Accelerator Physics
Particle Acceleration

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Lecture 8

Radial Equation

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\omega_L^2 = qr\dot{\theta}B_z = -2\gamma m r\omega_L^2$$

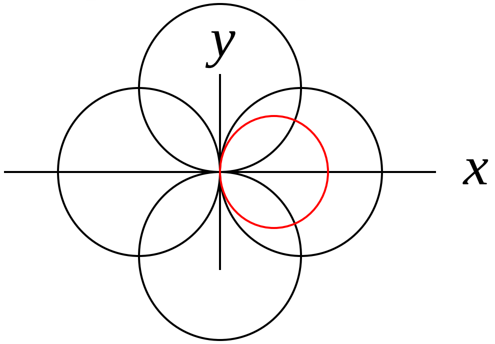
$$\therefore k = \frac{\omega_L^2}{\beta_z^2 c^2}$$

thin lens focal length

$$\frac{1}{f} = \frac{e^2 \int_{-\infty}^{\infty} B_z^2 dz}{4\beta_z^2\gamma^2 m^2 c^2}$$

weak compared to quadrupole for high γ

If go to full ¼ oscillation inside the magnetic field in the "thick" lens case, all particles end up at r = 0! Non-zero emittance spreads out perfect focusing!



Larmor's Theorem

This result is a special case of a more general result. If go to frame that rotates with the local value of Larmor's frequency, then the transverse dynamics including the magnetic field are simply those of a harmonic oscillator with frequency equal to the Larmor frequency. Any force from the magnetic field linear in the field strength is "transformed away" in the Larmor frame. And the motion in the two transverse degrees of freedom are now decoupled. Pf: The equations of motion are

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}^2 = qr\dot{\theta}B_z$$

$$\gamma m r^2\dot{\theta} + qA_\theta = cons = P_\theta$$

$$\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}'^2 + 2\gamma m r\theta'\omega_L - \gamma m r\omega_L^2 = qr\dot{\theta}'B_z - qr\omega_L B_z$$

$$\gamma m r^2\dot{\theta}' = P_\theta$$

$$\left.\begin{aligned}\frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}'^2 &= -\gamma m r\omega_L^2 \\ \gamma m r^2\dot{\theta}' &= P_\theta\end{aligned}\right\}\text{ 2-D Harmonic Oscillator}$$

Transfer Matrix

- For solenoid of length L, transfer matrix is

$$M_{sol} = M_{end}^{-1} M_{from\ rot} M_{dec} M_{to\ rot} M_{end}$$

- Decoupled matrix in rotating coordinate system (Eq. 17.34)

$$M_{dec} = \begin{pmatrix} \cos(\omega_L L/v_z) & (v_z/\omega_L)\sin(\omega_L L/v_z) & 0 & 0 \\ -(\omega_L/v_z)\sin(\omega_L L/v_z) & \cos(\omega_L L/v_z) & 0 & 0 \\ 0 & 0 & \cos(\omega_L L/v_z) & (v_z/\omega_L)\sin(\omega_L L/v_z) \\ 0 & 0 & -(\omega_L/v_z)\sin(\omega_L L/v_z) & \cos(\omega_L L/v_z) \end{pmatrix}$$

- Matrix from Rotating Coordinates (Eq. 17.36, corrected)

$$M_{from\ rot} = \begin{pmatrix} \cos(\omega_L L/v_z) & 0 & \sin(\omega_L L/v_z) & 0 \\ -(\omega_L/v_z)\sin(\omega_L L/v_z) & \cos(\omega_L L/v_z) & (\omega_L/v_z)\cos(\omega_L L/v_z) & \sin(\omega_L L/v_z) \\ -\sin(\omega_L L/v_z) & 0 & \cos(\omega_L L/v_z) & 0 \\ -(\omega_L/v_z)\cos(\omega_L L/v_z) & -\sin(\omega_L L/v_z) & -(\omega_L/v_z)\sin(\omega_L L/v_z) & \cos(\omega_L L/v_z) \end{pmatrix}$$

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To/from rotating coordinates

$$v(z) = x(z)\cos(\omega_L z/v_z) - y(z)\sin(\omega_L z/v_z)$$

$$w(z) = x(z)\sin(\omega_L z/v_z) + y(z)\cos(\omega_L z/v_z)$$

$$\frac{dv}{dz} = \left(\frac{dx}{dz} - \frac{\omega_L}{v_z}y\right)\cos(\omega_L z/v_z) - \left(\frac{dy}{dz} + \frac{\omega_L}{v_z}x\right)\sin(\omega_L z/v_z)$$

$$\frac{dw}{dz} = \left(\frac{dx}{dz} - \frac{\omega_L}{v_z}y\right)\sin(\omega_L z/v_z) + \left(\frac{dy}{dz} + \frac{\omega_L}{v_z}x\right)\cos(\omega_L z/v_z)$$

$$\begin{pmatrix} v(z) \\ v'(z) \\ w(z) \\ w'(z) \end{pmatrix} = M_{to\ rot}\begin{pmatrix} x(z) \\ x'(z) \\ y(z) \\ y'(z) \end{pmatrix} = \begin{pmatrix} \cos(\omega_L z/v_z) & 0 & -\sin(\omega_L z/v_z) & 0 \\ -(\omega_L/v_z)\sin(\omega_L z/v_z) & \cos(\omega_L z/v_z) & -(\omega_L/v_z)\cos(\omega_L z/v_z) & -\sin(\omega_L z/v_z) \\ \sin(\omega_L z/v_z) & 0 & \cos(\omega_L z/v_z) & 0 \\ (\omega_L/v_z)\cos(\omega_L z/v_z) & \sin(\omega_L z/v_z) & -(\omega_L/v_z)\sin(\omega_L z/v_z) & \cos(\omega_L z/v_z) \end{pmatrix}\begin{pmatrix} x(z) \\ x'(z) \\ y(z) \\ y'(z) \end{pmatrix}$$

$$\begin{pmatrix} x'(z) \\ y'(z) \end{pmatrix} = \begin{pmatrix} (\omega_L z/v_z)y \\ -(\omega_L z/v_z)x \end{pmatrix} + \begin{pmatrix} \cos(\omega_L z/v_z) & \sin(\omega_L z/v_z) \\ -\sin(\omega_L z/v_z) & \cos(\omega_L z/v_z) \end{pmatrix}\begin{pmatrix} v'(z) \\ w'(z) \end{pmatrix}$$

$$\begin{pmatrix} x(z) \\ x'(z) \\ y(z) \\ y'(z) \end{pmatrix} = M_{from\ rot}\begin{pmatrix} v(z) \\ v'(z) \\ w(z) \\ w'(z) \end{pmatrix} = \begin{pmatrix} \cos(\omega_L z/v_z) & 0 & \sin(\omega_L z/v_z) & 0 \\ -(\omega_L/v_z)\sin(\omega_L z/v_z) & \cos(\omega_L z/v_z) & (\omega_L/v_z)\cos(\omega_L z/v_z) & \sin(\omega_L z/v_z) \\ -\sin(\omega_L z/v_z) & 0 & \cos(\omega_L z/v_z) & 0 \\ -(\omega_L/v_z)\cos(\omega_L z/v_z) & -\sin(\omega_L z/v_z) & -(\omega_L/v_z)\sin(\omega_L z/v_z) & \cos(\omega_L z/v_z) \end{pmatrix}\begin{pmatrix} v(z) \\ v'(z) \\ w(z) \\ w'(z) \end{pmatrix}$$

- Fringe effect by conservation cannonical momentum

$$M_{end} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & v_z/\omega_L & 0 \\ 0 & 0 & 1 & 0 \\ -v_z/\omega_L & 0 & 0 & 1 \end{pmatrix} \qquad M_{end}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -v_z/\omega_L & 0 \\ 0 & 0 & 1 & 0 \\ v_z/\omega_L & 0 & 0 & 1 \end{pmatrix}$$

- Match to Boundary Conditions at $z = 0$

$$M_{to\ rot}(z=0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\omega_L/v_z & 0 \\ 0 & 0 & 1 & 0 \\ \omega_L/v_z & 0 & 0 & 1 \end{pmatrix} = M_{end}^{-1}$$

Total Solenoid Transfer

$$M_{sol} = \begin{pmatrix} \cos^2\Phi & (1/S)\sin 2\Phi & (1/2)\sin 2\Phi & (2/S)\sin^2\Phi \\ -(S/4)\sin 2\Phi & \cos^2\Phi & -(S/2)\sin^2\Phi & (1/2)\sin 2\Phi \\ -(1/2)\sin 2\Phi & -(2/S)\sin^2\Phi & \cos^2\Phi & (1/S)\sin 2\Phi \\ (S/2)\sin^2\Phi & -(1/2)\sin 2\Phi & -(S/4)\sin 2\Phi & \cos^2\Phi \end{pmatrix}$$

Wiedemann 17.39 $\quad \Phi = \omega_L L / v_z \quad S = 2\omega_L / v_z$

$$M_{sol} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/f_{sol} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/f_{sol} & 1 \end{pmatrix} \quad \text{Thin Lens}$$

Easy Calculation that Works

- Wiedemann points out the following simple calculation is OK

$$M_{sol} = \begin{pmatrix} \cos\Phi & 0 & \sin\Phi & 0 \\ 0 & \cos\Phi & 0 & \sin\Phi \\ -\sin\Phi & 0 & \cos\Phi & 0 \\ 0 & -\sin\Phi & 0 & \cos\Phi \end{pmatrix} \begin{pmatrix} \cos\Phi & 2\sin\Phi/S & 0 & 0 \\ -\frac{S}{2}\sin\Phi & \cos\Phi & 0 & 0 \\ 0 & 0 & \cos\Phi & 2\sin\Phi/S \\ 0 & 0 & -\frac{S}{2}\sin\Phi & \cos\Phi \end{pmatrix}$$

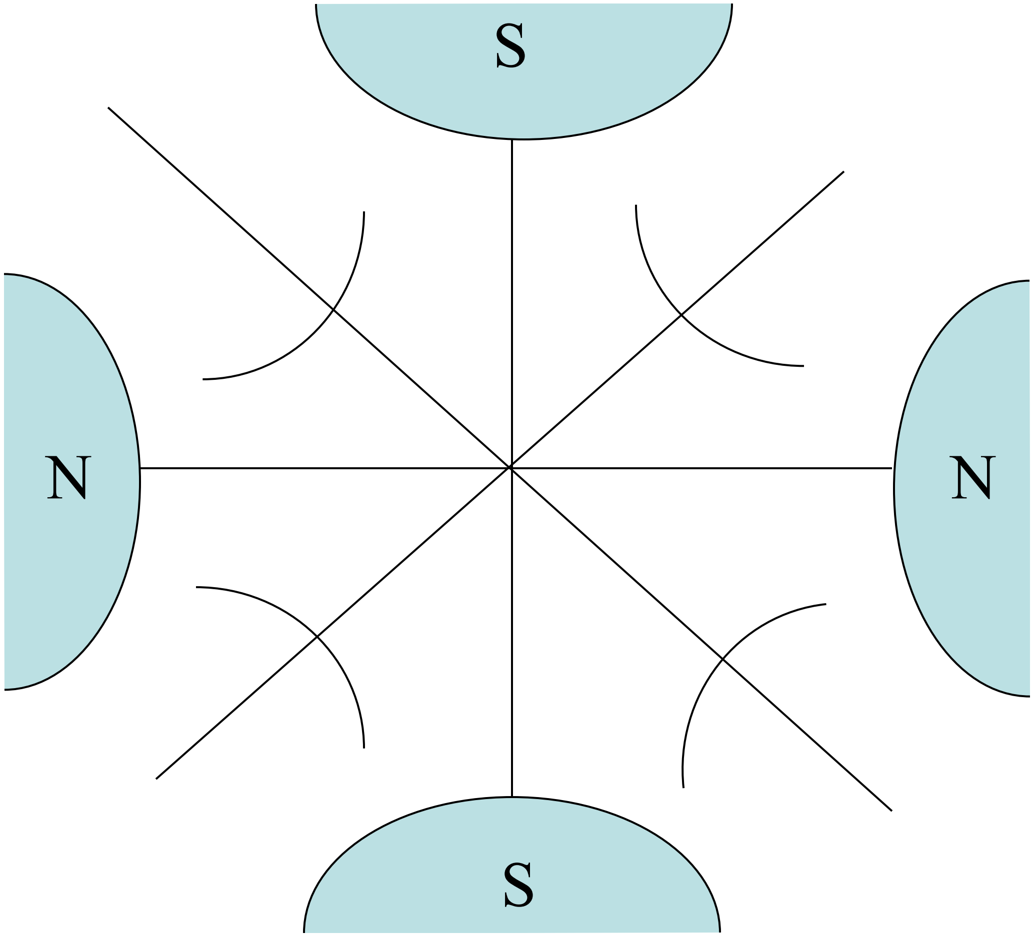
- Works because

$$M_{to\ rot} M_{end} = I$$

$$M_{end}^{-1} M_{from\ rot} = \begin{pmatrix} \cos\Phi & 0 & -\sin\Phi & 0 \\ 0 & \cos\Phi & 0 & -\sin\Phi \\ -\sin\Phi & 0 & \cos\Phi & 0 \\ 0 & -\sin\Phi & 0 & \cos\Phi \end{pmatrix}$$

- Explanation hard to follow

Skew Quadrupole



Equations

$$x'' + \frac{B'}{B\rho} y = 0$$

$$y'' + \frac{B'}{B\rho} x = 0$$

$$(x+y)'' + \frac{B'}{B\rho}(x+y) = 0$$

$$(x-y)'' - \frac{B'}{B\rho}(x-y) = 0$$

Focusing in x + y, defocusing in x - y

Transfer Matrix

$$k = \sqrt{\frac{B'}{B\rho}}$$

$$\begin{pmatrix} x+y \\ x'+y' \\ x-y \\ x'-y' \end{pmatrix}_{after} = \begin{pmatrix} \cos kL & \frac{1}{k}\sin kL & 0 & 0 \\ -k\sin kL & \cos kL & 0 & 0 \\ 0 & 0 & \cosh kL & \frac{1}{k}\sinh kL \\ 0 & 0 & k\sinh kL & \cosh kL \end{pmatrix} \begin{pmatrix} x+y \\ x'+y' \\ x-y \\ x'-y' \end{pmatrix}_{before}$$

$$M_{skew} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos kL & \frac{1}{k}\sin kL & 0 & 0 \\ -k\sin kL & \cos kL & 0 & 0 \\ 0 & 0 & \cosh kL & \frac{1}{k}\sinh kL \\ 0 & 0 & k\sinh kL & \cosh kL \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

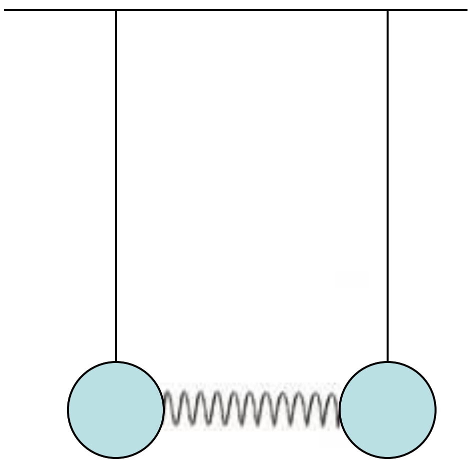
In terms of the usual variables

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{after} = \frac{1}{2}\begin{pmatrix} C^+ & S^+/k & C^- & S^-/k \\ -kS^- & C^+ & -kS^+ & C^- \\ C^- & S^-/k & C^+ & S^+/k \\ -kS^+ & C^- & -kS^- & C^+ \end{pmatrix}\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{before}$$

$$C^{\pm} = \cos kL \pm \cosh kL \qquad S^{\pm} = \sin kL \pm \sinh kL$$

$$M_{skew,thin} = \begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & -1/f & 0 \\ 0 & 0 & 1 & L \\ -1/f & 0 & 0 & 1 \end{pmatrix} \text{ Thin Lens}$$

Coupled Pendula



$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega_g^2 x^2}{2} + \frac{m\omega_g^2 y^2}{2} + \frac{m\omega_s^2 (x-y)^2}{2}$$

Equations of motion

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \omega_g^2 + \omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_g^2 + \omega_s^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{i\omega t}$$

$$\mathrm{Det}\begin{pmatrix} \omega_g^2 + \omega_s^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_g^2 + \omega_s^2 - \omega^2 \end{pmatrix} = 0$$

Solutions

Eigenvalues

$$\left(\omega^2-\omega_g^2-\omega_s^2\right)^2-\omega_s^2=0$$

$$\omega^2=\omega_g^2+\omega_s^2\pm\omega_s^2$$

Eigenvectors

$$\omega_-=\omega_g \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{symmetric mode}$$

$$\omega_+=\sqrt{\omega_g^2+2\omega_s^2} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \quad \text{antisymmetric mode}$$

Time Dependence

For an oscillation starting in *x*-direction

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{x_0}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\left(\omega_g t\right) + \frac{x_0}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos\left(\sqrt{\omega_g^2 + 2\omega_s^2}\, t\right)$$

$$\omega_g = \frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} + \frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2}$$

$$\sqrt{\omega_g^2 + 2\omega_s^2} = \frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} - \frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2}$$

$$\therefore x(t) = x_0 \cos\left[\frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right] \cos\left[\frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right]$$

$$\therefore y(t) = -x_0 \sin\left[\frac{\omega_g + \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right] \sin\left[\frac{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}{2} t\right]$$

Qualitatively

- Oscillation energy migrates $x \to y \to x$
- Period for a complete cycle is

$$\frac{4\pi}{\omega_g - \sqrt{\omega_g^2 + 2\omega_s^2}}$$

Becomes longer the weaker the coupling (→compensation)

- If un-coupled oscillation periods different

$$\omega^2 = \frac{\omega_1^2 + \omega_2^2 + 2\omega_s^2 \pm \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 - 4\omega_s^4}}{2}$$

Eigenvectors no longer pure symmetric and antisymmetric but "migration" is fairly generic behavior