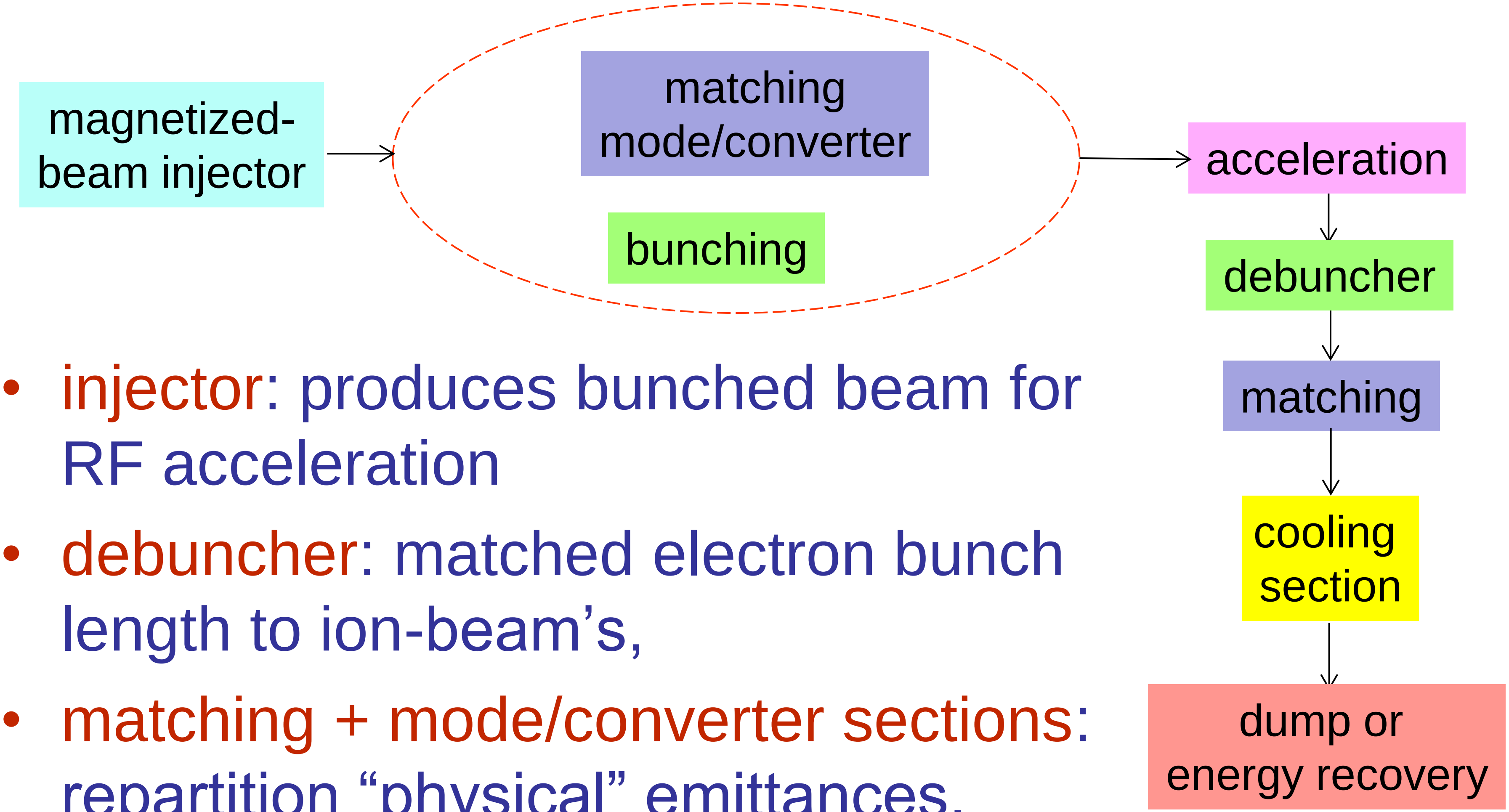
- high-energy coolers:
  - medium energies required (50-100 MeV),
  - acceleration in SCRF linac → bunching
  - lumped solenoidal fields → matching



early concept for RHIC e-cooling

e.g. see I. Ben-Zvi, et al., Proc. PAC2001, p. 48 (2001)

High-energy coolers

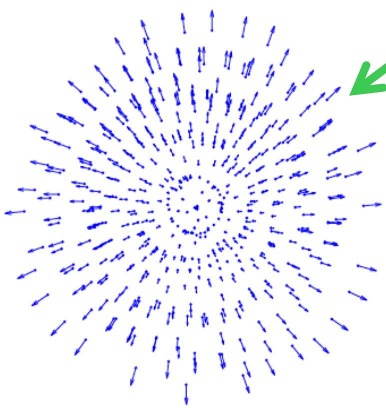


- injector: produces bunched beam for RF acceleration
- debuncher: matched electron bunch length to ion-beam's,
- matching + mode/converter sections: repartition "physical" emittances, match in cooling-solenoid section.

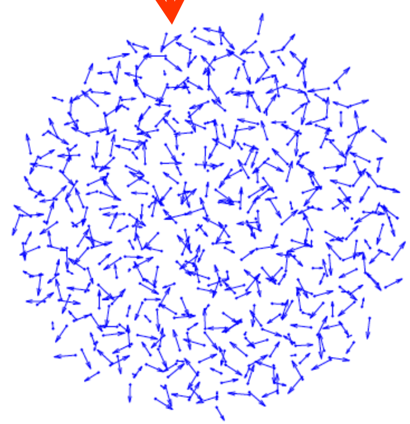
Beam dynamics regimes (round beams)

- Radial envelope (σ) equation in a drift (Lawson):

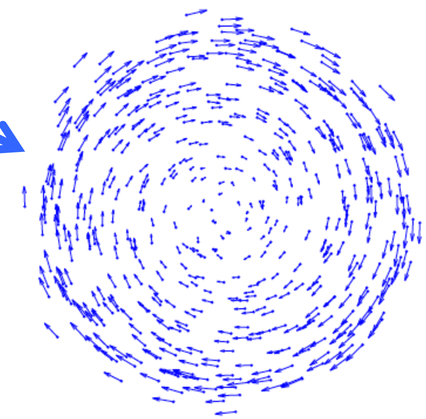
$$\sigma'' - \frac{K}{4\sigma} - \frac{\epsilon_u^2}{\sigma^3} - \frac{\mathcal{L}^2}{\sigma^3} = 0,$$



space charge



emittance “pressure”



angular momentum contribution

K : generalized perveance

ϵ_u : uncorrelated geometric emittance

$\mathcal{L}$: magnetization

adapted from Y.-E Sun, Dissertation U. Chicago (2005)

Features & Parameterization

- possible parameterization of coupled motion between 2 degrees of freedom has been extensively discussed; see:
  - D.A. Edwards and L.C. Teng, IEEE Trans. Nucl. Sci. 20, 3, pp. 885-889 (1973).
  - I. Borchardt, E. Karantzoulis, H. Mais, G. Ripken, DESY 87-161 (1987).
  - V. Lebedev, S. A. Bogacz, ArXiV:1207.5526 (2007).
  - A. Burov, S. Nagaitsev, A. Shemyakin, Ya. Derbenev, PRSTAB 3, 094002 (2000).
  - A. Burov, S. Nagaitsev, Ya. Derbenev, PRE 66, 016503 (2002).
- Simpler description that provides the necessary insights..

A simple description of coupled motion

- Consider the 4x4 beam matrix

$$\Sigma \equiv \begin{bmatrix} \langle \mathbf{X}\widetilde{\mathbf{X}} \rangle & \langle \mathbf{X}\widetilde{\mathbf{Y}} \rangle \\ \langle \mathbf{Y}\widetilde{\mathbf{X}} \rangle & \langle \mathbf{Y}\widetilde{\mathbf{Y}} \rangle \end{bmatrix} \quad \text{where} \quad \begin{array}{l} \widetilde{\mathbf{X}} \equiv (x, x') \\ \widetilde{\mathbf{Y}} \equiv (y, y') \end{array}$$

- Introduce the "correlation" matrix: $C \equiv \langle \mathbf{Y}\widetilde{\mathbf{X}} \rangle \langle \mathbf{X}\widetilde{\mathbf{X}} \rangle^{-1}$
- Beam matrix takes the form:

$$\Sigma = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & C^{-1} \\ C & 0 \end{bmatrix} \right) \begin{bmatrix} \langle \mathbf{X}\widetilde{\mathbf{X}} \rangle & 0 \\ 0 & \langle \mathbf{Y}\widetilde{\mathbf{Y}} \rangle \end{bmatrix}$$

- The correlation subjects to $R = \begin{bmatrix} H & G \\ U & V \end{bmatrix}$ transforms as $C_0 \rightarrow C$

$$C = (U + VC_0)(H + GC_0)^{-1}$$

- C provides information on the coupling only.

Beam matrix for a round magnetized beam

- At a waist, the matrix of a magnetized (round) beam is

$$\Sigma_0 = \begin{bmatrix} \varepsilon T_0 & \mathcal{L} J \\ -\mathcal{L} J & \varepsilon T_0 \end{bmatrix}.$$

where $T_0 = \begin{bmatrix} \beta & -\alpha \\ -\alpha & \frac{1+\alpha^2}{\beta} \end{bmatrix}$

and the magnetization is

$$\mathcal{L} = \langle xy' \rangle = -\langle x'y \rangle = \frac{L}{2p_z}$$

- The eigen-emittances of this beam matrix are:

$$\varepsilon_\pm = \varepsilon \pm \mathcal{L}.$$ where $\varepsilon^2 = \mathcal{L}^2 + \varepsilon_u{}^2 = |\Sigma|$

- the eigen-emittances can be mapped into "physical" emittances using a skewed beamline $\begin{bmatrix} M_+ & M_- \\ M_- & M_+ \end{bmatrix}$

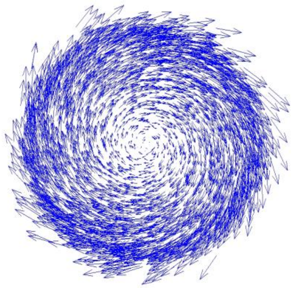
decoupling when

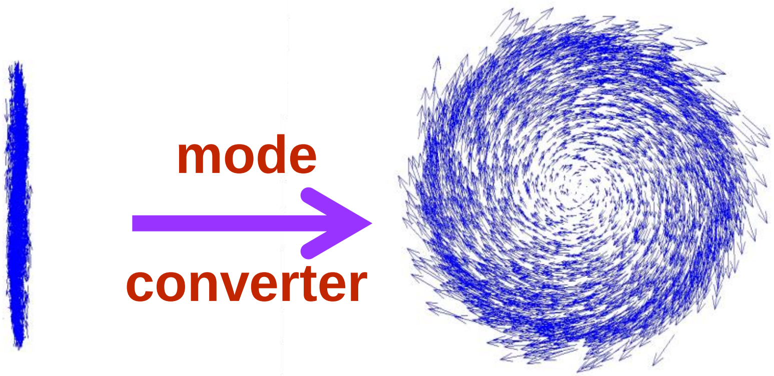
$$M_- + M_+ C_0 = 0.$$

K.-J. Kim, PRSTAB 6, 104002 (2003)
D. A. Edwards, unpublished (2001)

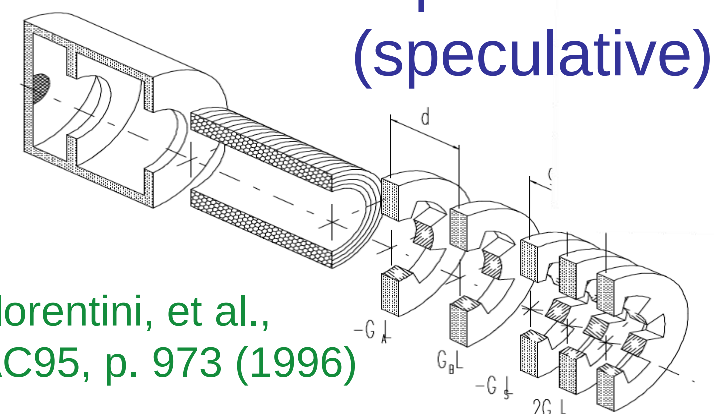
Formation of magnetized bunches

- Cathode immersed in an axial **B** field
- Sheet beams at birth (with subsequent flat-to-round beam converter)
  - shaped cathode,
  - line-laser focus
  - Nonlinear optics (speculative)





Y. Derbenev, University of Michigan report UM-HE-98-04 (1998)



G. Florentini, et al., Proc. PAC95, p. 973 (1996)

Cathode in a magnetic field

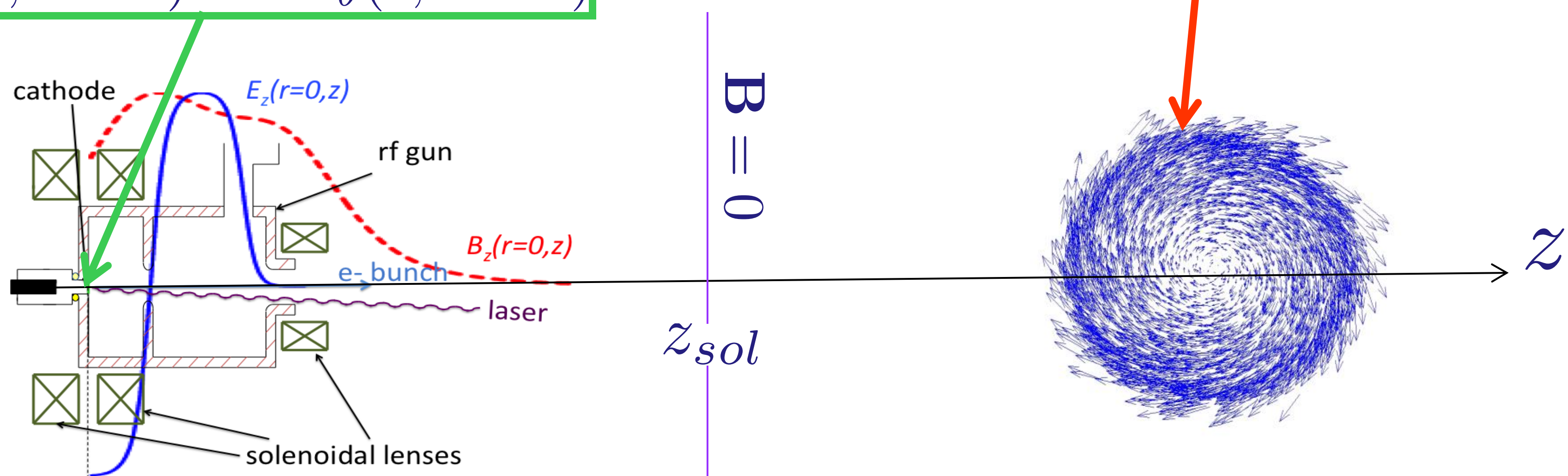
- electrons born in an axial B field $B_z \rightarrow$ CAM

$$L(r) = erA_\theta \simeq \frac{er^2}{2}B_{z,0} + \mathcal{O}(r^4)$$

- upon exit of solenoid field ($A_\theta = 0$): CAM becomes purely kinetic.

$$P_\theta(r, z = 0) = eA_\theta(r, z = 0)$$

$$p_\theta(r, z > z_{sol}) = P_\theta(r, z = 0)$$



Emittance vs magnetization

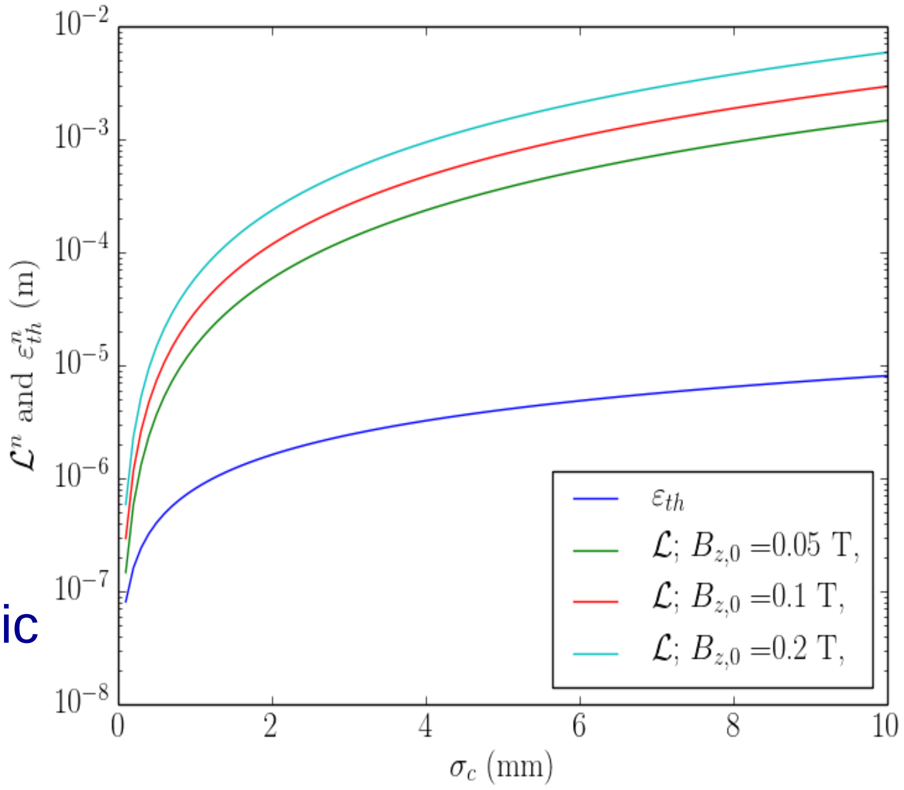
- "effective emittance" $\varepsilon^2 = \mathcal{L}^2 + {\varepsilon_u}^2$
- magnetization

$$\mathcal{L} = \frac{eB_0}{2mc}\sigma_c^2$$

- The emittance has a lower-bound value :

$$\varepsilon_u^n \equiv \beta\gamma\varepsilon_u \geq \varepsilon_{th} = \sigma_c \left(\frac{2\delta E}{3mc^2}\right)^{1/2}$$

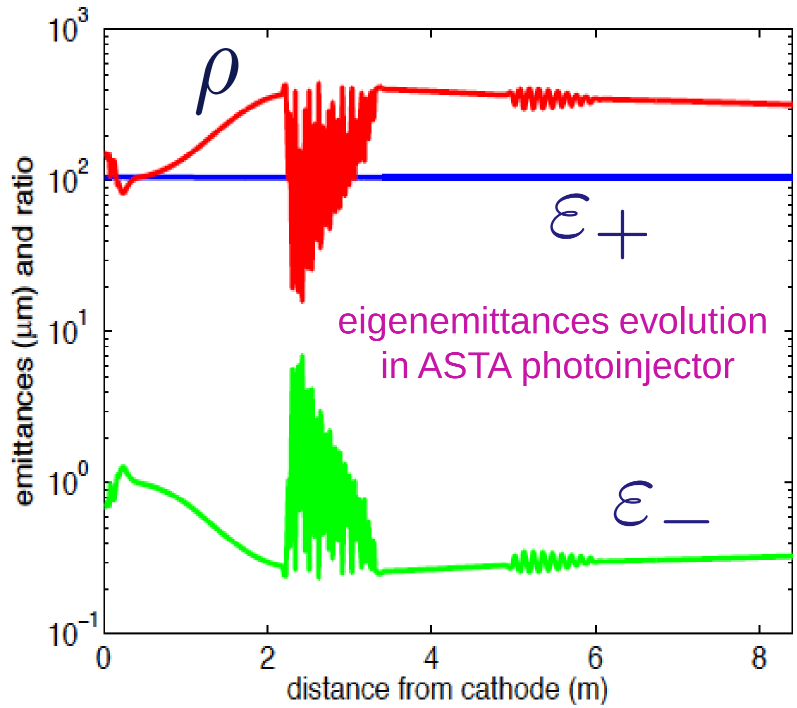
where δE is the excess in kinetic energy during emission



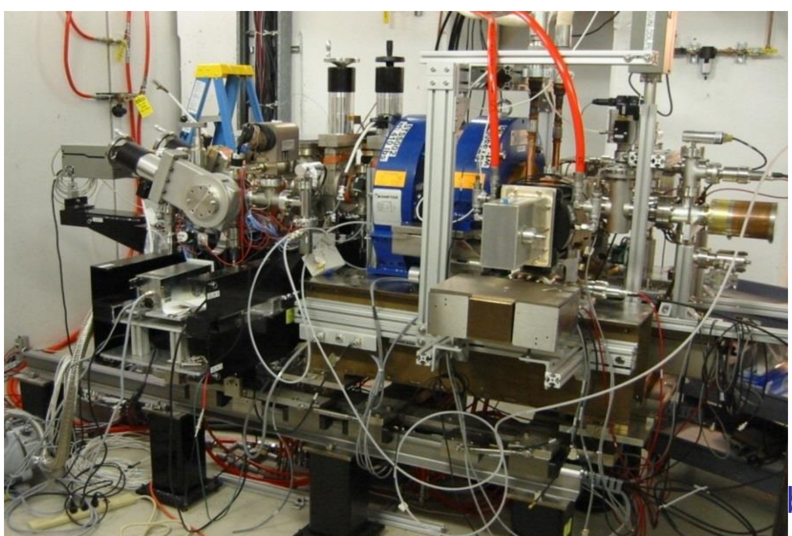
- Practically, ε_u includes other contributions.

Example of 3.2-nC magnetized bunch

- high-charge bunch subject to emittance degradation
- proper optimization (emittance compensation) → 4-D emittance com- parable to round beams.



parameter	flat-beam configuration	round-beam configuration	units
Q	3.2	3.2	nC
E	47.18	48.77	MeV
ε_x	105.04	5.43	μm
ε_y	0.31	5.44	μm
ε_{4D}	5.53	5.44	μm
ρ	$\simeq 334$	$\simeq 1$	–

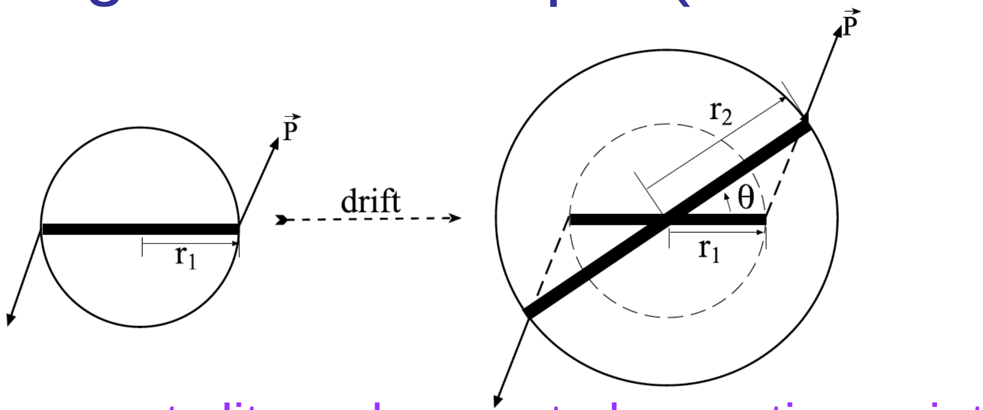


P. Piot et al. IPAC13; C-X. Wang, FEL06, 721 (2006)
X. Chang, I. Ben-Zvi, J. Kewisch AAC04 (2004)

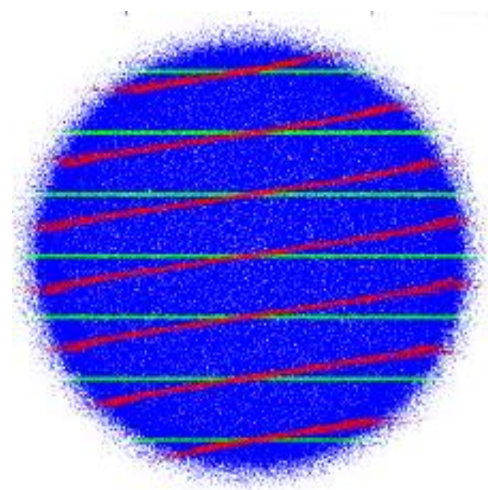
Measuring (kinetic) angular momentum

Y.-E Sun et al, PRSTAB 7, 123501 (2004)

- Kinetic angular momentum can be measured using a slit technique (similar to emittance)



beam at slits beam at observation point



- The beam's average angular momentum is given by

$$\langle L \rangle = 2P_z \frac{\sigma_1 \sigma_2 \sin\theta}{D}$$

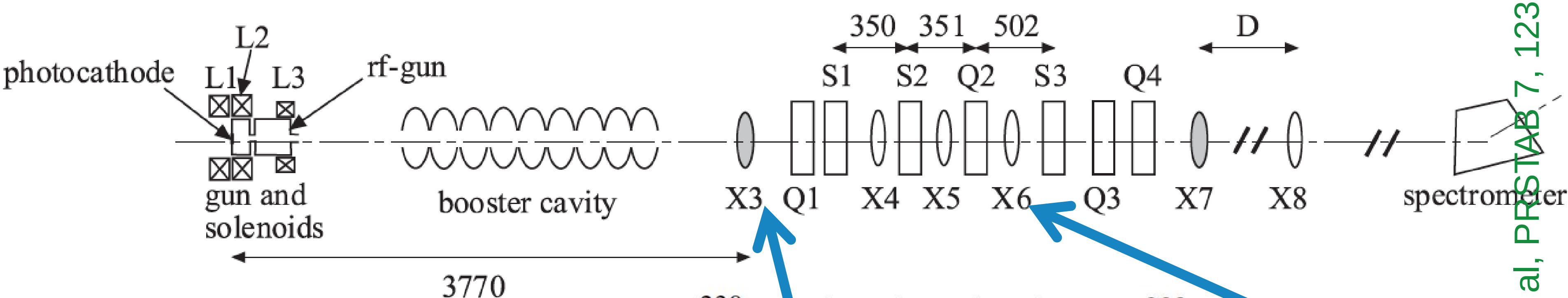
$\sigma_{1,2}$: rms beam size at slit (1) and observation screen (2),

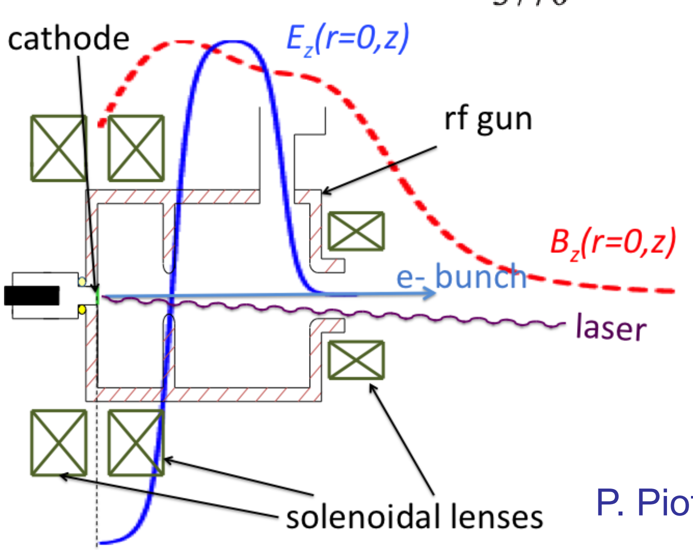
P_z : axial momentum

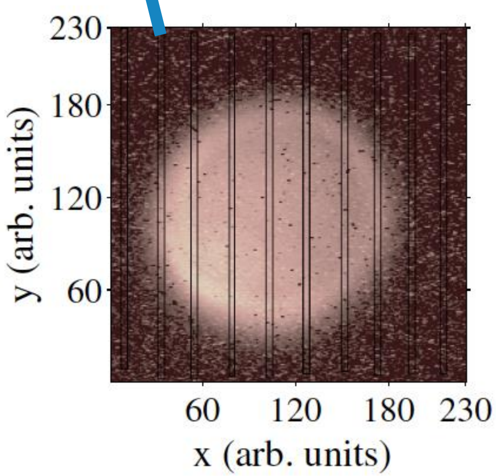
D : drift length between locations (1) and (2).

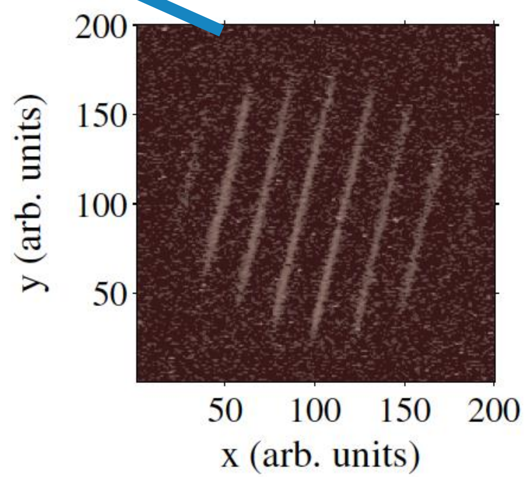
Experimental generation in a photoinjector

- Fermilab A0 normal-conducting photoinjector (decommissioned),
- 15 MeV, charge up to 2 nC,~3-10 ps bunch





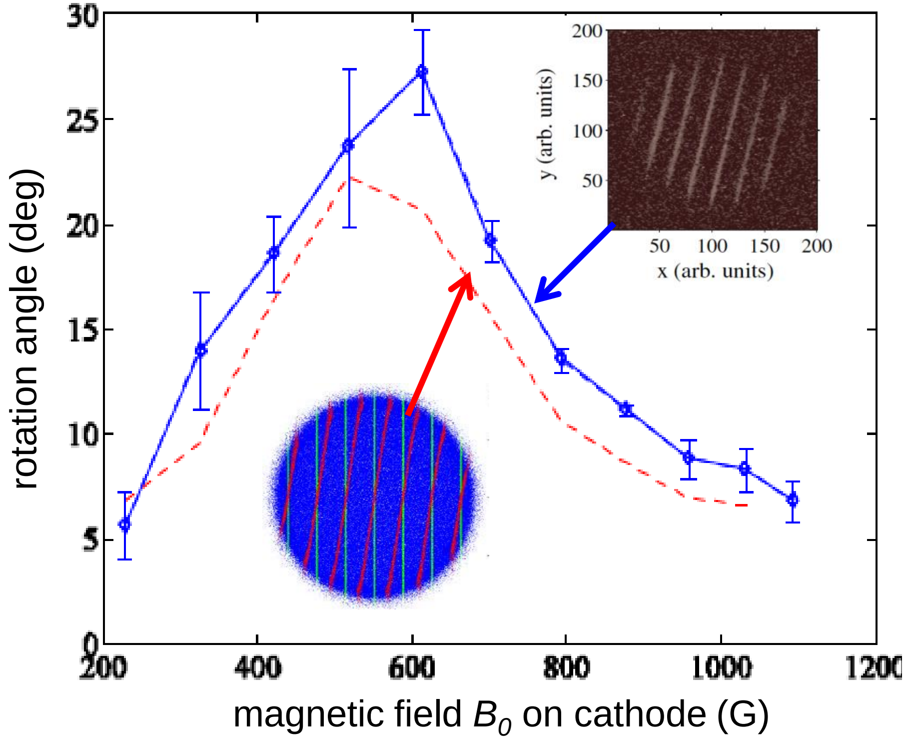


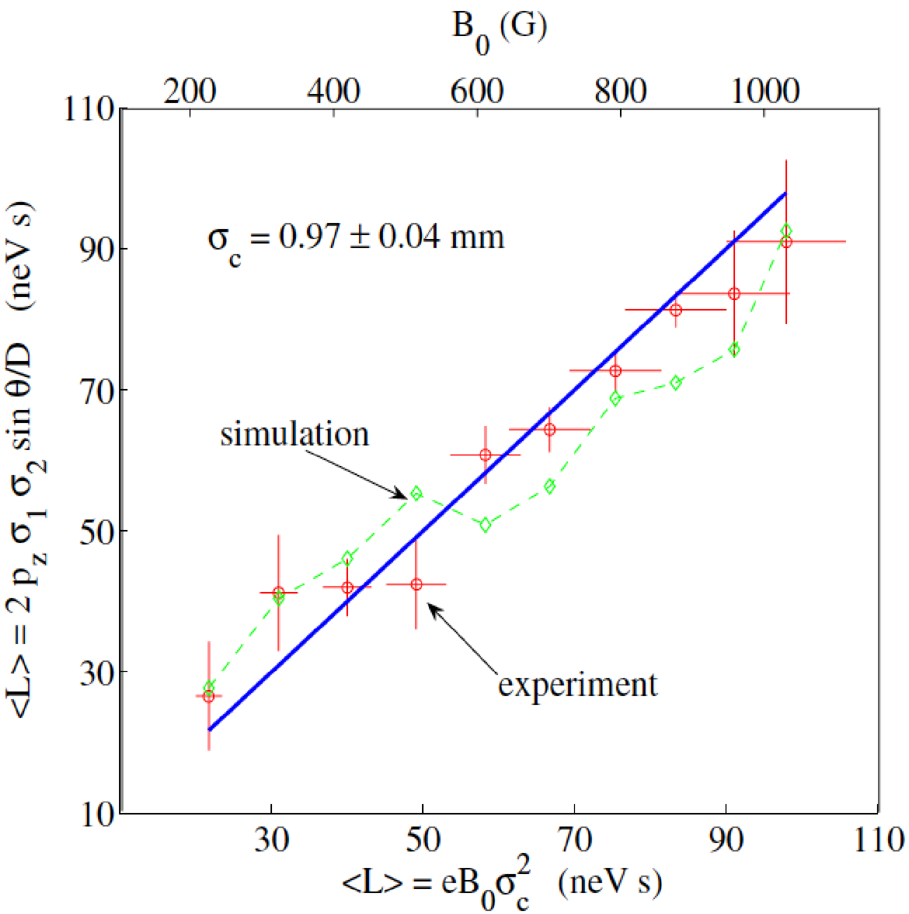


Y.-E Sun et al, PRSTAB 7, 123501 (2004)

Experimental generation in a photoinjector

- linear scaling with B field on photocathode

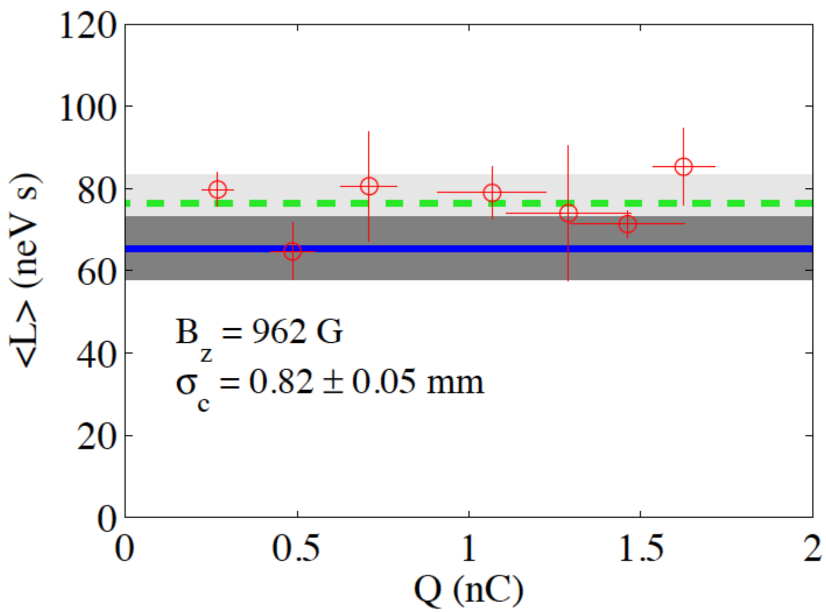


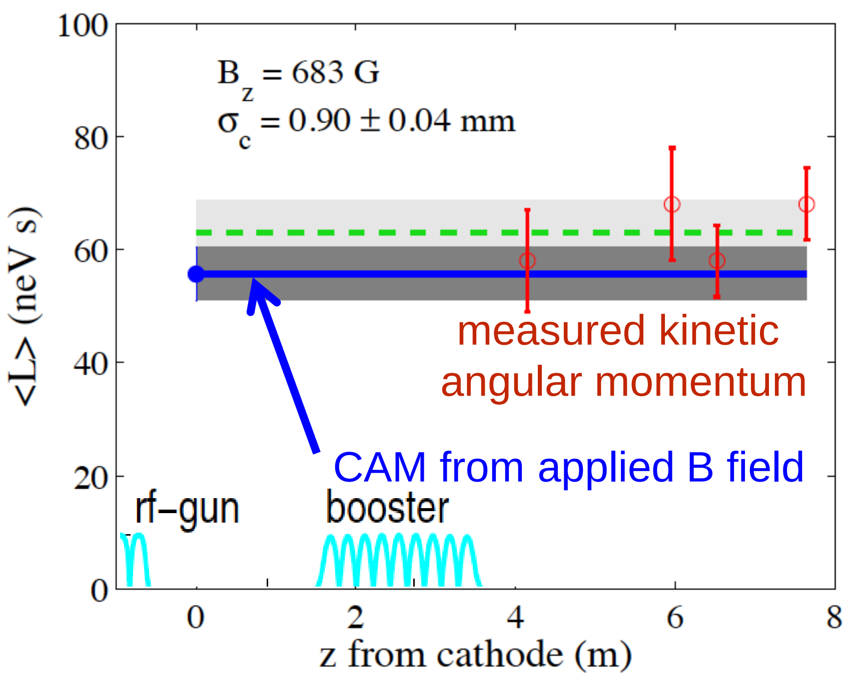


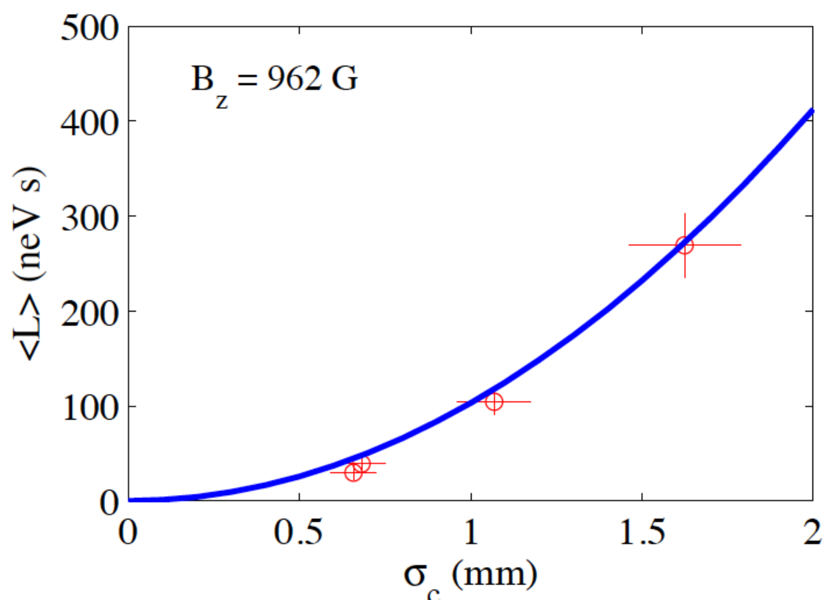
Y.-E Sun et al, PRSTAB 7, 123501 (2004)

Experimental generation in a photoinjector

- weak Q dependence,
- quadratic scaling with laser spot size σ_c on photocathode.







Y.-E Sun et al, PRSTAB 7, 123501 (2004)

Decoupling into flat ($\varepsilon_x/\varepsilon_y \neq 1$) beam

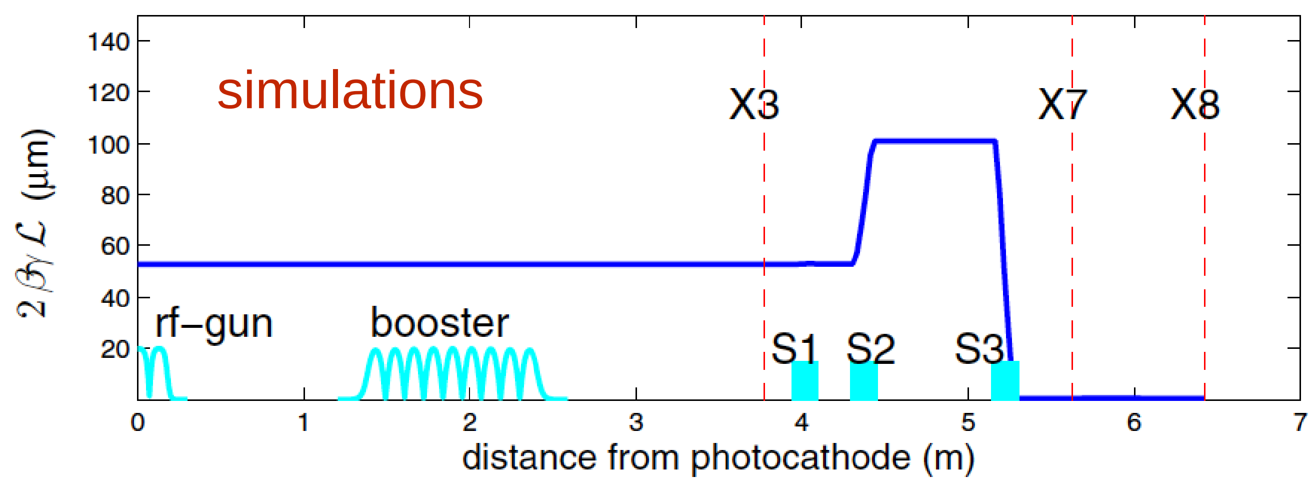
- Transport of magnetized bunches while preserving $\mathcal{L}$ is challenging,
- Use of round-to-flat beam transformer to convert into uncoupled (flat) beam
  $\rightarrow$ eigen-emittances maps into "physical" transverse emittances:

$$\varepsilon_n^{\pm} = \sqrt{(\varepsilon_n^u)^2 + (\beta\gamma\mathcal{L})^2} \pm (\beta\gamma\mathcal{L}) \xrightarrow{\beta\gamma\mathcal{L} \gg \varepsilon_n^u} \begin{cases} \varepsilon_n^+ \simeq 2\beta\gamma\mathcal{L}, \\ \varepsilon_n^- \simeq \frac{(\varepsilon_n^u)^2}{2\beta\gamma\mathcal{L}}, \end{cases}$$

R. Brinkmann et al., PRSTAB 4, 053501 (2001)
Y. Derbenev, University of Michigan report UM-HE-98-04 (1998)

Decoupling into flat beam: experiments (1)

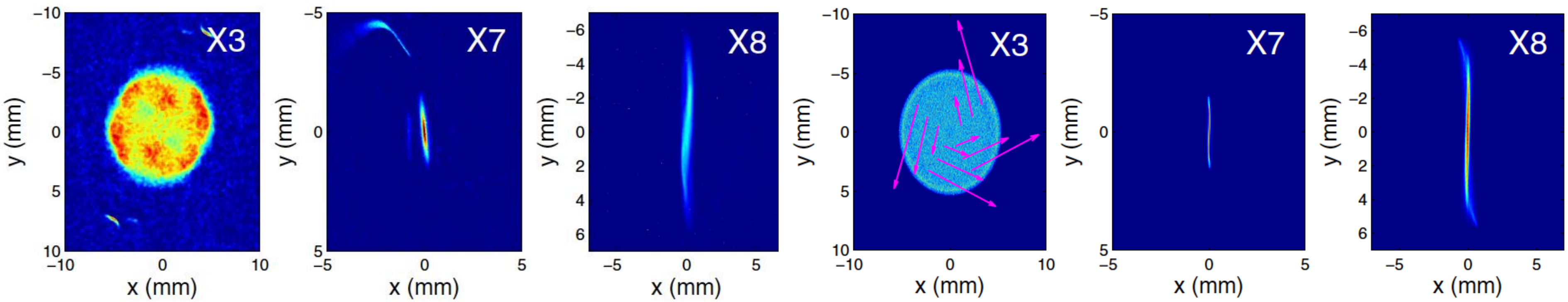
- Same experimental setup as used for generation of CAM-dominated beams



0.5 nC

P. Piot, et al, PRSTAB 9, 031001 (2006)

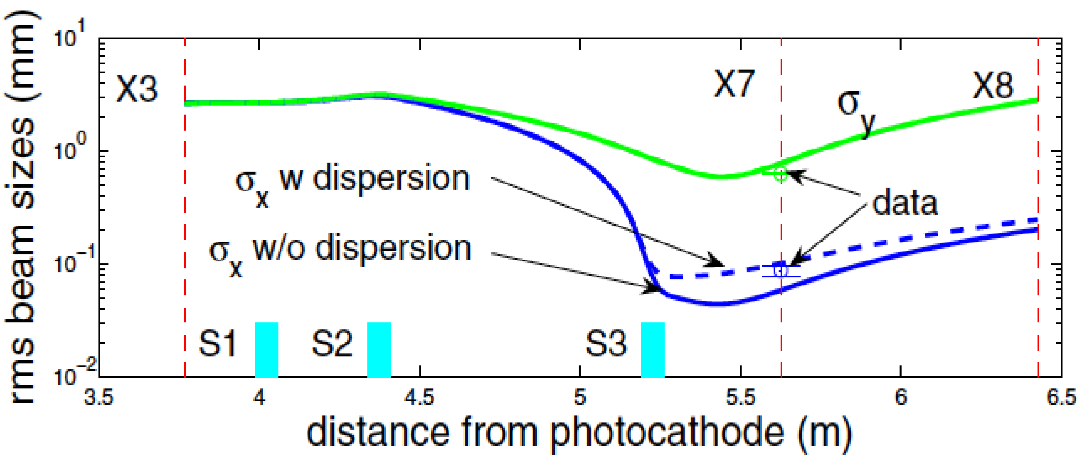
experiments | simulations

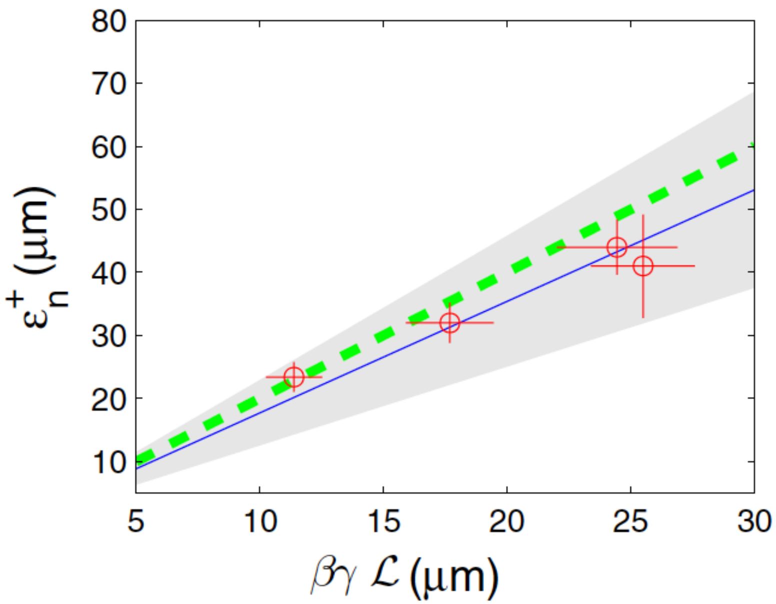


Decoupling into flat beam: experiments (2)

- normal emittances map into the flat-beam emittance
- large experimental uncertainties for smallest emittance meas.

Parameter	Experiment	Simulation	Unit
σ_x^{X7}	$0.088 \pm 0.01\ (\pm 0.01)$	0.058	mm
σ_y^{X7}	$0.63 \pm 0.01\ (\pm 0.01)$	0.77	mm
$\sigma_x^{X8,v}$	$0.12 \pm 0.01\ (\pm 0.01)$	0.11	mm
$\sigma_y^{X8,h}$	$1.68 \pm 0.09\ (\pm 0.01)$	1.50	mm
ε_n^x	$0.41 \pm 0.06\ (\pm 0.02)$	0.27	μm
ε_n^y	$41.1 \pm 2.5\ (\pm 0.54)$	53	μm
$\varepsilon_n^y/\varepsilon_n^x$	$100.2 \pm 20.2\ (\pm 5.2)$	196	





Outlook + open questions

- magnetized beam from a SCRF gun:
  - flux concentrator around cathode?
  - flat beam at cathode
    [J. Rosenzweig, PAC93 showed $(\varepsilon_+, \varepsilon_-)$=(95,4.5) μm]
- needed ϵ_u and $\mathcal{L}$? and limit on 4-D emittance?
- planned future experiment at ASTA