Accelerator Physics
NMI and Synchrotron Radiation

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Lecture 16

Oscillation Frequency

$$\Delta\Omega^2 = \left(\Omega - n\omega_0\right)^2 = -i\,\frac{nq\eta_c\omega_0^2 I_0}{2\pi\beta^2 E_0} Z_{||}$$

$\mathrm{Re}\, Z_{||} \neq 0 \rightarrow 1$ mode has positive imaginary part

$\rightarrow$ instability

Resistive impedance has positive real part

"Resistive wall instability"

If $\mathrm{Re}\, Z_{||} = 0$ (e.g. space charge impedance at long wavelengths)

stability/instability depends on sign of RHS

$\mathrm{Im}\, Z_{||} < 0$ (inductive, stable if $\eta_c < 0$, unstable if $\eta_c > 0$)

$\mathrm{Im}\, Z_{||} > 0$ (capacitive, space charge is this way,

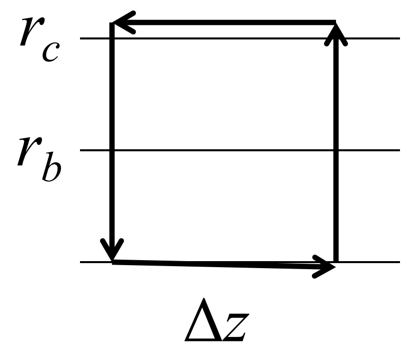
stable if $\eta_c > 0$, unstable if $\eta_c < 0$)

Later case is negative mass instability

NMI Growth time

Impedance?

$$E_r = \begin{cases} \dfrac{e\lambda r}{2\pi\varepsilon_0 r_b^2} & r < r_b \\ \dfrac{e\lambda}{2\pi\varepsilon_0 r} & r > r_b \end{cases} \qquad B_\theta = \begin{cases} \beta c \dfrac{\mu_0 e\lambda r}{2\pi r_b^2} & r < r_b \\ \beta c \dfrac{\mu_0 e\lambda}{2\pi r} & r > r_b \end{cases}$$



$$E_z = -\frac{e}{4\pi\varepsilon_0}\left(1-\beta^2\right)\frac{\partial\lambda}{\partial z}\left(1+2\ln\left(r_c/r_b\right)\right)$$

$$\lambda \propto \lambda_n e^{i(n\theta-\Omega t)}$$

$$V_{SC} = \frac{-in}{2\varepsilon_0\gamma^2}\lambda_n\left(1+2\ln\left(r_c/r_b\right)\right) = \frac{-in}{2\varepsilon_0\gamma^2\beta c} I_n\left(1+2\ln\left(r_c/r_b\right)\right)$$

$$\left(\Omega-n\omega_0\right)^2 = -i\frac{nq\eta_c\omega_0^2 I_0}{2\pi\beta^2 E_0} Z_{SC} = -\frac{n^2\omega_0^2}{\beta^2\gamma^2}\left(\frac{q\eta_c I_0}{4\pi\varepsilon_0 c\beta E_0}\left(1+2\ln\left(r_c/r_b\right)\right)\right)$$

Stabilization by Beam Temperature?

Canonical variables $\theta, \delta \equiv \Delta p / p_0$

$$\left[\frac{\partial}{\partial t} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\delta}\frac{\partial}{\partial \delta}\right]\psi = 0$$

$$\psi = \psi_0 + \psi_n e^{i(n\theta - \omega_n t)}$$

$$i(\omega_n - n\omega)\psi_n = \frac{\dot{\delta}}{e^{i(n\theta - \omega_n t)}}\frac{\partial \psi_0}{\partial \delta}$$

$$\frac{\partial \psi_0}{\partial \delta} = \frac{\partial \psi_0}{\partial \omega}\frac{\partial \omega}{\partial \delta} = \eta_c \omega_0 \frac{\partial \psi_0}{\partial \omega}$$

current perturbation is

$$I_n = q\omega_0 \int_{-\infty}^{\infty} \psi_n d\delta$$

Dispersion Relation

$$\psi_0(\delta) = \eta_c \omega_0 \Phi_0(\omega)$$

$$\dot{\delta} = \frac{1}{\eta_c \omega_0} \dot{\omega} = \left(\frac{dE}{dt}\right) / \left(\beta^2 E_0\right)$$

$$1 = i \frac{q^2 \omega_0^3 \eta_c Z_{||}}{2\pi\beta^2 E_0} \int \frac{\partial \Phi_0 / \partial \omega}{\omega_n - n\omega} d\omega$$

recover before

$$\Phi_0 = N_b \delta(\omega - \omega_0) / 2\pi$$

$$\int \frac{\partial \Phi_0 / \partial \omega}{\omega_n - n\omega} d\omega = -\frac{N_b n}{2\pi\left(\omega_n - n\omega_0\right)^2}$$

Landau Damping

Use our favorite analytic distribution

$$\psi_0(\delta) \propto \frac{1}{\pi} \frac{\delta_0}{\delta_0^2 + \delta^2} \qquad \Phi_0(\omega) \propto \frac{1}{\pi} \frac{\hat{\omega}}{\hat{\omega}^2 + (\omega - \omega_0)^2}$$

$$\hat{\omega} = \delta_0 \eta_c \omega_0$$

$$1 = -i \frac{q\omega_0^2 \eta_c Z_{\|} I_0 n}{2\pi\beta^2 E_0} \int \frac{\hat{\omega}}{(\omega_n - n\omega)^2 \pi\left(\hat{\omega}^2 + (\omega - \omega_0)^2\right)} d\omega$$

$$1 = -i \frac{q\eta_c Z_{\|} I_0 n}{2\pi\beta^2 E_0} \frac{\omega_0^2}{(\omega_n - n\omega_0 + ni\hat{\omega})^2}$$

$$\omega_n = n\omega_0 - ni\hat{\omega} + \sqrt{V + iU}$$

$$\ddot{u}+\Omega_2^2 u = Fe^{i\omega t}$$

$$u = F\frac{e^{i\omega t}}{2\omega}\left(\frac{1}{\Omega-\omega}-\frac{1}{\Omega+\omega}\right)$$

$$\psi(\omega)=\frac{1}{N_b}\frac{dN_b}{d\Omega}$$

$$\ddot{u} = F\frac{e^{i\omega t}}{2\omega}\int_{-\infty}^{\infty}\left[\frac{\psi(\Omega)}{\Omega-\omega}-\frac{\psi(\Omega)}{\Omega+\omega}\right]d\Omega$$

$$\ddot{u} = F\frac{e^{i\omega t}}{\omega}\int_{-\infty}^{\infty}\frac{\psi(\Omega)}{\Omega-\omega}d\Omega$$

LD from another view

Single Oscillator

$$\ddot{u} + \Omega^2 u = F e^{-i\omega t}$$

$$u(t) = \frac{F e^{-i\omega t}}{2\omega}\left(\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right)$$

Many oscillators distributed in frequency

$$\psi(\Omega) = \frac{1}{N}\frac{dN}{d\Omega}$$

$$U = \frac{\sum_{i=1}^{N} u_i}{N}$$

$$U = \frac{F e^{-i\omega t}}{2\omega}\int\left[\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right] d\Omega$$

for $\psi(\Omega) = \psi(-\Omega)$

$$U = \frac{F e^{-i\omega t}}{\omega}\int \frac{\psi(\Omega)}{\Omega - \omega} d\Omega$$

Resonance Effect

$$U = \frac{Fe^{-i\omega t}}{\omega}\left[+i\pi\psi(\omega) + P.V.\int \frac{\psi(\Omega)}{\Omega-\omega} d\Omega \right]$$

$$\dot{U} = Fe^{-i\omega t}\left[\pi\psi(\omega) - iP.V.\int \frac{\psi(\Omega)}{\Omega-\omega} d\Omega \right]$$

For our analytic Lorentzian

$$\psi(\Omega) = \frac{\Delta}{\pi\left(\Delta^2 + \Omega^2\right)}$$

$$\dot{U} = \frac{Fe^{-i\omega t}}{-i\Delta - \omega} = \frac{Fe^{-i\omega t}}{\Delta^2 + \omega^2}(\Delta - i\omega)$$

Energy goes in!

Where does it go?

Inhomogeneous Solution

$$u(t) = a\sin\Omega t + \frac{F}{\Omega^2 - \omega^2}\sin\omega t$$

Solution with zero initial excitation

$$a = -\frac{\omega}{\Omega}\frac{F}{\Omega^2 - \omega^2}$$

$$\therefore u_{\Omega\neq\omega} = \frac{F}{\Omega^2 - \omega^2}\left(\sin\omega t - \frac{\omega}{\Omega}\sin\Omega t\right)$$

No energy flow

$$\therefore u_{\Omega=\omega} = \frac{F}{\Omega^2 - \omega^2}\left(t\cos\omega t - \frac{\sin\omega t}{\omega}\right)$$

Resonant particles capture energy and oscillation generated out of phase

Oscillators Similtaneously Excited

$$u_i(t) = 1$$

$$\ddot{u} + \Omega^2 u = F e^{-i\omega t}$$

$$u(t) = \frac{F e^{-i\omega t}}{2\omega}\left(\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right)$$

Many oscillators distributed in frequency

$$\psi(\Omega) = \frac{1}{N}\frac{dN}{d\Omega}$$

$$U = \frac{\sum_{i=1}^{N} u_i}{N}$$

$$U = \frac{F e^{-i\omega t}}{2\omega}\int\left[\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right] d\Omega$$

for $\psi(\Omega) = \psi(-\Omega)$

$$U = \frac{F e^{-i\omega t}}{\omega}\int \frac{\psi(\Omega)}{\Omega - \omega} d\Omega$$

Synchrotron Radiation

Accelerated particles emit electromagnetic radiation. Emission from very high energy particles has unique properties for a radiation source. As such radiation was first observed at one of the earliest electron synchrotrons, radiation from high energy particles (mainly electrons) is known generically as synchrotron radiation by the accelerator and HENP communities.

The radiation is highly collimated in the beam direction

From relativity

$$ct' = \gamma ct - \gamma\beta z$$

$$x' = x$$

$$y' = y$$

$$z' = -\gamma\beta ct + \gamma z$$

Lorentz invariance of wave phase implies $k^\mu = (\omega/c, k_x, k_y, k_z)$ is a Lorentz 4-vector

$$\omega' = \gamma\omega - \gamma\beta k_z c$$
$$k'_x = k_x$$
$$k'_y = k_y$$
$$k'_z = -\gamma\beta\omega / c + \gamma k_z$$

$$\sin\theta = \frac{\sqrt{k_x^2 + k_y^2}}{\omega / c} \qquad \sin\theta' = \frac{\sqrt{k_x'^2 + k_y'^2}}{\omega' / c} \qquad \cos\theta' = \frac{k'_z}{\omega' / c}$$

$$\omega / c = \gamma\omega' / c + \gamma\beta k'_z = \gamma\left(1 + \beta\cos\theta'\right)\left(\omega' / c\right)$$
$$\omega' / c = \gamma\omega / c - \gamma\beta k_z = \gamma\left(1 - \beta\cos\theta\right)\left(\omega / c\right)$$

$$\theta \approx \sin\theta = \frac{\sin\theta'}{\gamma\left(1+\beta\cos\theta'\right)}$$

Therefore all radiation with $\theta' < \pi / 2$, which is roughly ½ of the photon emission for dipole emission from a transverse acceleration in the beam frame, is Lorentz transformed into an angle less than $1/\gamma$. Because of the strong Doppler shift of the photon energy, higher for $\theta \rightarrow 0$, most of the energy in the photons is within a cone of angular extent $1/\gamma$ around the beam direction.

Larmor's Formula

For a particle executing non-relativistic motion, the total power emitted in electromagnetic radiation is (Larmor, verified later)

$$P(t)=\frac{1}{6\pi\varepsilon_0}\frac{q^2}{c^3}\left|\vec{a}\right|^2=\frac{1}{6\pi\varepsilon_0}\frac{e^2}{m^2c^3}\left|\dot{\vec{p}}\right|^2$$

Lienard's relativistic generalization: Note both *dE* and *dt* are the fourth component of relativistic 4-vectors when one is dealing with photon emission. Therefore, their ratio must be an Lorentz invariant. The invariant that reduces to Larmor's formula in the non-relativistic limit is

$$P=-\frac{e^2}{6\pi\varepsilon_0 c}\frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau}$$

$$P(t)=\frac{e^2}{6\pi\varepsilon_0 c}\gamma^6\left(\dot{\vec{\beta}}^2-\left[\vec{\beta}\times\dot{\vec{\beta}}\right]^2\right)$$

For acceleration along a line, second term is zero and first term for the radiation reaction is small compared to the acceleration as long as gradient less than 10^{14} MV/m. Technically impossible.

For transverse bend acceleration $\dot{\vec{\beta}}=-\frac{\beta^2 c}{\rho}\hat{r}$

$$P(t)=\frac{e^2 c}{6\pi\varepsilon_0\rho^2}\beta^4\gamma^4$$

Fractional Energy Loss

$$\delta E = \frac{e^2}{6\pi\varepsilon_0 \rho} \Theta \beta^3 \gamma^4$$

For one turn with isomagnetic bending fields

$$\frac{\delta E}{E_{beam}} = \frac{4\pi r_e}{3\rho} \beta^3 \gamma^3$$

r_e is the classical electron radius: 2.82×10^{-13} cm

Radiation Power Distribution

Consulting your favorite Classical E&M text (Jackson, Schwinger, Landau and Lifshitz Classical Theory of Fields)

$$\frac{dP}{d\omega} = \frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)dx$$

Critical Frequency

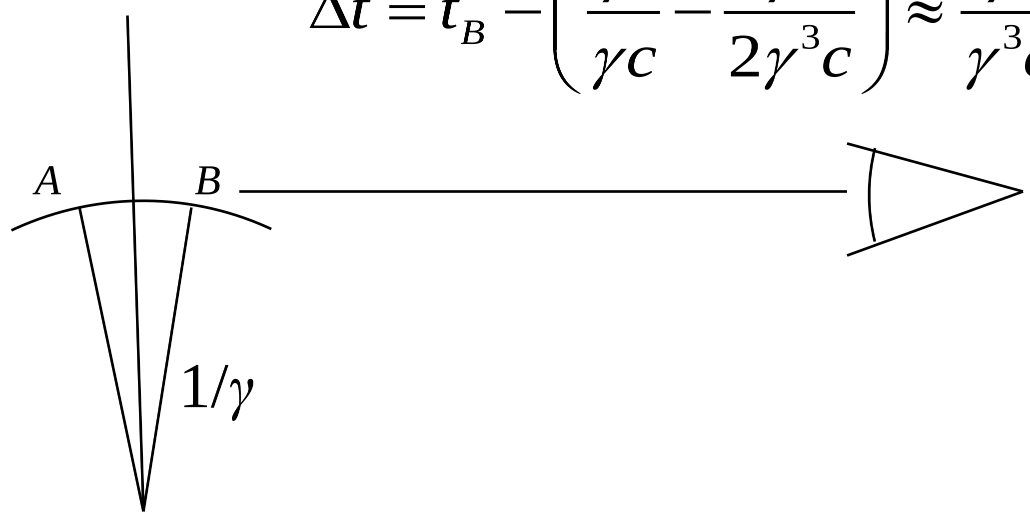
Critical (angular) frequency is

$$\omega_c = \frac{3}{2}\gamma^3 \frac{c}{\rho}$$

Energy scaling of critical frequency is understood from $1/\gamma$ emission cone and fact that $1 - \beta \sim 1/(2\gamma^2)$

$$t_A = -\frac{\rho}{\gamma\beta c}$$

$$t_B = \frac{\rho}{\gamma\beta c} \approx \frac{\rho}{\gamma c} + \frac{\rho}{2\gamma^3 c}$$



$$\Delta t = t_B - \left(\frac{\rho}{\gamma c} - \frac{\rho}{2\gamma^3 c} \right) \approx \frac{\rho}{\gamma^3 c}$$

Photon Number

$$P=\int_0^\infty \frac{dP}{d\omega}d\omega=\frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\omega_c\gamma\int_0^\infty \xi\int_\xi^\infty K_{5/3}(x)\,dx\,d\xi=\frac{e^2c}{6\pi\varepsilon_0\rho^2}\gamma^4$$

$$\frac{d\dot{n}}{d\omega}=\frac{1}{\hbar\omega}\frac{dP}{d\omega}$$

$$\langle\hbar\omega\rangle=\frac{\int_0^\infty \hbar\omega\frac{d\dot{n}}{d\omega}d\omega}{\int_0^\infty \frac{d\dot{n}}{d\omega}d\omega}=\frac{8}{15\sqrt{3}}\hbar\omega_c$$

$$\dot{n}=\frac{5\alpha}{2\sqrt{3}}\frac{c}{\rho}\gamma \qquad \delta n=\frac{5\alpha}{2\sqrt{3}}\Theta\gamma \qquad \alpha=\frac{e^2}{4\pi\varepsilon_0\hbar c}\approx\frac{1}{137}$$

Insertion Devices (ID)

Often periodic magnetic field magnets are placed in beam path of high energy storage rings. The radiation generated by electrons passing through such insertion devices has unique properties.

Field of the insertion device magnet

$$\vec{B}(x,y,z) = B(z)\hat{y} \qquad B(z) \approx B_0 \cos(2\pi z/\lambda_{ID})$$

Vector potential for magnet (1 dimensional approximation)

$$\vec{A}(x,y,z) = A(z)\hat{x} \qquad A(z) \approx \frac{B_0 \lambda_{ID}}{2\pi} \sin(2\pi z/\lambda_{ID})$$

Electron Orbit

Uniformity in *x*-direction means that canonical momentum in the *x*-direction is conserved

$$\mathrm{v}_x(z) = \frac{eA(z)}{\gamma m} = \frac{K}{\gamma} c \sin\left(2\pi z / \lambda_{ID}\right)$$

$$x(z) = \int \frac{\mathrm{v}_x}{\mathrm{v}_z} dz \approx -\frac{1}{\langle \beta_z \rangle} \frac{K}{\gamma} \frac{\lambda_{ID}}{2\pi} \cos\left(2\pi z / \lambda_{ID}\right)$$

Field Strength Parameter

$$K \equiv \frac{eB_0 \lambda_{ID}}{2\pi mc}$$

Average Velocity

Energy conservation gives that γ is a constant of the motion

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2(z)} = \sqrt{\beta_{z0}^2 - \beta_x^2(z)}$$

Average longitudinal velocity in the insertion device is

$$\beta^{*2} = \langle \beta_z \rangle^2 = 1 - \frac{1}{\gamma^2} - \frac{K^2}{2\gamma^2}$$

Average rest frame has

$$\gamma^{*2} = \frac{1}{1 - \beta^{*2}} = \frac{\gamma^2}{1 + K^2/2}$$

Relativistic Kinematics

In average rest frame the insertion device is Lorentz contracted, and so its wavelength is

$$\lambda^* = \lambda_{ID} / \beta^* \gamma^*$$

The sinusoidal wiggling motion emits with angular frequency

$$\omega^* = 2\pi c / \lambda^*$$

Lorentz transformation formulas for the wave vector of the emitted radiation

$$k^* = \gamma^* k \left(1 - \beta^* \cos\theta\right)$$

$$k_x^* = k_x = k \sin\theta \cos\varphi$$

$$k_y^* = k_y = k \sin\theta \sin\varphi$$

$$k_z^* = \gamma^* k \left(\cos\theta - \beta^*\right)$$

ID (or FEL) Resonance Condition

Angle transforms as

$$\cos\theta^* = \frac{k_z^*}{k^*} = \frac{\left(\cos\theta - \beta^*\right)}{\left(1 - \beta^* \cos\theta\right)}$$

Wave vector in lab frame has

$$k = \frac{k^*}{\gamma^*\left(1 - \beta^* \cos\theta\right)} = \frac{2\pi\beta^* c}{\lambda_{ID}\left(1 - \beta^* \cos\theta\right)}$$

In the forward direction $\cos\theta = 1$

$$\lambda_e \approx \frac{\lambda_{ID}}{2\gamma^{*2}} = \frac{\lambda_{ID}}{2\gamma^2}\left(1 + K^2/2\right)$$