Accelerator Physics Synchrotron Radiation

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Lecture 17

Relativistic Kinematics

In average rest frame the insertion device is Lorentz contracted, and so its wavelength is

$$\lambda^* = \lambda_{ID} / \beta^* \gamma^*$$

The sinusoidal wiggling motion emits with angular frequency

$$\omega^* = 2\pi c / \lambda^*$$

Lorentz transformation formulas for the wave vector of the emitted radiation

$$k^* = \gamma^* k\left(1 - \beta^* \cos\theta\right)$$

$$k_x^* = k_x = k \sin\theta \cos\varphi$$

$$k_y^* = k_y = k \sin\theta \sin\varphi$$

$$k_z^* = \gamma^* k\left(\cos\theta - \beta^*\right)$$

ID (or FEL) Resonance Condition

Angle transforms as

$$\cos\theta^* = \frac{k_z^*}{k^*} = \frac{\left(\cos\theta - \beta^*\right)}{\left(1 - \beta^* \cos\theta\right)}$$

Wave vector in lab frame has

$$k = \frac{k^*}{\gamma^*\left(1 - \beta^* \cos\theta\right)} = \frac{2\pi\beta^* c}{\lambda_{ID}\left(1 - \beta^* \cos\theta\right)}$$

In the forward direction $\cos\theta = 1$

$$\lambda_e \approx \frac{\lambda_{ID}}{2\gamma^{*2}} = \frac{\lambda_{ID}}{2\gamma^2}\left(1 + K^2/2\right)$$

Power Emitted Lab Frame

Larmor/Lienard calculation in the lab frame yields

$$\langle P \rangle = \frac{e^2}{6\pi\varepsilon_0}\gamma^4\beta_{z0}^2 c\left(\frac{K}{\gamma}\right)^2\left(\frac{2\pi}{\lambda_{ID}}\right)^2\frac{1}{2}$$

Total energy radiated after one passage of the insertion device

$$\delta E = 2\pi^2\frac{e^2}{6\pi\varepsilon_0\lambda_{ID}}\gamma^2\frac{\beta_{z0}^2}{\beta^*}NK^2$$

Power Emitted Beam Frame

Larmor/Lienard calculation in the beam frame yields

$$\left\langle P^* \right\rangle = \frac{e^2}{6\pi\varepsilon_0} cK^2 \left(\frac{2\pi}{\lambda^*} \right)^2 \frac{1}{2}$$

Total energy of each photon is $\hbar 2\pi c/\lambda^*$, therefore the total number of photons radiated after one passage of the insertion device

$$N_\gamma = \frac{2\pi}{3} \alpha N K^2$$

Spectral Distribution in Beam Frame

Begin with average power distribution in beam frame: dipole radiation pattern (single harmonic only when K<<1; replace γ* by γ, β* by β)

$$\frac{dP^*}{d\Omega^*} = \frac{e^2 c}{32\pi^2 \varepsilon_0} K^2 k^{*2} \sin^2 \Theta^*$$

Number distribution in terms of wave number

$$\frac{dN_\gamma}{d\Omega^*} = \frac{\alpha}{4} N K^2 \frac{k_y^{*2} + k_z^{*2}}{k^{*2}}$$

Solid angle transformation

$$d\Omega^* = \frac{d\Omega}{\gamma^2 \left(1 - \beta \cos\theta\right)^2}$$

Number distribution in beam frame

$$\frac{dN_\gamma}{d\Omega} = \frac{\alpha}{4} NK^2 \frac{\sin^2\theta\sin^2\varphi + \gamma^2\left(\cos\theta - \beta\right)^2}{\gamma^4\left(1-\beta\cos\theta\right)^4}$$

Energy is simply

$$E(\theta) = \hbar\frac{2\pi\beta c}{\lambda_{ID}\left(1-\beta\cos\theta\right)} \qquad \hat{E}(\theta) = \frac{1}{\left(1-\beta\cos\theta\right)}$$

Number distribution as a function of normalized lab-frame energy

$$\frac{dN_\gamma}{d\hat{E}} = \frac{\alpha\pi}{4\gamma^2\beta^3} NK^2\left[\left(\frac{\hat{E}}{\gamma^2} - 1\right)^2 + \beta^2\right]$$

Average Energy

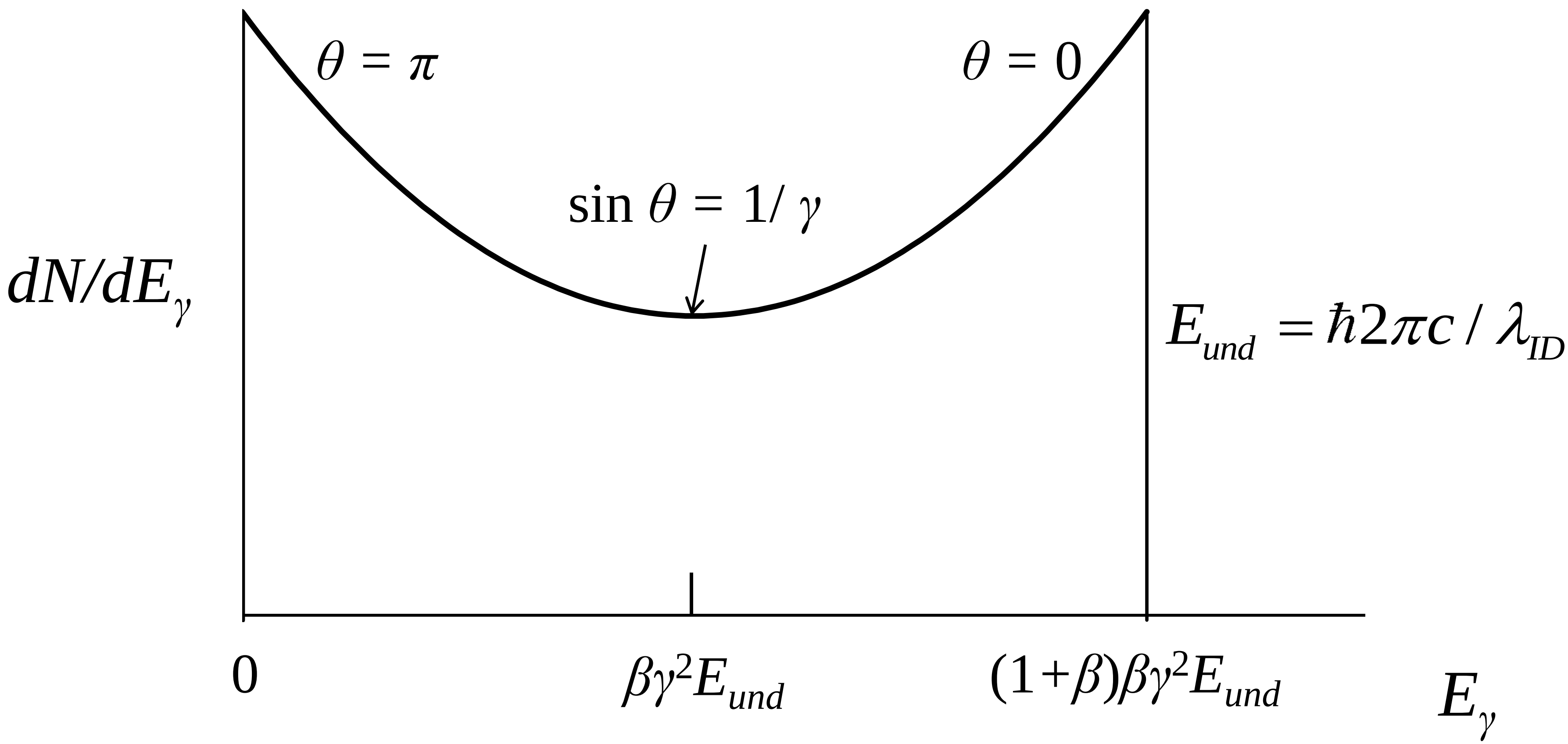
Limits of integration

$$\cos\theta = 1 \qquad \hat{E} = \frac{1}{1-\beta} \qquad \cos\theta = -1 \qquad \hat{E} = \frac{1}{1+\beta}$$

Average energy is also analytically calculable

$$\langle E \rangle = \frac{\int_0^\infty E \frac{dN_\gamma}{d\hat{E}} d\hat{E}}{\int_0^\infty \frac{dN_\gamma}{d\hat{E}} d\hat{E}} = \gamma^2 \hbar 2\pi\beta c / \lambda_{ID} \approx \frac{E_{\max}}{2}$$

Number Spectrum



Nonlinear Thomson Scattering

- Many of the the newer Thomson Sources are based on a PULSED laser (e.g. all of the high-energy lasers are pulsed by their very nature)
- Have developed a general theory to cover radiation distribution calculations in the general case of a pulsed, high field strength laser interacting with electrons in a Thomson scattering arrangement.
- The new theory shows that in many situations the estimates people do to calculate flux and brilliance, based on a constant amplitude models, need to be modified.
- The new theory is general enough to cover all "1-D" undulator calculations and all "1-D" pulsed laser Thomson scattering calculations.
- The main "new physics" that the new calculations include properly is the fact that the electron motion changes based on the local value of the field strength squared. Such ponderomotive forces (i.e., forces proportional to the field strength squared), lead to a red-shift detuning of the emission, angle dependent Doppler shifts of the emitted scattered radiation, and additional transverse dipole emission that this theory can calculate.

Ancient History

- Early 1960s: Laser Invented
- Brown and Kibble (1964): Earliest definition of the field strength parameters *K* and/or *a* in the literature that I'm aware of

$$a = \frac{eE_0\lambda_0}{2\pi mc^2} \quad \text{Thomson Sources} \qquad K = \frac{eB_0\lambda_0}{2\pi mc} \quad \text{Undulators}$$

  Interpreted frequency shifts that occur at high fields as a "relativistic mass shift".
- Sarachik and Schappert (1970): Power into harmonics at high *K* and/or *a* . Full calculation for CW (monochromatic) laser. Later referenced, corrected, and extended by workers in fusion plasma diagnostics.
- Alferov, Bashmakov, and Bessonov (1974): Undulator/Insertion Device theories developed under the assumption of constant field strength. Numerical codes developed to calculate "real" fields in undulators.
- Coisson (1979): Simplified undulator theory, which works at low *K* and/or *a,* developed to understand the frequency distribution of "edge" emission, or emission from "short" magnets, i.e., including pulse effects

Coisson's Spectrum of a Short Magnet

Coisson low-field strength undulator spectrum*

$$\frac{dE}{d\nu d\Omega} = \frac{r_e^2 c}{\pi} \gamma^2 \left(1+\gamma^2\theta^2\right)^2 f^2 \left| \tilde{B}\left(\nu\left(1+\gamma^2\theta^2\right)/2\gamma^2\right) \right|^2$$

$$f^2 = f_\sigma^2 + f_\pi^2$$

$$f_\sigma = \frac{1}{\left(1+\gamma^2\theta^2\right)^2} \sin\phi$$

$$f_\pi = \frac{1}{\left(1+\gamma^2\theta^2\right)^2} \left(\frac{1-\gamma^2\theta^2}{1+\gamma^2\theta^2}\right) \cos\phi$$

*R. Coisson, Phys. Rev. A 20, 524 (1979)

Dipole Radiation

Assume a single charge moves in the *x* direction

$$\rho(x,y,z,t) = e\delta(x-d(t))\delta(y)\delta(z)$$

$$\vec{J}(x,y,z,t) = e\dot{d}(t)\hat{x}\delta(x-d(t))\delta(y)\delta(z)$$

Introduce scalar and vector potential for fields.
Retarded solution to wave equation (Lorenz gauge), $R = \left|\vec{r} - \vec{r}\,'(t')\right|$

$$\phi(\vec{r},t) = \int \frac{1}{R}\rho\left(\vec{r}\,',t-\frac{R}{c}\right)dx'dy'dz' = \frac{e}{4\pi\varepsilon_0}\int \frac{\delta(t'-t+R/c)}{R}dt'$$

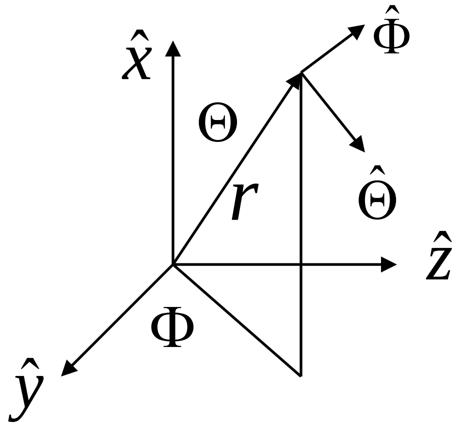
$$A_x(\vec{r},t) = \int \frac{1}{R}J_x\left(\vec{r}\,',t-\frac{R}{c}\right)dx'dy'dz' = \frac{\mu_0 e}{4\pi}\int \frac{\dot{d}(t')\delta(t'-t+R/c)}{R}dt'$$

$$\vec{B} = \vec{\nabla}\times\vec{A},\quad \lim_{\substack{r\to\infty \\ \beta\ll 1}}\vec{B} = \lim_{r\to\infty}\frac{\mu_0 e}{4\pi R}\vec{\nabla}\times\dot{d}(t-R/c) \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

Dipole Radiation

$$\lim_{\substack{r\to\infty \\ \beta\ll 1}} \vec{B} = \frac{\mu_0 e\ddot{d}\left(t-r/c\right)}{4\pi c r}\sin\Theta\hat{\Phi}$$

$$\lim_{\substack{r\to\infty \\ \beta\ll 1}} \vec{E} = \frac{e\ddot{d}\left(t-r/c\right)}{4\pi\varepsilon_0 c^2 r}\sin\Theta\hat{\Theta}$$



$$I = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{1}{16\pi^2\varepsilon_0}\frac{e^2\ddot{d}^2\left(t-r/c\right)}{c^3 r^2}\sin^2\Theta\hat{r}$$

$$\frac{dI}{d\Omega} = \frac{1}{16\pi^2\varepsilon_0}\frac{e^2\ddot{d}^2\left(t-r/c\right)}{c^3}\sin^2\Theta$$

Polarized in the plane containing $\hat{r}=\vec{n}$ and $\hat{x}$

Dipole Radiation

Define the Fourier Transform

$$\tilde{d}(\omega) = \int d(t) e^{-i\omega t} dt \qquad d(t) = \frac{1}{2\pi}\int \tilde{d}(\omega) e^{i\omega t} d\omega$$

With these conventions Parseval's Theorem is

$$\int d^2(t)\, dt = \frac{1}{2\pi}\int \left|\tilde{d}\right|^2(\omega)\, d\omega$$

$$\frac{dE}{d\Omega} = \frac{e^2}{16\pi^2\varepsilon_0 c^3}\int \ddot{d}^2(t - r/c)\ dt = \frac{e^2}{32\pi^3\varepsilon_0 c^3}\int \omega^4 \left|\tilde{d}\right|^2(\omega)\ d\omega$$

$$\frac{dE}{d\omega d\Omega} = \frac{1}{32\pi^3\varepsilon_0}\frac{e^2\omega^4\left|\tilde{d}(\omega)\right|^2}{c^3}\sin^2\Theta$$

Blue Sky!

This equation does not follow the typical (see Jackson) convention that combines both positive and negative frequencies together in a single positive frequency integral. The reason is that we would like to apply Parseval's Theorem easily. By symmetry, the difference is a factor of two.

Dipole Radiation

For a motion in three dimensions

$$\frac{dE}{d\omega d\Omega}=\frac{1}{32\pi^3\varepsilon_0}\frac{e^2\omega^4\left|\tilde{\vec{d}}(\omega)\times\vec{n}\right|^2}{c^3}$$

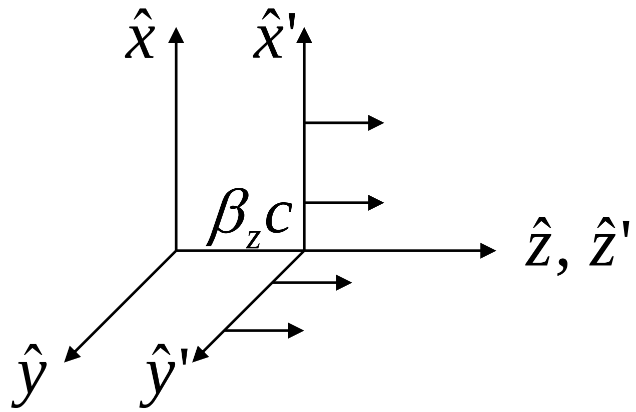
Vector inside absolute value along the magnetic field

$$\frac{dE}{d\omega d\Omega}=\frac{1}{32\pi^3}\frac{e^2\omega^4\left|\left(\tilde{\vec{d}}(\omega)\times\vec{n}\right)\times\vec{n}\right|^2}{\varepsilon_0 c^3}=\frac{1}{32\pi^3}\frac{e^2\omega^4\left|\left(\vec{n}\cdot\tilde{\vec{d}}(\omega)\right)\vec{n}-\tilde{\vec{d}}(\omega)\right|^2}{\varepsilon_0 c^3}$$

Vector inside absolute value along the electric field. To get energy into specific polarization, take scalar product with the polarization vector

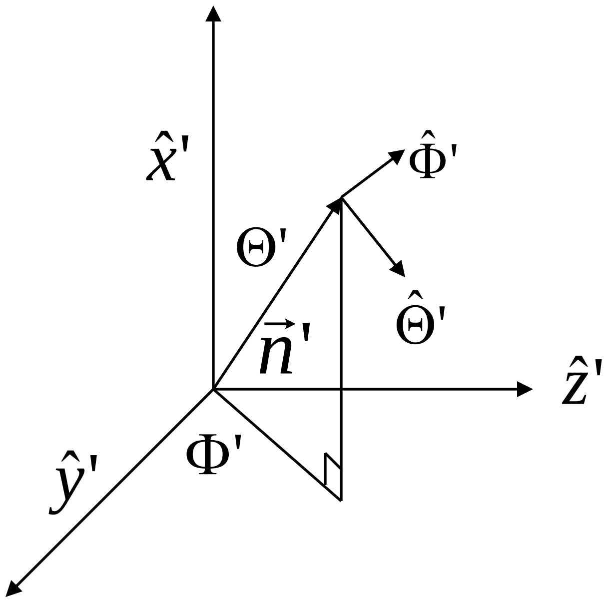
Co-moving Coordinates

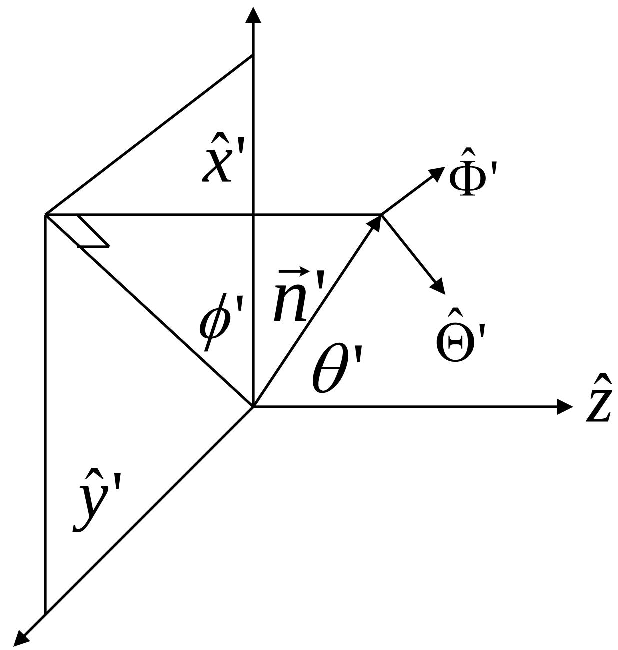
- Assume radiating charge is moving with a velocity close to light in a direction taken to be the *z* axis, and the charge is on average at rest in this coordinate system
- For the remainder of the presentation, quantities referred to the moving coordinates will have primes; unprimed quantities refer to the lab system



- In the co-moving system the dipole radiation pattern applies

New Coordinates





Resolve the polarization of scattered energy into that perpendicular (σ) and that parallel (π) to the scattering plane

$$\vec{n}' = \sin\theta'\cos\phi'\,\hat{x}' + \sin\theta'\sin\phi'\,\hat{y}' + \cos\theta'\,\hat{z}'$$

$$\hat{e}'_\sigma = \vec{n}'\times\hat{z}' / \left|\vec{n}'\times\hat{z}'\right| = \sin\phi'\,\hat{x}' - \cos\phi'\,\hat{y}' = -\hat{\phi}'$$

$$\hat{e}'_\pi = \vec{n}'\times\hat{e}'_\sigma = \cos\theta'\cos\phi'\,\hat{x}' + \cos\theta'\sin\phi'\,\hat{y}' - \sin\theta'\,\hat{z}' = \hat{\theta}'$$

Polarization

It follows that

$$\tilde{\vec{d}}'(\omega')\cdot\hat{e}'_{\sigma}=\tilde{d}'_x(\omega')\sin\phi'-\tilde{d}'_y(\omega')\cos\phi'$$

$$\tilde{\vec{d}}'(\omega')\cdot\hat{e}'_{\pi}=\tilde{d}'_x(\omega')\cos\theta'\cos\phi'+\tilde{d}'_y(\omega')\cos\theta'\sin\phi'-\sin\theta'\tilde{d}'_z(\omega')$$

So the energy into the two polarizations in the beam frame is

$$\frac{dE'_{\sigma}}{d\omega'd\Omega'}=\frac{1}{32\pi^3\varepsilon_0}\frac{e^2\omega'^4}{c^3}\left|\tilde{d}'_x(\omega')\sin\varphi'-\tilde{d}'_y(\omega')\cos\varphi'\right|^2$$

$$\frac{dE'_{\pi}}{d\omega'd\Omega'}=\frac{1}{32\pi^3\varepsilon_0}\frac{e^2\omega'^4}{c^3}\left|\begin{array}{c}\tilde{d}'_x(\omega')\cos\theta'\cos\varphi'+\tilde{d}'_y(\omega')\cos\theta'\sin\varphi'\\ -\sin\theta'\tilde{d}'_z(\omega')\end{array}\right|^2$$

Comments/Sum Rule

- There is no radiation parallel or anti-parallel to the *x*-axis for *x*-dipole motion
- In the forward direction $\theta' \to 0$, the radiation polarization is parallel to the *x*-axis for an *x*-dipole motion
- One may integrate over all angles to obtain a result for the total energy radiated

$$\frac{dE'_\sigma}{d\omega'} = \frac{1}{32\pi^3\varepsilon_0}\frac{e^2\omega'^4}{c^3}\left(\left|\tilde{d}'_x(\omega')\right|^2 + \left|\tilde{d}'_y(\omega')\right|^2\right)2\pi$$

$$\frac{dE'_\pi}{d\omega'} = \frac{1}{32\pi^3\varepsilon_0}\frac{e^2\omega'^4}{c^3}\left[\left(\left|\tilde{d}'_x(\omega')\right|^2 + \left|\tilde{d}'_y(\omega')\right|^2\right)\frac{2\pi}{3} + \left|\tilde{d}'_z(\omega')\right|^2\frac{8\pi}{3}\right]$$

$$\frac{dE'_{tot}}{d\omega'} = \frac{1}{32\pi^3\varepsilon_0}\frac{e^2\omega'^4\left|\vec{\tilde{d}}'(\omega')\right|^2}{c^3}\frac{8\pi}{3}$$

Generalized Larmor

Sum Rule

Total energy sum rule

$$E'_{tot} = \int_{-\infty}^{\infty} \frac{1}{12\pi^2\varepsilon_0} \frac{e^2\omega'^4 \left|\tilde{\vec{d}}'(\omega')\right|^2}{c^3} d\omega'$$

Parseval's Theorem again gives "standard" Larmor formula

$$P' = \frac{dE'_{tot}}{dt'} = \frac{1}{6\pi\varepsilon_0} \frac{e^2\ddot{\vec{d}}'^2(t')}{c^3} = \frac{1}{6\pi\varepsilon_0} \frac{e^2\vec{a}'^2(t')}{c^3}$$

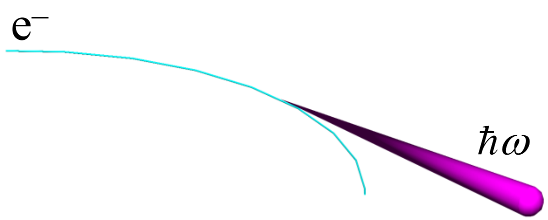
Energy Distribution in Lab Frame

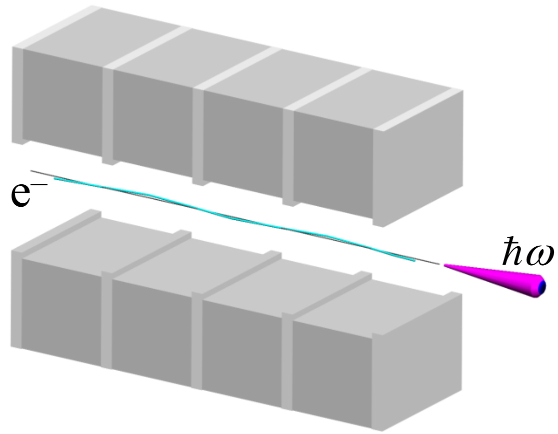
$$\frac{dE_\sigma}{d\omega d\Omega} = \frac{e^2\omega^4\gamma^2\left(1-\beta\cos\theta\right)^2}{32\pi^3c^3}\left|\begin{array}{l}\tilde{d}'_x\left(\omega\gamma\left(1-\beta\cos\theta\right)\right)\sin\phi\\ -\tilde{d}'_y\left(\omega\gamma\left(1-\beta\cos\theta\right)\right)\cos\phi\end{array}\right|^2$$

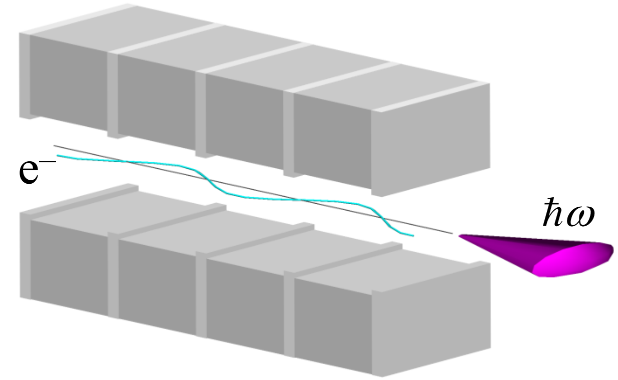
$$\frac{dE_\pi}{d\omega d\Omega} = \frac{e^2\omega^4\gamma^2\left(1-\beta\cos\theta\right)^2}{32\pi^3c^3}\left|\begin{array}{l}\tilde{d}'_x\left(\omega\gamma\left(1-\beta\cos\theta\right)\right)\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}\cos\phi\\ +\tilde{d}'_y\left(\omega\gamma\left(1-\beta\cos\theta\right)\right)\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}\sin\phi\\ -\tilde{d}'_z\left(\omega\gamma\left(1-\beta\cos\theta\right)\right)\dfrac{\sin\theta}{\gamma\left(1-\beta\cos\theta\right)}\end{array}\right|^2$$

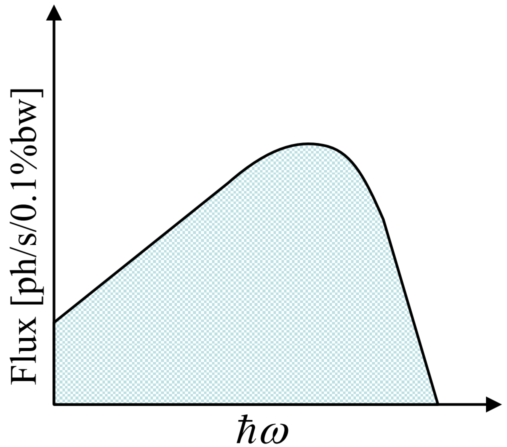
By placing the expression for the Doppler shifted frequency and angles inside the transformed beam frame distribution. Total energy radiated from d'_z is the same as d'_x and d'_y for same dipole strength.

Bend   Undulator   Wiggler









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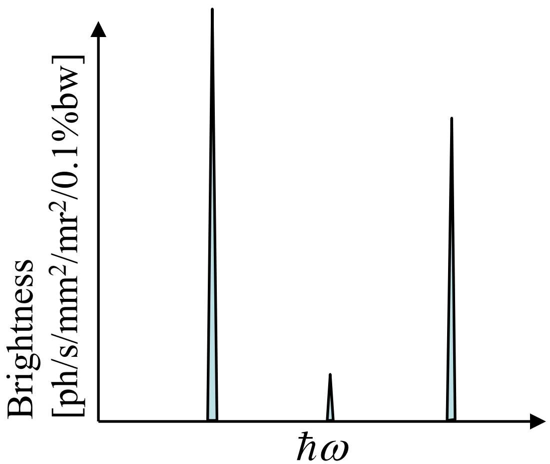

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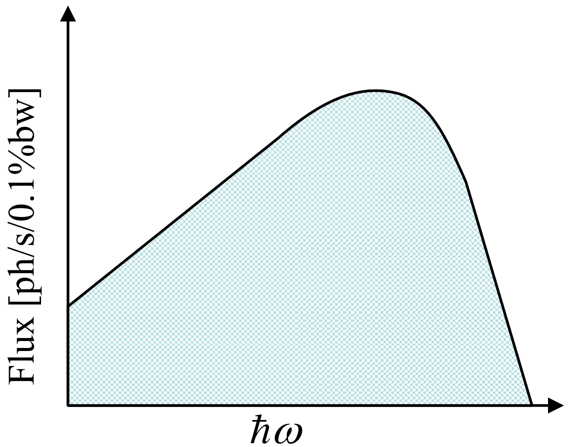


powerful white source

Weak Field Undulator Spectrum

$$\tilde{\vec{d}}'(\omega') = \tilde{d}'(\omega')\,\hat{x} = -\frac{ec}{mc}\frac{\tilde{B}(\omega'/c\beta_z\gamma)}{\omega'^2}\,\hat{x} \qquad \tilde{B}(k) = \int B(z)e^{-ikz}dz$$

$$\frac{dE_\sigma}{d\omega d\Omega} = \frac{\varepsilon_0 r_e^2 c}{2\pi}\frac{\left|\tilde{B}\left(\omega(1-\beta_z\cos\theta)/c\beta_z\right)\right|^2}{\gamma^2(1-\beta_z\cos\theta)^2}\sin^2\phi \qquad r_e^2 \equiv \frac{e^4}{16\pi^2\varepsilon_0^2 m^2 c^4}$$

$$\frac{dE_\pi}{d\omega d\Omega} = \frac{\varepsilon_0 r_e^2 c}{2\pi}\frac{\left|\tilde{B}\left(\omega(1-\beta_z\cos\theta)/c\beta_z\right)\right|^2}{\gamma^2(1-\beta_z\cos\theta)^2}\left(\frac{\cos\theta-\beta_z}{1-\beta_z\cos\theta}\right)^2\cos^2\phi$$

$$(1-\beta_z\cos\theta)(1+\beta_z) \approx \frac{1}{\gamma^2} + \theta^2 + \ldots \approx \frac{1+\gamma^2\theta^2}{\gamma^2} \qquad \lambda \approx \frac{\lambda_0}{2\gamma^2}$$

Generalizes Coisson to arbitrary observation angles

Strong Field Case

Why is the FEL resonance condition?

$$\lambda = \frac{\lambda_0}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$

$$\frac{d}{dt}\gamma = 0$$

$$\frac{d}{dt}\gamma m\vec{\beta}c = -ec\vec{\beta}\times\vec{B}$$

$$\beta_x(z) = \frac{e}{\gamma mc}\int_{-\infty}^{z} B(z')dz'$$

High K

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2(z)}$$

$$\beta_z(z) \approx 1 - \frac{1}{2\gamma^2} - \frac{1}{2}\left(\frac{e}{\gamma mc}\int_{-\infty}^{z} B(z')dz'\right)^2 = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) - \frac{K^2}{4\gamma^2}\cos(2k_0 z)$$

Inside the insertion device the average (z) velocity is

$$\beta^*_z = 1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

and the radiation emission frequency redshifts by the $1+K^2/2$ factor

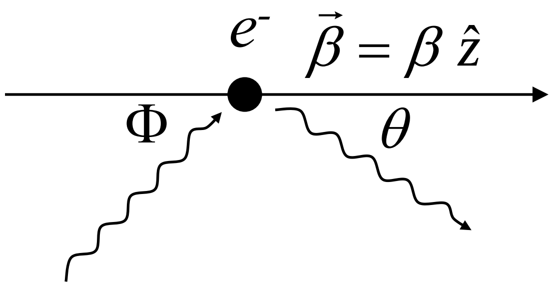
Thomson Scattering

- Purely "classical" scattering of photons by electrons
- Thomson regime defined by the photon energy in the electron rest frame being small compared to the rest energy of the electron, allowing one to neglect the quantum mechanical "Dirac" recoil on the electron
- In this case electron radiates at the same frequency as incident photon for low enough field strengths
- Classical dipole radiation pattern is generated in beam frame
- Therefore radiation patterns, at low field strength, can be largely copied from textbooks

- Note on terminology: Some authors call any scattering of photons by free electrons Compton Scattering. Compton observed (the so-called Compton effect) frequency shifts in X-ray scattering off (resting!) electrons that depended on scattering angle. Such frequency shifts arise only when the energy of the photon in the rest frame becomes comparable with 0.511 MeV.

Krafft, G. A. and G. Priebe, *Rev. Accel. Sci. and Tech.*, **3**, 147 (2010)

Simple Kinematics



Beam Frame

$$p'^{\mu}_{e} = \left(mc^2, 0\right)$$

$$p'^{\mu}_{p} = \left(E'_L, \vec{E}'_L\right)$$

Lab Frame

$$p^{\mu}_{e} = mc^2\left(\gamma, \gamma\beta\,\hat{z}\right)$$

$$p^{\mu}_{p} = E_L\left(1, \sin\Phi\hat{y} + \cos\Phi\hat{z}\right)$$

$$p_e \cdot p_p = mc^2 E'_L = mc^2 E_L \gamma\left(1 - \beta\cos\Phi\right)$$

$$E'_L = E_L \gamma \left(1 - \beta \cos \Phi\right)$$

In beam frame scattered photon radiated with wave vector

$$k'_\mu = \frac{E'_L}{c}\left(1, \sin\theta' \cos\phi', \sin\theta' \sin\phi', \cos\theta'\right)$$

Back in the lab frame, the scattered photon energy E_s is

$$E_s = E'_L \gamma \left(1 + \beta \cos\theta'\right) = \frac{E'_L}{\gamma\left(1 - \beta\cos\theta\right)}$$

$$E_s = E_L \frac{\left(1 - \beta\cos\Phi\right)}{\left(1 - \beta\cos\theta\right)}$$

Electron in a Plane Wave

Assume linearly-polarized pulsed laser beam moving in the direction (electron charge is $-e$)

$$\vec{n}_{inc} = \sin\Phi\hat{y} + \cos\Phi\hat{z}$$

$$\vec{A}_{inc}(\vec{x},t) = A_x(ct - \sin\Phi y - \cos\Phi z)\hat{x} \equiv A(\xi)\hat{x}$$

Polarization 4-vector (linearly polarized)

$$\varepsilon^{\mu} = (0,1,0,0)$$

Light-like incident propagation 4-vector

$$n^{\mu}_{inc} = (1,0,\sin\Phi,\cos\Phi)$$

$$\varepsilon \cdot n_{inc} = \varepsilon_{\mu} n^{\mu}_{inc} = \vec{\varepsilon} \cdot \vec{n}_{inc} = 0$$

Krafft, G. A., *Physical Review Letters,* 92, 204802 (2004), Krafft, Doyuran, and Rosenzweig , *Physical Review E,* 72, 056502 (2005)

Electromagnetic Field

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \varepsilon^\nu \frac{\partial A}{\partial x_\mu} - \varepsilon^\mu \frac{\partial A}{\partial x_\nu}$$

$$= \left(\varepsilon^\nu n^\mu_{inc} - \varepsilon^\mu n^\nu_{inc}\right)\frac{dA}{d\xi}(\xi)$$

Our goal is to find $x^\mu(\tau)=(ct(\tau),x(\tau),y(\tau),z(\tau))$ when the 4-velocity $u^\mu(\tau)=(cdt/d\tau,dx/d\tau,dy/d\tau,dz/d\tau)(\tau)$ satisfies $du^\mu/d\tau= -eF^{\mu\nu}u_\nu/mc$ where τ is proper time. For any solution to the equations of motion.

$$\frac{d\left(n_{inc\mu}u^\mu\right)}{d\tau} = n_{inc\mu}F^{\mu\nu}u_\nu = 0 \qquad \therefore n_{inc\mu}u^\mu = n_{inc\mu}u^\mu(-\infty)$$

Proportional to amount frequencies up-shifted in going to beam frame

ξ is proportional to the proper time

On the orbit

$$\xi(\tau) = ct(\tau) - \vec{n}_{inc} \cdot \vec{x}(\tau) \qquad d\xi / d\tau = n_{inc\mu} u^{\mu}$$

Integrate with respect to ξ instead of τ. Now

$$\frac{d(\varepsilon_{\mu} u^{\mu})}{d\tau} = c \frac{df}{d\xi} n_{inc\mu} u^{\mu} = c \frac{d}{d\tau} f(\xi(\tau))$$

where the unitless vector potential is $f(\xi) = -eA(\xi)/mc$.

$$\therefore \varepsilon_{\mu} u^{\mu} - cf = \varepsilon_{\mu} u^{\mu}(-\infty)$$

Electron Orbit

$$u^\mu(\xi) = u^\mu(-\infty) + cf(\xi)\left\{\frac{\varepsilon_\nu u^\nu(-\infty)}{n_{inc\nu}u^\nu(-\infty)}n^\mu_{inc} - \varepsilon^\mu\right\} + \frac{c^2 f^2(\xi)}{2\left(n_{inc\nu}u^\nu(-\infty)\right)}n^\mu_{inc}$$

Direct Force from Electric Field

Ponderomotive Force

$$x^\mu(\xi) = \frac{u^\mu(-\infty)\xi}{n_{inc\nu}u^\nu(-\infty)} + \left\{\frac{c\varepsilon_\nu u^\nu(-\infty)}{\left(n_{inc\nu}u^\nu(-\infty)\right)^2}n^\mu_{inc} - \frac{c\varepsilon^\mu}{n_{inc\nu}u^\nu(-\infty)}\right\}\int_{-\infty}^{\xi} f(\xi')d\xi'$$

$$+ \frac{c^2 n^\mu_{inc}}{\left(n_{inc\nu}u^\nu(-\infty)\right)^2}\int_{-\infty}^{\xi}\frac{f^2(\xi')}{2}d\xi'$$

Energy Distribution

$$\frac{dE_\sigma}{d\omega d\Omega} = \frac{e^2\omega^2}{32\pi^3 c^3}\left| \begin{matrix} D_t(\omega;\theta,\varphi)\sin\varphi \\ -\dfrac{\sin\Phi}{\gamma(1-\beta\cos\Phi)} D_p(\omega;\theta,\varphi)\cos\varphi \end{matrix} \right|^2$$

$$\frac{dE_\pi}{d\omega d\Omega} = \frac{e^2\omega^2}{32\pi^3 c^3}\left| \begin{matrix} D_t(\omega;\theta,\varphi)\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}\cos\varphi \\ +\dfrac{\sin\Phi}{\gamma(1-\beta\cos\Phi)} D_p(\omega;\theta,\varphi)\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}\sin\varphi \\ +\dfrac{\beta-\cos\Phi}{1-\beta\cos\Phi} D_p(\omega;\theta,\varphi)\dfrac{\sin\theta}{\gamma(1-\beta\cos\theta)} \end{matrix} \right|^2$$

Effective Dipole Motions: Lab Frame

ODU

$$D_t(\omega;\theta,\varphi)=\frac{1}{\gamma(1-\beta\cos\Phi)}\int\frac{eA(\xi)}{mc}e^{i\phi(\omega,\xi;\theta,\varphi)}d\xi$$

$$D_p(\omega;\theta,\varphi)=\frac{1}{\gamma(1-\beta\cos\Phi)}\int\frac{e^2A^2(\xi)}{2m^2c^2}e^{i\phi(\omega,\xi;\theta,\varphi)}d\xi$$

And the (Lorentz invariant!) phase is

$$\varphi(\omega,\xi;\theta,\phi)=\frac{\omega}{c}\left(\begin{array}{l}\xi\dfrac{(1-\beta\cos\theta)}{(1-\beta\cos\Phi)}-\dfrac{\sin\theta\cos\phi}{\gamma(1-\beta\cos\Phi)}\displaystyle\int_{-\infty}^{\xi}\frac{eA(\xi')}{mc}d\xi'\\+\dfrac{1-\sin\theta\sin\phi\sin\Phi-\cos\theta\cos\Phi}{\gamma^2(1-\beta\cos\Phi)^2}\displaystyle\int_{-\infty}^{\xi}\frac{e^2A^2(\xi')}{2m^2c^2}d\xi'\end{array}\right)$$

Summary

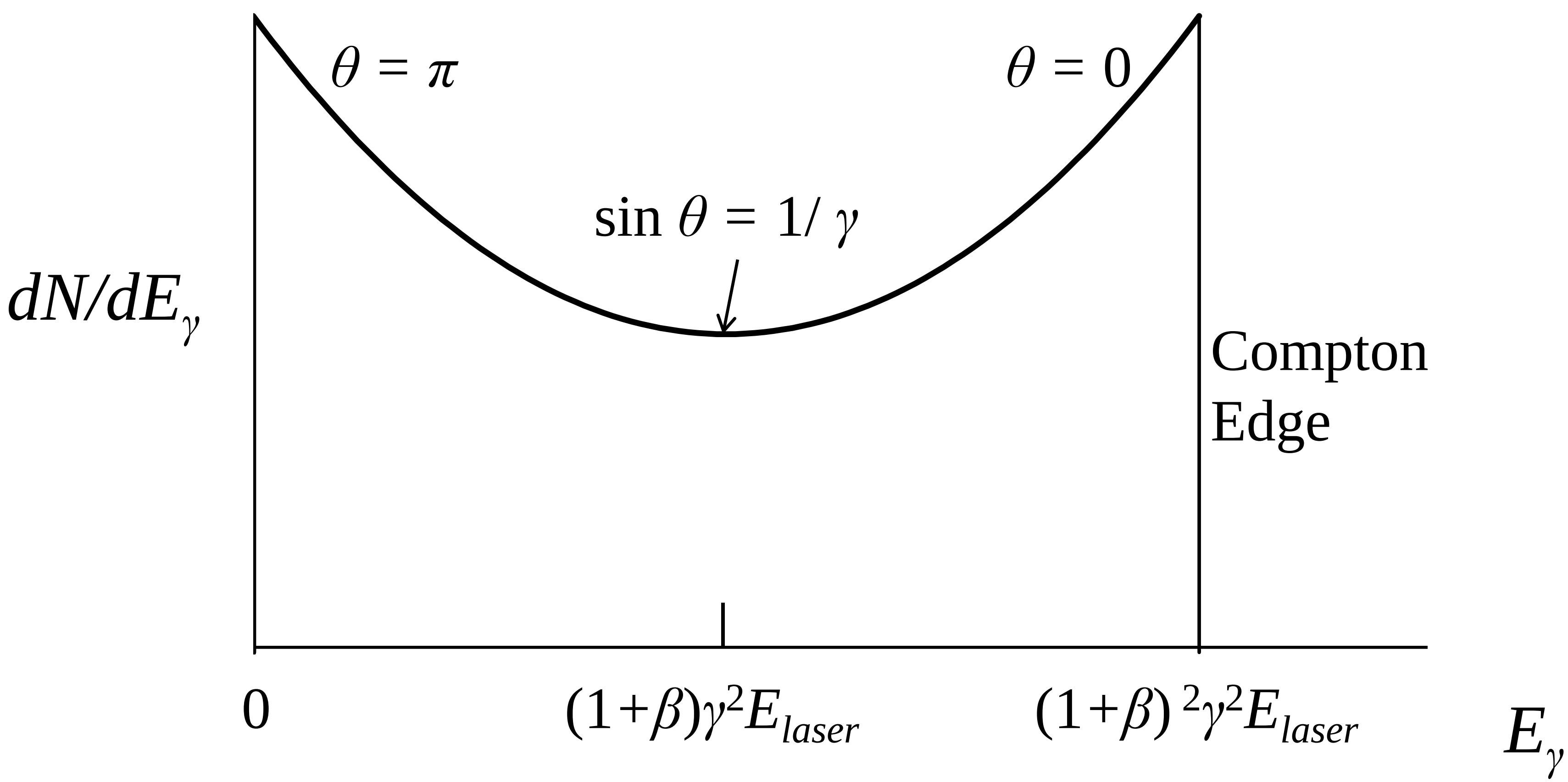
- Overall structure of the distributions is very like that from the general dipole motion, only the effective dipole motion, including physical effects such as the relativistic motion of the electrons and retardation, must be generalized beyond the straight Fourier transform of the field
- At low field strengths ($f \ll 1$), the distributions reduce directly to the classical Fourier transform dipole distributions
- The effective dipole motion from the ponderomotive force involves a simple projection of the incident wave vector in the beam frame onto the axis of interest, multiplied by the general ponderomotive dipole motion integral
- The radiation from the two transverse dipole motions are compressed by the same angular factors going from beam to lab frame as appears in the simple dipole case. The longitudinal dipole radiation is also transformed between beam and lab frame by the same fraction as in the simple longitudinal dipole motion. Thus the usual compression into a $1/\gamma$ cone applies

Weak Field Thomson Backscatter

With $\Phi = \pi$ and $f \ll 1$ the result is identical to the weak field undulator result with the replacement of the magnetic field Fourier transform by the electric field Fourier transform

	Undulator	Thomson Backscatter
Driving Field	$c\tilde{B}_y\left(\omega\left(1-\beta_z \cos\theta\right)/c\beta_z\right)$	$\tilde{E}_x\left(\omega\left(1-\beta_z\cos\theta\right)/\left(c\left(1+\beta_z\right)\right)\right)$
Forward Frequency	$\lambda \approx \frac{\lambda_0}{2\gamma^2}$	$\lambda \approx \frac{\lambda_0}{4\gamma^2}$
	Lorentz contract + Doppler	Double Doppler

Photon Number Spectrum



High Field Thomson Backscatter

ODU

For a flat incident laser pulse the main results are very similar to those from undulaters with the following correspondences

	Undulator	Thomson Backscatter
Field Strength	K	a
Forward Frequency	$\lambda \approx \frac{\lambda_0}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$	$\lambda \approx \frac{\lambda_0}{4\gamma^2}\left(1+\frac{a^2}{2}\right)$
Transverse Pattern	$\beta^*_z + \cos\theta'$	$1+\cos\theta'$

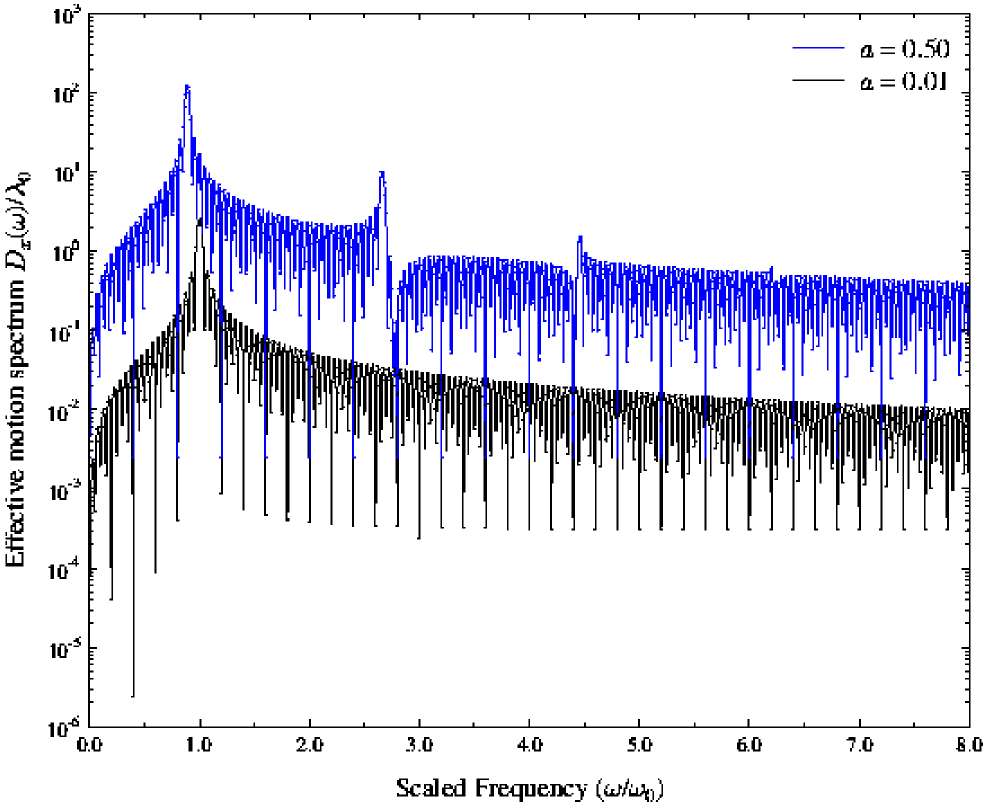
NB, be careful with the radiation pattern, it is the same at small angles, but quite a bit different at large angles

Forward Direction: Flat Laser Pulse

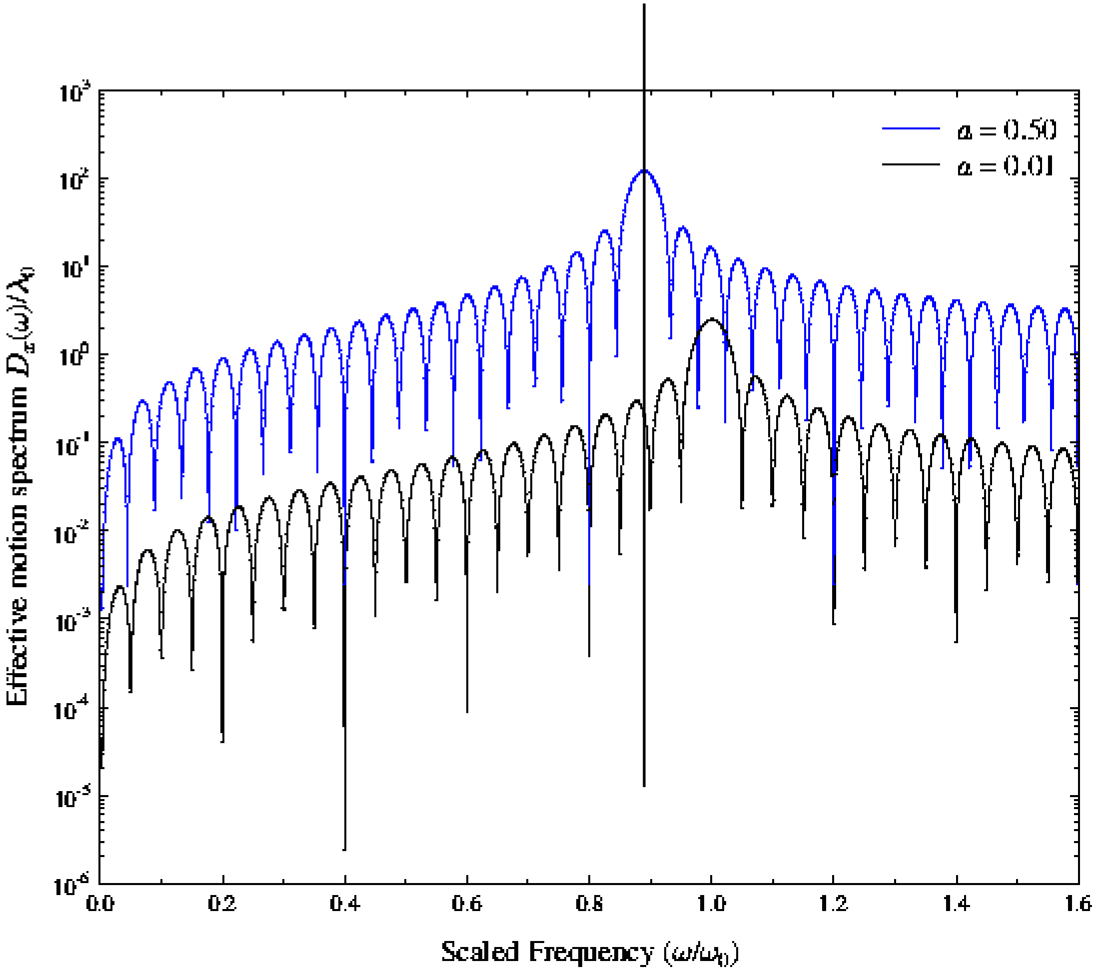
20-period

equivalent undulator: $A_x(\xi) = A_0 \cos(2\pi\xi/\lambda_0)[\Theta(\xi) - \Theta(\xi - 20\lambda_0)]$

$$\omega_0 \equiv (1+\beta_z)^2 \gamma^2 2\pi c/\lambda_0 \approx 4\gamma^2 2\pi c/\lambda_0, \quad a = eA_0/mc$$



$$1/\left(1+a^2/2\right)$$



Realistic Pulse Distribution at High *a*

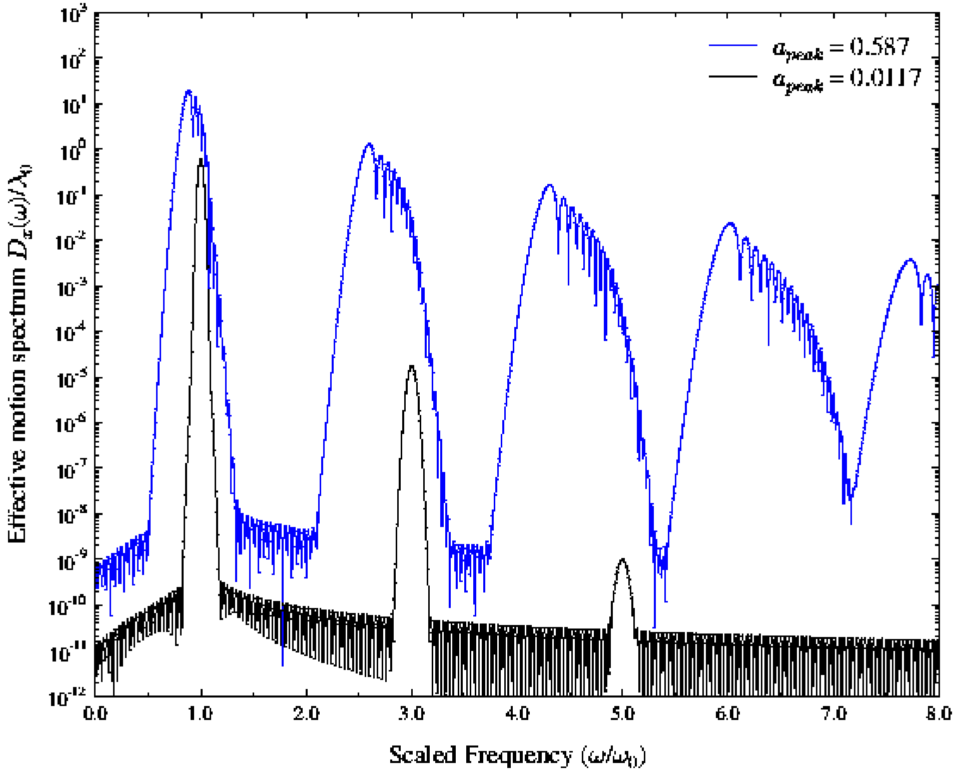
In general, it's easiest to just numerically integrate the lab-frame expression for the spectrum in terms of D_t and D_p. A 10^5 to 10^6 point Simpson integration is adequate for most purposes. Flat pulses reproduce previously known results and to evaluate numerical error, and Gaussian amplitude modulated pulses.

One may utilize a two-timing approximation (i.e., the laser pulse is a slowly varying sinusoid with amplitude $a(\xi)$), and the fundamental expressions, to write the energy distribution at any angle in terms of Bessel function expansions and a ξ integral over the modulation amplitude. This approach actually has a limited domain of applicability ($K, a < 0.1$)

Forward Direction: Gaussian Pulse

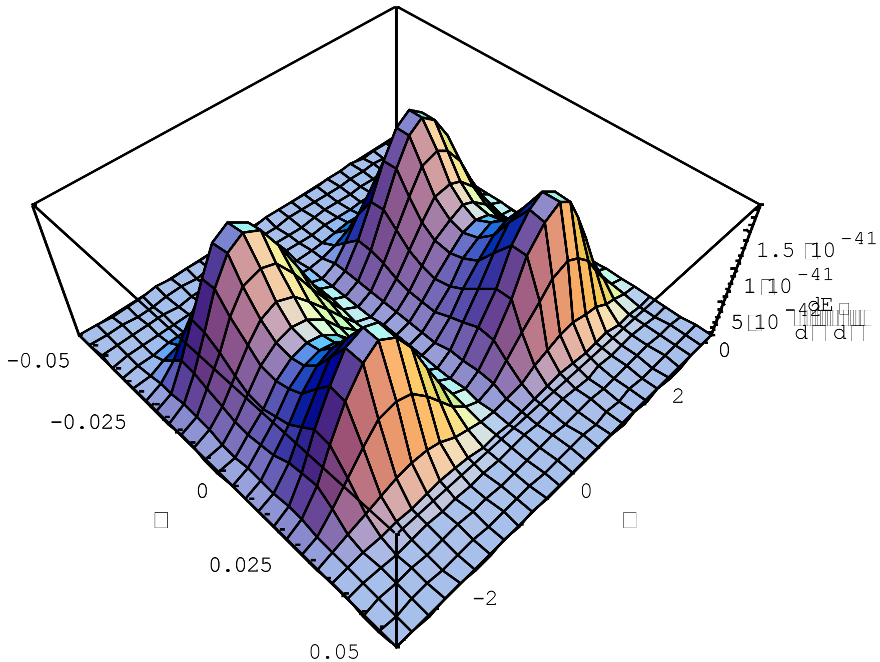
$$A_x(\xi) = A_{peak} \exp\left(-z^2 / 2(8.156\lambda_0)^2\right)\cos(2\pi\xi/\lambda_0) \qquad a_{peak} = eA_{peak}/mc$$

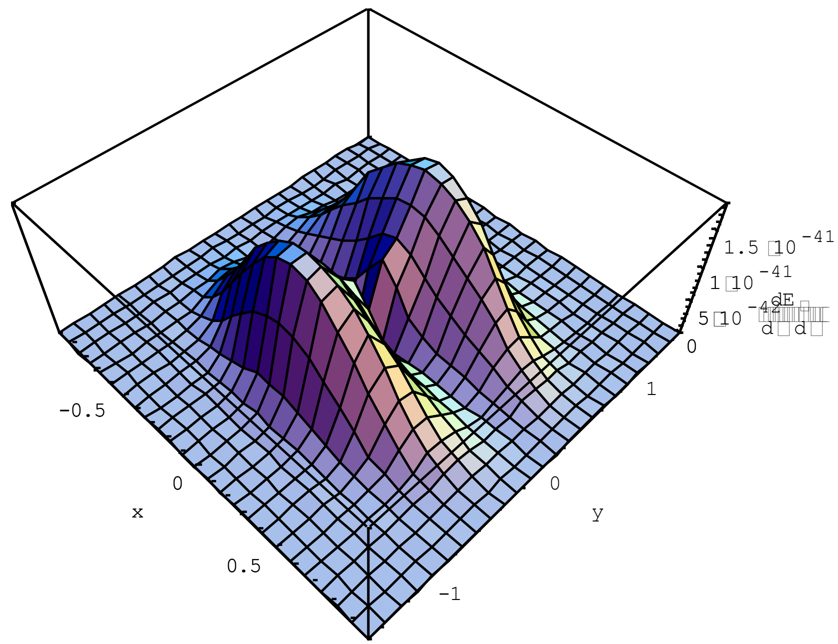
A_{peak} and λ_0 chosen for same intensity and same *rms* pulse length as previously



Radiation Distributions: Backscatter

Gaussian Pulse σ at first harmonic peak

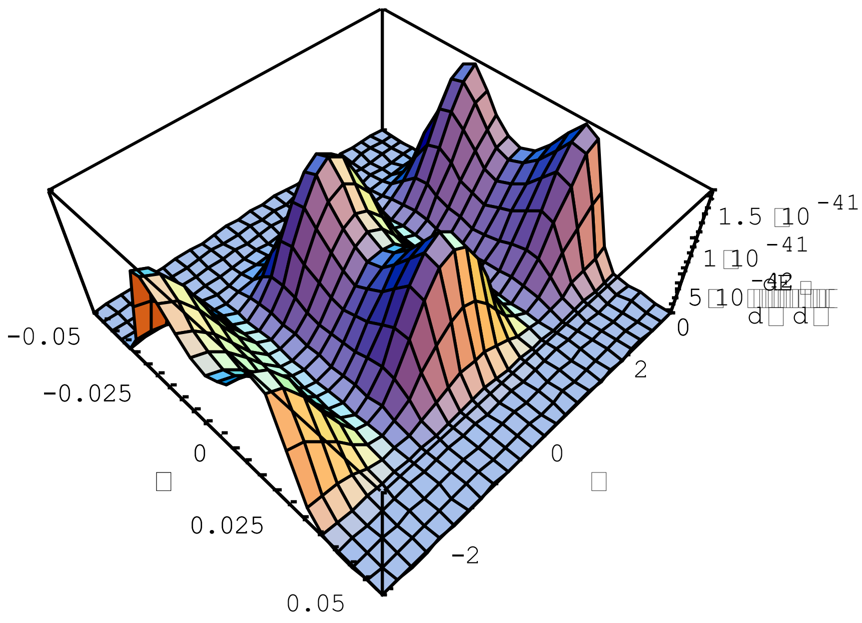


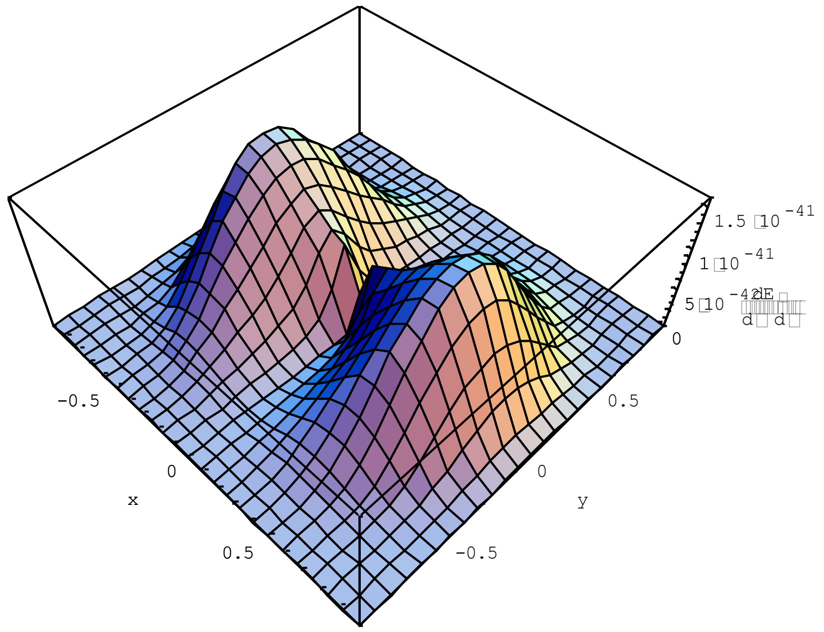


Courtesy: Adnan Doyuran (UCLA)

Radiation Distributions: Backscatter

Gaussian π at first harmonic peak

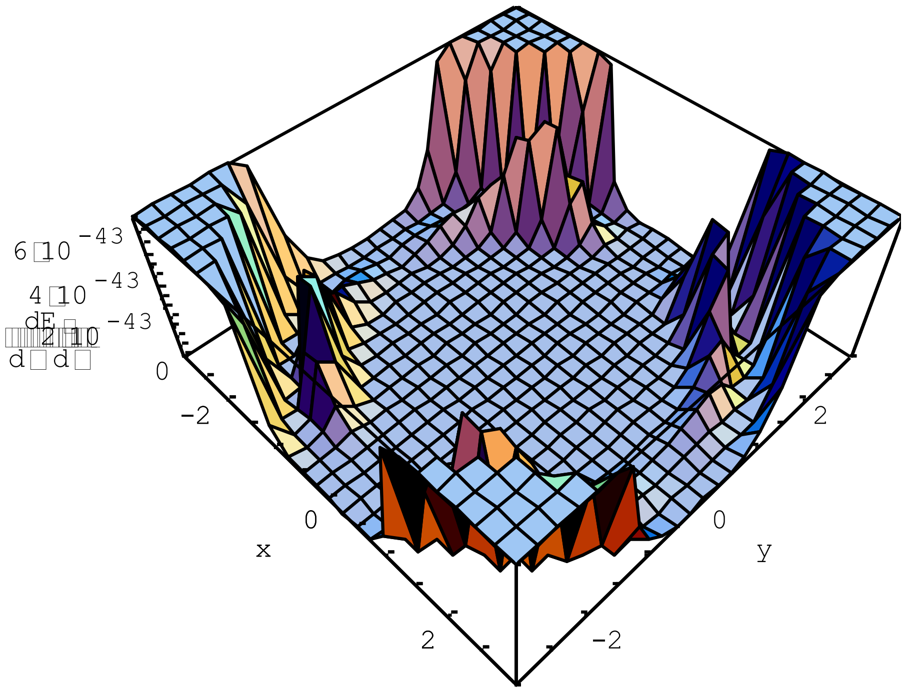




Courtesy: Adnan Doyuran (UCLA)

Radiation Distributions: Backscatter

Gaussian σ at second harmonic peak



Courtesy: Adnan Doyuran (UCLA)

90 Degree Scattering

$$\frac{dE_\sigma}{d\omega d\Omega} = \frac{e^2\omega^2}{32\pi^3 c^3}\left| D_t(\omega;\theta,\varphi)\sin\varphi - \frac{1}{\gamma}D_p(\omega;\theta,\varphi)\cos\varphi \right|^2$$

$$\frac{dE_\pi}{d\omega d\Omega} = \frac{e^2\omega^2}{32\pi^3 c^3}\left| \begin{array}{c} D_t(\omega;\theta,\varphi)\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}\cos\varphi \\ +\dfrac{1}{\gamma}D_p(\omega;\theta,\varphi)\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}\sin\varphi \\ +D_p(\omega;\theta,\varphi)\dfrac{\beta\sin\theta}{\gamma(1-\beta\cos\theta)} \end{array} \right|^2$$

90 Degree Scattering

$$D_t(\omega;\theta,\varphi)=\frac{1}{\gamma}\int\frac{eA(\xi)}{mc}e^{i\phi(\omega,\xi;\theta,\varphi)}d\xi$$

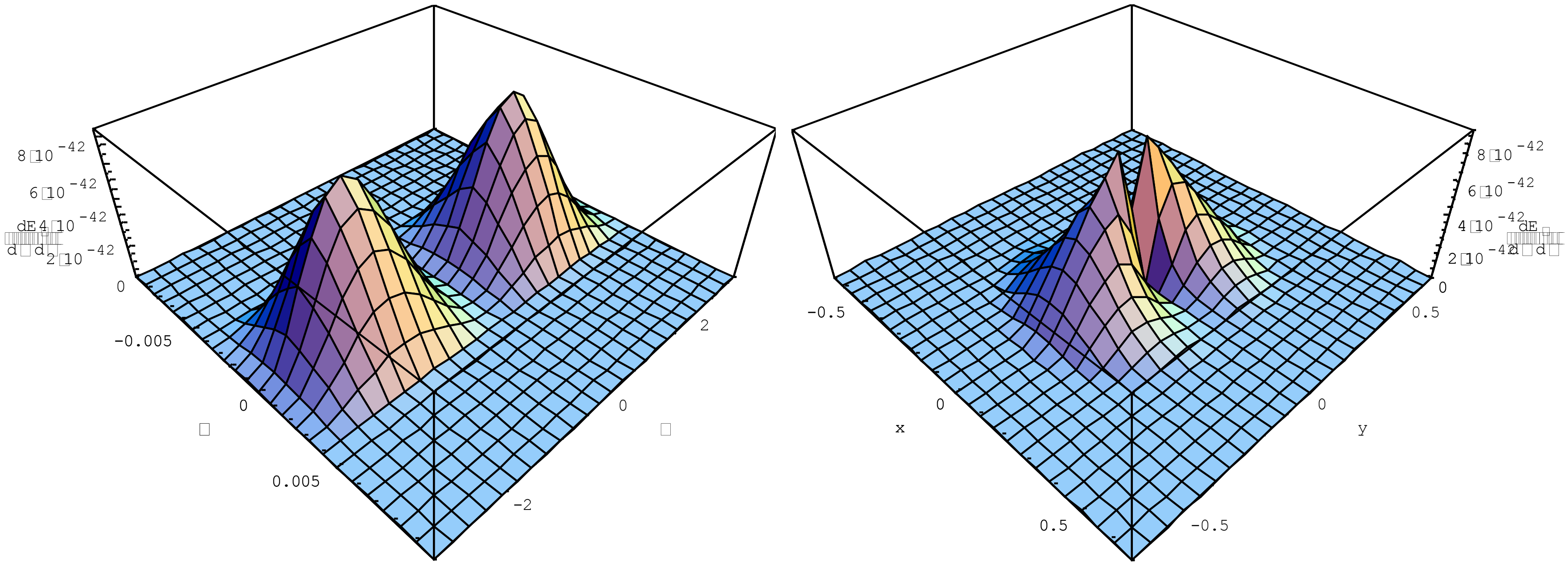
$$D_p(\omega;\theta,\varphi)=\frac{1}{\gamma}\int\frac{e^2A^2(\xi)}{2m^2c^2}e^{i\phi(\omega,\xi;\theta,\varphi)}d\xi$$

And the phase is

$$\phi(\omega,\xi;\theta,\varphi)=\frac{\omega}{c}\left(\begin{array}{c}\xi(1-\beta\cos\theta)-\dfrac{\sin\theta\cos\varphi}{\gamma}\displaystyle\int_{-\infty}^{\xi}\frac{eA(\xi')}{mc}d\xi' \\ +\dfrac{1-\sin\theta\sin\varphi}{\gamma^2}\displaystyle\int_{-\infty}^{\xi}\frac{e^2A^2(\xi')}{2m^2c^2}d\xi'\end{array}\right)$$

Radiation Distribution: 90 Degree

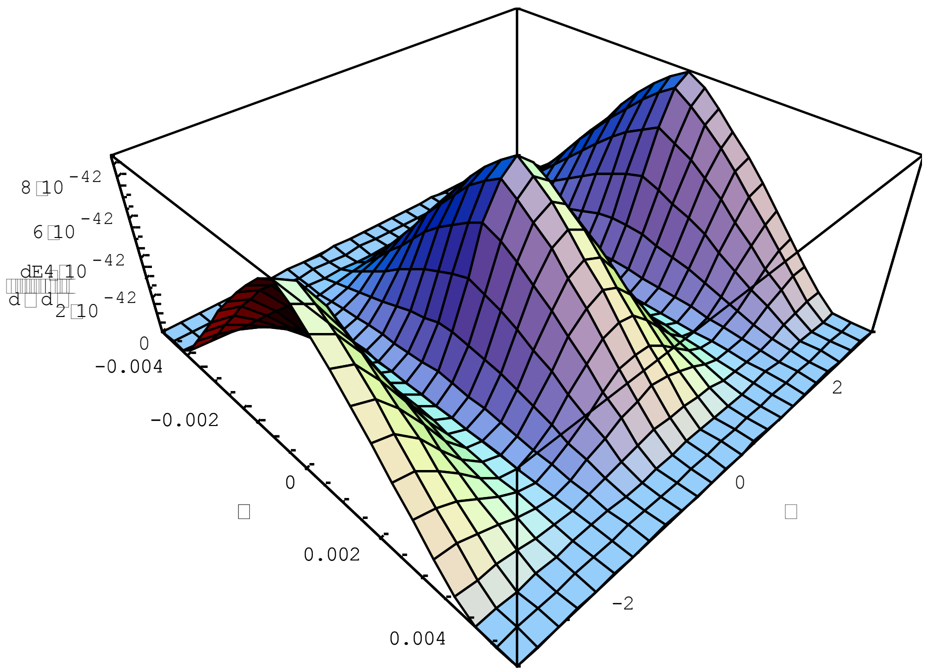
Gaussian Pulse σ at first harmonic peak

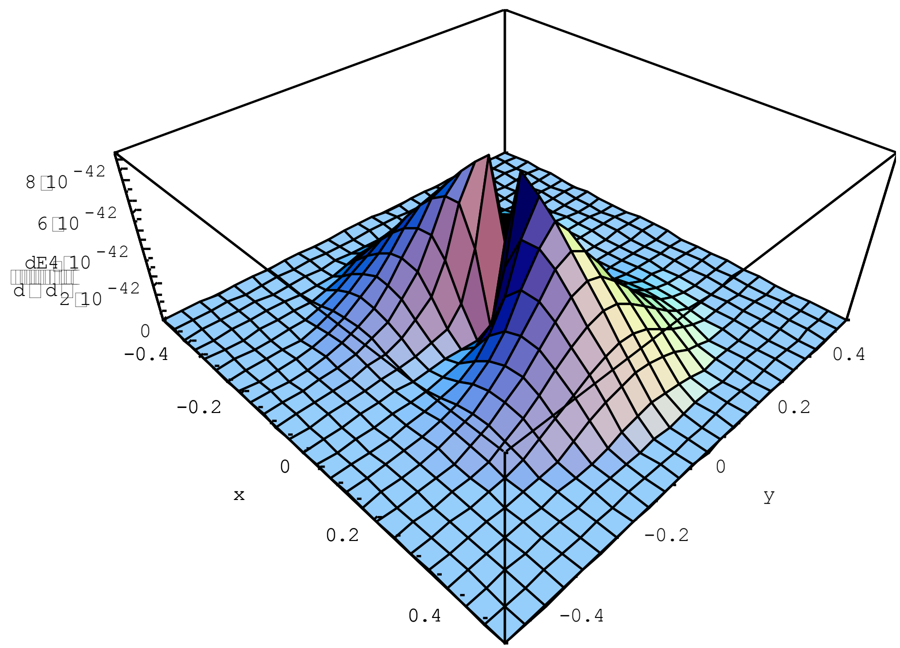


Courtesy: Adnan Doyuran (UCLA)

Radiation Distributions: 90 Degree

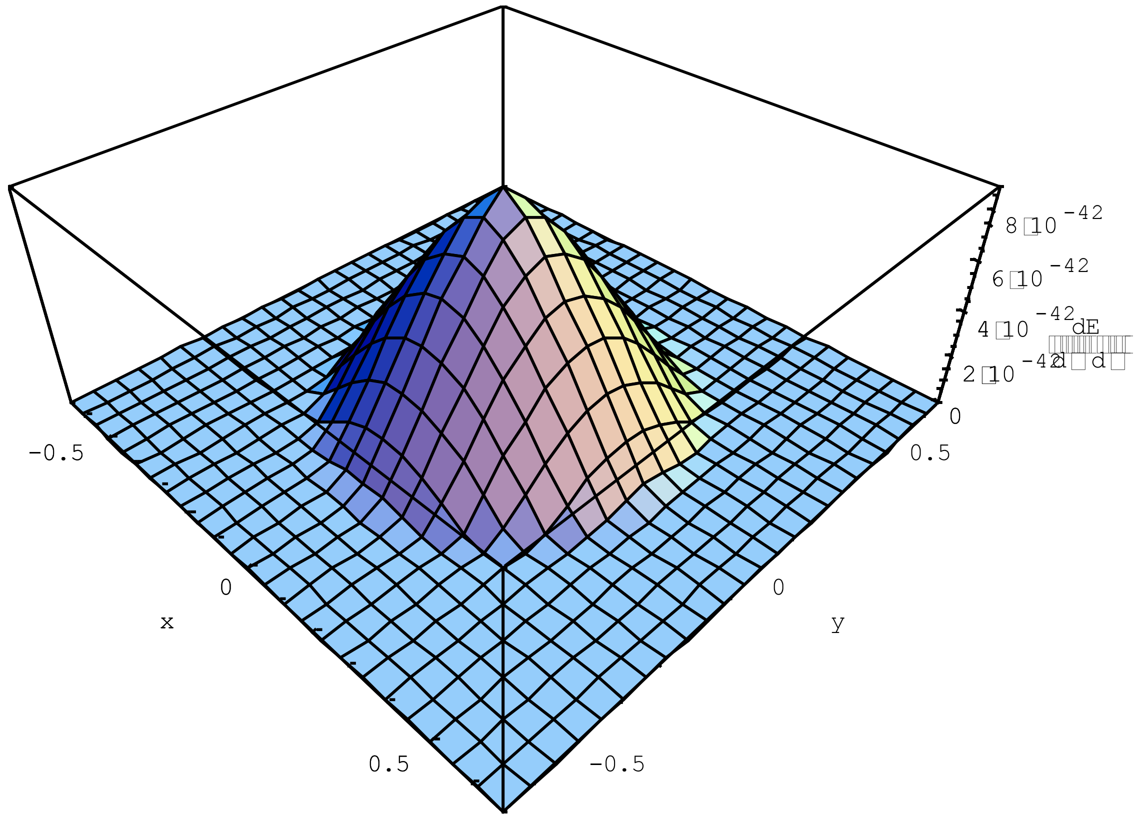
Gaussian Pulse π at first harmonic peak





Courtesy: Adnan Doyuran (UCLA)

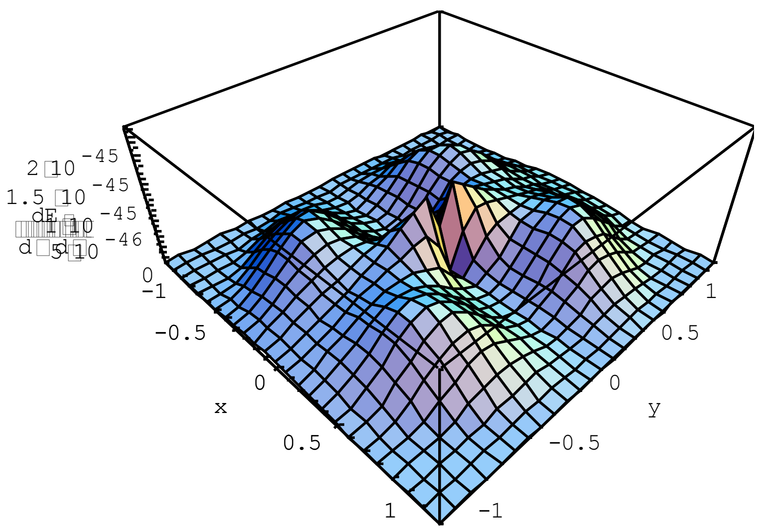
Polarization Sum: Gaussian 90 Degree



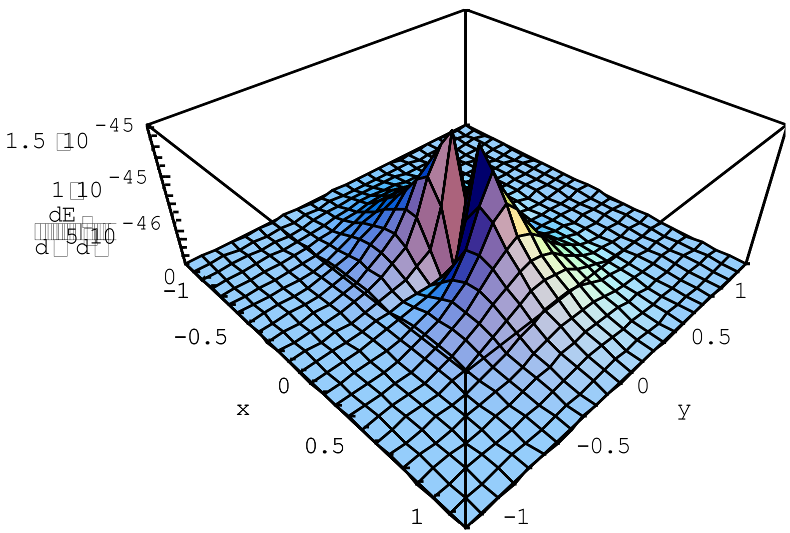
Courtesy: Adnan Doyuran (UCLA)

Radiation Distributions: 90 Degree

Gaussian Pulse second harmonic peak



σ



π

Second harmonic emission on axis from ponderomotive dipole!

Courtesy: Adnan Doyuran (UCLA)

Compensating the Red Shift

- Add possibility of frequency modulation (Ghebregziabher, et al.)

$$A_x(\xi) = a(\xi)\cos\left(2\pi\xi f(\xi)/\lambda\right)$$

- Wave phase

$$\Phi = 2\pi\xi f(\xi)/\lambda$$

- Wave (angular) frequency and wave number

$$\omega(\xi) = \frac{\partial\Phi}{\partial t} \qquad k(\xi) = \frac{\partial\Phi}{\partial z}$$

Double Doppler to Constant Frequency

- In beam frame

$$\omega' = \left(1+\beta^*\right)\gamma^*\omega \qquad k' = \left(1+\beta^*\right)\gamma^* k$$

$$\beta^{*2} = 1 - \frac{1}{\gamma^2}\left(1+\frac{a^2(\xi)}{2}\right) \qquad \gamma^{*2} = \gamma^2 \Big/ \left(1+\frac{a^2(\xi)}{2}\right)$$

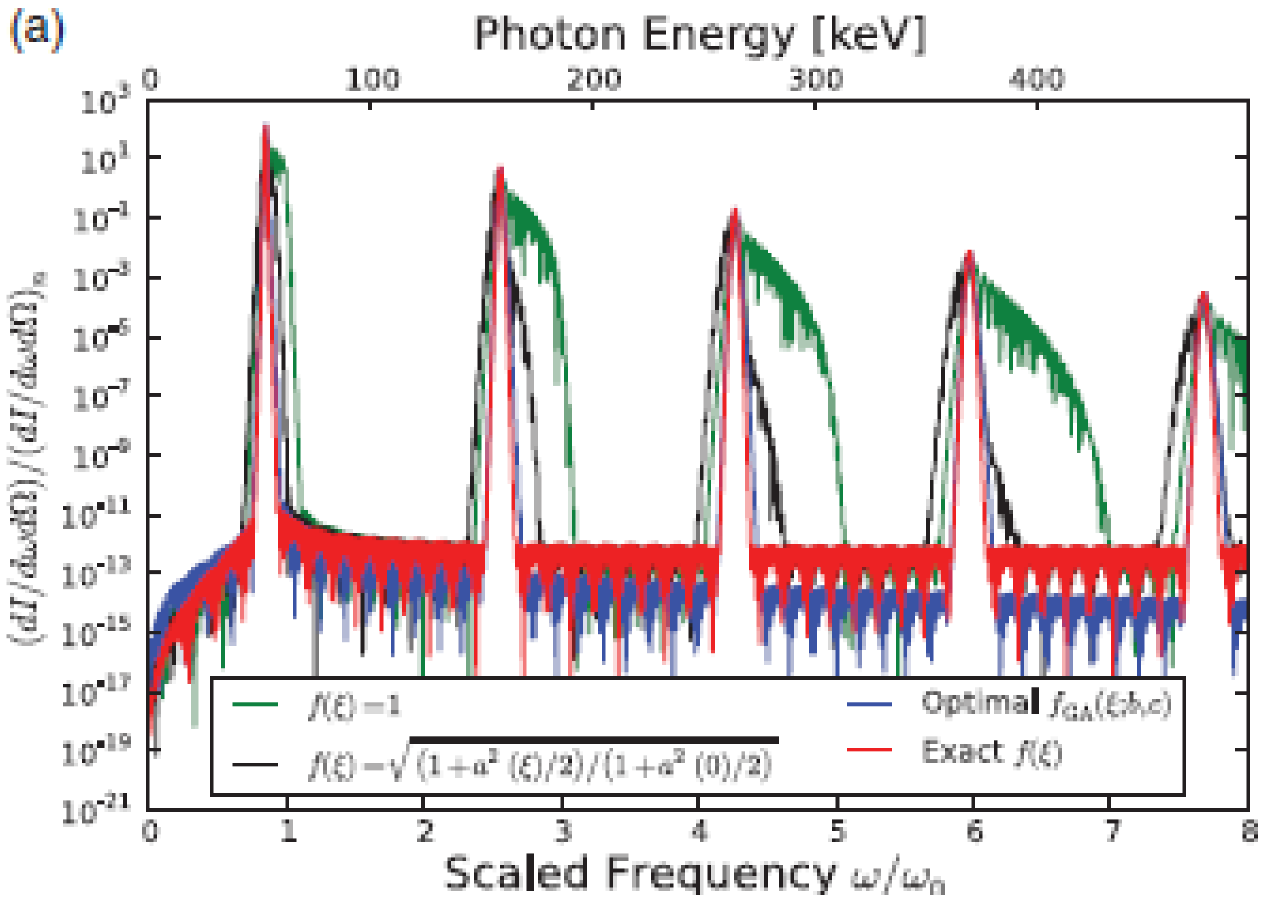
- Doppler shift to lab frame (θ=0)

$$\left(1+\beta^*\right)^2 \gamma^{*2}\frac{d\Phi}{d\xi} = C = \left(1+\beta\right)^2\gamma^2\frac{d\Phi}{d\xi}\left(\xi=-\infty\right)$$

- Compensation occurs when

$$\xi f(\xi) = \frac{1}{1+a_0^2/2}\left(\xi + \int_0^{\xi}\left[a^2(\xi')/2\right]d\xi'\right)$$

Compensation By "Chirping"



Terzic, Deitrick, Hofler, and Krafft,
Phys. Rev. Lett., **112**, 074801 (2014)

$$f(\xi) = \frac{1}{1+a^2(0)/2}\left(1 + \frac{\int_0^\xi a^2(\xi')d\xi'}{2\xi}\right)$$

Hajima, et al.

Uranium Detection

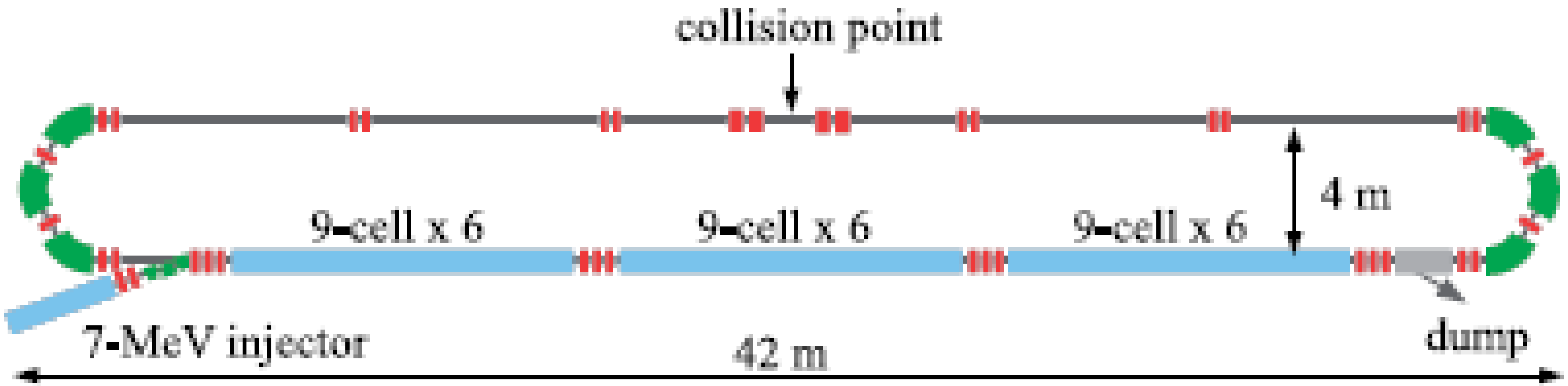
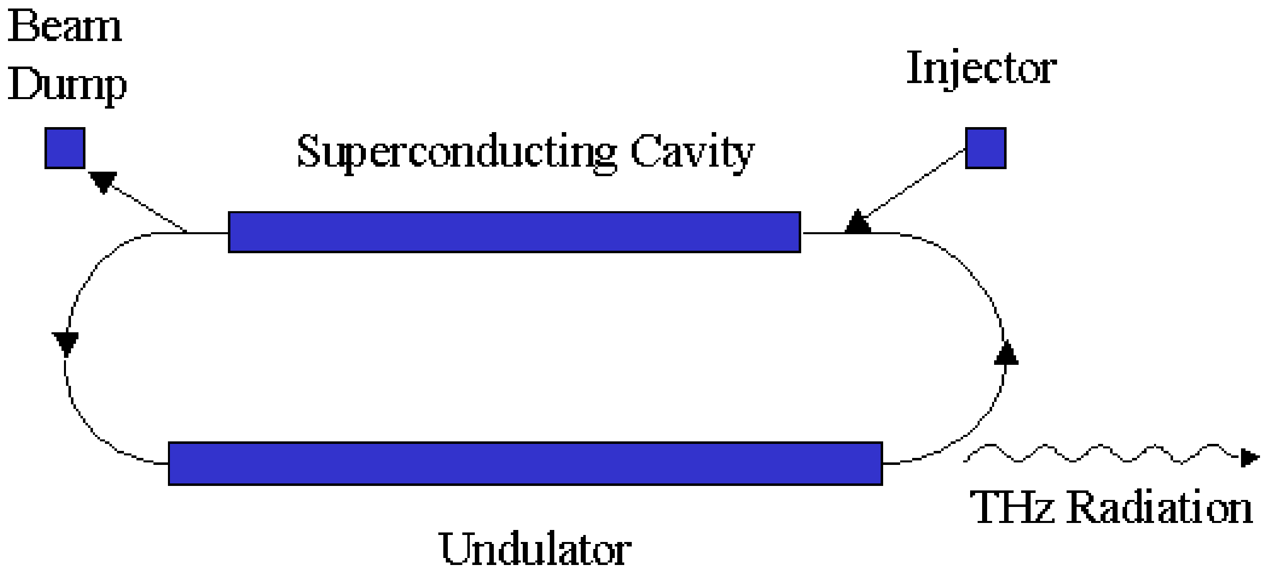


Fig. 3. Layout of the 350-MeV ERL designed for a high-flux γ-ray source. An electron beam generated by the 7-MeV injector is accelerated up to 350 MeV by the main linac and transported to the recirculation loop. The collision point for LCS γ-ray generation is located in the middle of the straight section.

Hajima, *et al.*, *NIM A*, **608**, S57 (2009)

TRIUMF Moly Source?

THZ Source



Beam
Dump
Injector
Superconducting Cavity
THz Radiation
Undulator

Radiation Distributions for Short High-Field Magnets

$$D_t(\omega;\theta,\phi)=\int_{-\infty}^{\infty}\frac{a(z)/\gamma_0}{\sqrt{\beta_{z0}^2-a^2(z)/\gamma_0^2}}\exp i\varphi(\omega,z;\theta,\phi)\,dz$$

$$D_p(\omega;\theta,\phi)=\int_{-\infty}^{\infty}\left[\frac{1}{\beta_z(z)}-\frac{1}{\beta_{z0}}\right]\exp i\varphi(\omega,z;\theta,\phi)\,dz$$

And the phase is

$$\varphi(\omega,z;\theta,\phi)=\frac{\omega(1-\beta_{z0}\cos\theta)z}{\beta_{z0}c}+\frac{\omega}{c}\int_{-\infty}^{z}\left[\frac{1}{\beta_z(z')}-\frac{1}{\beta_{z0}}\right]dz'$$

$$-\frac{\omega\sin\theta\cos\phi}{c}\int_{-\infty}^{z}\frac{a(z')/\gamma_0}{\sqrt{\beta_{z0}^2-a^2(z')/\gamma_0^2}}\,dz'.$$

Krafft, G. A., *Phys. Rev. ST-AB,* **9**, 010701 (2006)

Wideband THz Undulater

Primary requirements: wide bandwidth and no motion and deflection. Implies generate *A* and *B* by simple motion. "One half" an oscillation is highest bandwidth!

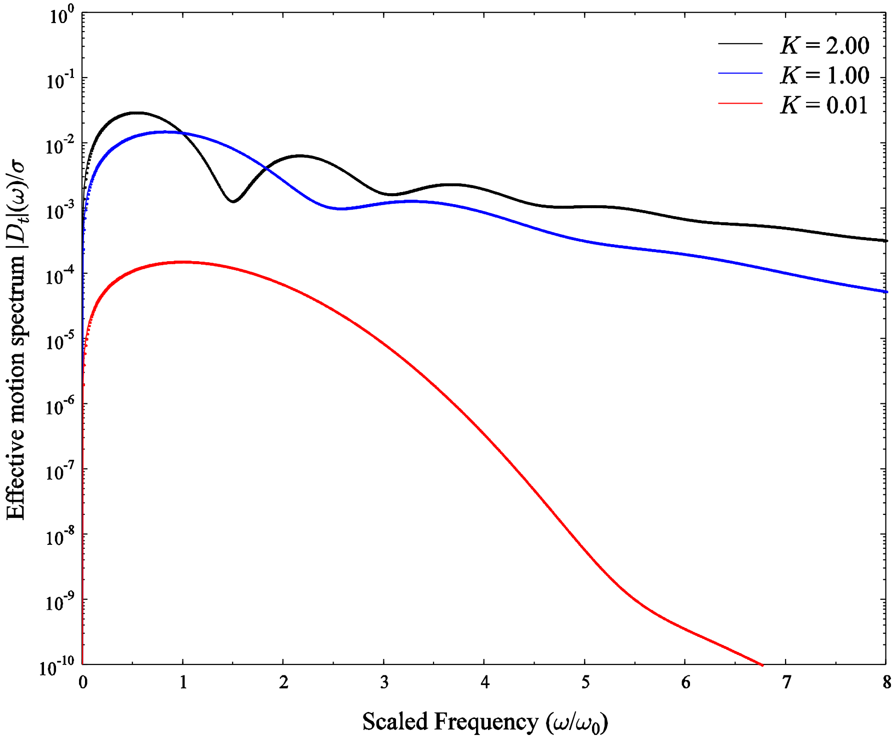
$$x(z) = -\sigma \exp\left(-z^2/2\sigma^2\right)$$

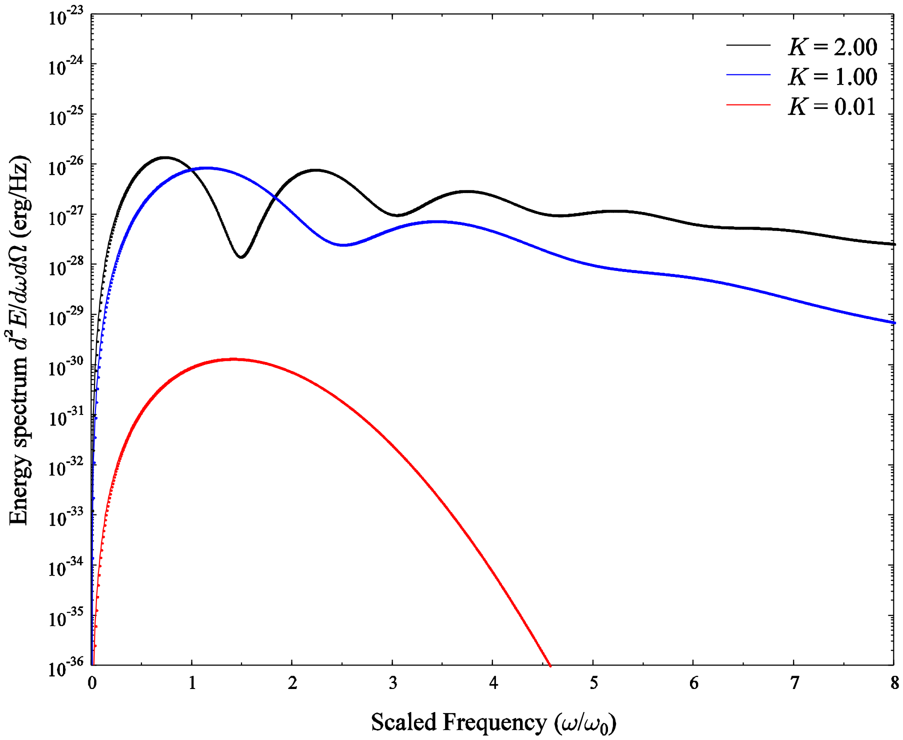
$$a(z) = \frac{eA(z)}{mc} = \left(\frac{z}{\sigma}\right)\exp\left(-z^2/2\sigma^2\right)$$

$$B(z) = \frac{dA}{dz} = B_{peak}\left(1-\left(\frac{z}{\sigma}\right)^2\right)\exp\left(-z^2/2\sigma^2\right)$$

$$K = \frac{eB_{peak}\sigma}{mc}$$

THz Undulator Radiation Spectrum





Total Energy Radiated

Lienard's Generalization of Larmor Formula (1898!)

$$\frac{dE}{dt} = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c}\gamma^6\left[\left(\dot{\vec{\beta}}\right)^2 + \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right] = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c}\gamma^4\left[\left(\dot{\vec{\beta}}\right)^2 + \gamma^2\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2\right]$$

Barut's Version

$$\frac{dE}{d\tau} = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c^3}\frac{dt}{d\tau}\frac{d^2x^\mu}{d\tau^2}\frac{d^2x_\mu}{d\tau^2}$$

$$E = \frac{e^2}{6\pi\varepsilon_0}\int_{-\infty}^{\infty}\left[\gamma^2(1-\beta\cos\Phi)\left(\frac{df}{d\xi}\right)^2 + \frac{f^2}{2}\left(\frac{df}{d\xi}\right)^2\right]d\xi$$

Usual Larmor term

From ponderomotive dipole

Some Cases

Total radiation from electron initially at rest

$$E = \frac{e^2}{6\pi\varepsilon_0} \int_{-\infty}^{\infty} \left[\left(\frac{df}{d\xi} \right)^2 + \frac{f^2}{2} \left(\frac{df}{d\xi} \right)^2 \right] d\xi$$

For a flat pulse exactly (Sarachik and Schappert)

$$\frac{dE}{dt} = \frac{1}{12\pi\varepsilon_0} \frac{e^2\omega^2}{c} a^2 \left(1 + a^2 / 8 \right)$$

For Circular Polarization

Only other specific case I can find in literature completely calculated has usual circular polarization and flat pulses. The orbits are then pure circles

Sokolov and Ternov, in *Radiation from Relativistic Electrons*, give

$$\frac{dE'}{dt'} = \frac{1}{6\pi\varepsilon_0}\frac{e^2\omega'^2}{c}a^2\left(1+a^2\right)$$

(which goes back to Schott and the turn of the 20th century!) and the general formula checks out

Conclusions

- I’ve shown how dipole solutions to the Maxwell equations can be used to obtain and understand very general expressions for the spectral angular energy distributions for weak field undulators and general weak field Thomson Scattering photon sources
- A “new” calculation scheme for high intensity pulsed laser Thomson Scattering has been developed. This same scheme can be applied to calculate spectral properties of “short”, high-K wigglers.
- Due to ponderomotive broadening, it is simply wrong to use single-frequency estimates of flux and brilliance in situations where the square of the field strength parameter becomes comparable to or exceeds the $(1/N)$ spectral width of the induced electron wiggle
- The new theory is especially useful when considering Thomson scattering of Table Top TeraWatt lasers, which have exceedingly high field and short pulses. Any calculation that does not include ponderomotive broadening is incorrect.

Conclusions

- Because the laser beam in a Thomson scatter source can interact with the electron beam non-colinearly with the beam motion (a piece of physics that cannot happen in an undulator), ponderomotively driven transverse dipole motion is now possible
- This motion can generate radiation at the second harmonic of the up-shifted incident frequency on axis. The dipole direction is in the direction of laser incidence.
- Because of Doppler shifts generated by the ponderomotive displacement velocity induced in the electron by the intense laser, the frequency of the emitted radiation has an angular asymmetry.
- Sum rules for the total energy radiated, which generalize the usual Larmor/Lenard sum rule, have been obtained.

Conventions on Fourier Transforms

For the time dimensions

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega$$

Results on Dirac delta functions

$$\tilde{\delta}(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{i\omega t} dt = 1$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega$$

For the three spatial dimensions

$$\tilde{f}\left(\vec{k}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\vec{x}\right)e^{-i\vec{k}\cdot\vec{x}}d^3\vec{x}$$

$$f\left(\vec{x}\right)=\frac{1}{\left(2\pi\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}\left(\vec{k}\right)e^{+i\vec{k}\cdot\vec{x}}d^3\vec{k}$$

$$\delta^3\left(\vec{x}\right)=\delta\left(\vec{x}\right)=\frac{1}{\left(2\pi\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{+i\vec{k}\cdot\vec{x}}d^3\vec{k}$$

Green Function for Wave Equation

Solution to inhomogeneous wave equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G\left(\vec{x},t;\vec{x}',t'\right)$$

$$=-4\pi\delta\left(\vec{x}-\vec{x}'\right)\delta\left(t-t'\right)$$

Will pick out the solution with causal boundary conditions, i. e.

$$G\left(\vec{x},t;\vec{x}',t'\right)=0 \qquad\qquad t<t'$$

This choice leads automatically to the so-called *Retarded* Green Function

In general

$$G\left(\vec{x},t;\vec{x}',t'\right)=0 \qquad t<t'$$

$$G\left(\vec{x},t;\vec{x}',t'\right)=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[A\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}+B\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}+\omega t\right)}\right]d^3\vec{k} \qquad t>t'$$

because there are two possible signs of the frequency for each value of the wave vector. To solve the homogeneous wave equation it is necessary that

$$\omega\left(\vec{k}\right)=\left|\vec{k}\right|c$$

i.e., there is no dispersion in free space.

Continuity of G implies

$$A\left(\vec{k}\right)e^{-i\omega t'} = -B\left(\vec{k}\right)e^{i\omega t'}$$

Integrate the inhomogeneous equation between $t = t' + \varepsilon$ and $t = t' - \varepsilon$

$$-\frac{1}{c^2}\frac{\partial G\left(\vec{x},t;\vec{x}',t'\right)}{\partial t}\bigg|_{t'+\varepsilon} = -4\pi\delta\left(\vec{x}-\vec{x}'\right)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-i\omega A\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}-\omega t'\right)} + i\omega B\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}+\omega t'\right)}\right] d^3\vec{k}$$

$$= 4\pi c^2\delta\left(\vec{x}-\vec{x}'\right)$$

$$A\left(\vec{k}\right) = -\frac{c^2}{\left(2\pi\right)^2 i\omega}e^{-i\vec{k}\cdot\vec{x}'}e^{i\omega t'}$$

$$G\left(\vec{x},t;\vec{x}',t'\right)=$$

$$-\frac{c^2}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\omega}\left[e^{i\left(\vec{k}\cdot(\vec{x}-\vec{x}')-\omega(t-t')\right)}-e^{i\left(\vec{k}\cdot(\vec{x}-\vec{x}')+\omega(t-t')\right)}\right]\, d^3\vec{k}$$

$$t>t'$$

$$=\frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}e^{-i\omega(t-t')}dk-\frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}e^{+i\omega(t-t')}dk \quad t>t'$$

$$=\frac{\delta\left(|\vec{x}-\vec{x}'|/c-t+t'\right)}{|\vec{x}-\vec{x}'|}+0$$

Called retarded because the influence at time *t* is due to the source evaluated at the retarded time

$$t'=t-|\vec{x}-\vec{x}'|/c$$