Accelerator Physics
Statistical and Beam-Beam Effects

G. A. Krafft
Old Dominion University
Jefferson Lab

Lecture 14

Waterbag Distribution

- Lemons and Thode were first to point out SC field is solved as Bessel Functions for a certain equation of state. Later, others, including my advisor and I showed the equation of state was exact for the waterbag distribution.

$$H_T = \frac{p_z^2}{2m}\left(x'^2 + y'^2\right) + \frac{m\omega_0^2\left(x^2+y^2\right)}{2} + e\phi_{SC}$$

$$\psi = A\Theta\left(H_0 - H_T\right)$$

$$n(r) = \iint \psi\, dx' dy' = n_0\left(1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right) \qquad \phi_0 = H_0 / e$$

$$\sigma_v^2 = \frac{\iint \psi\, p_z^2\left(x'^2+y'^2\right) dx' dy'}{m^2 \iint \psi\, dx' dy'} = \frac{H_0}{m}\left(1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right)$$

Integrals

$$\iint \psi\, dx'dy' = A\frac{2m}{p_z^2}\iint \Theta\left(1-\tilde{x}^2-\tilde{y}^2-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)d\tilde{x}d\tilde{y}$$

$$A\frac{2m}{p_z^2}\int_0^{\sqrt{1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}}}\int_0^{2\pi} r dr d\theta = A\frac{2m}{p_z^2}H_0\pi\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)$$

$$\sigma_v^2 = \frac{\iint \psi\, p_z^2\left(x'^2+y'^2\right)dx'dy'}{m^2\iint \psi\, dx'dy'} = \frac{2H_0A}{m}\frac{\frac{2m}{p_z^2}H_0\int_0^{\sqrt{1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}}}\int_0^{2\pi} r^3 dr d\theta}{A\frac{2m}{p_z^2}H_0\pi\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)}$$

$$=\frac{H_0}{m}\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)^2\Bigg/\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)=\frac{H_0}{m}\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)$$

Self-consistent potential solves

$$\nabla^2\phi_{SC} - \frac{\phi_{SC}}{\lambda_D^2} = -\frac{en_0}{\varepsilon_0}\left[1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}\right]$$

$$\lambda_D = \frac{\sigma_v}{\omega_p} = \sqrt{\frac{\varepsilon_0 m}{e^2 n_0}\frac{H_0}{m}} = \sqrt{\frac{\varepsilon_0 H_0}{e^2 n_0}} \qquad \text{Debye Length}$$

Analytic solutions in terms of Modified Bessel Functions

$$e\phi(r) = -\frac{m\omega_0^2\left(x^2+y^2\right)}{2} + A\left(I_0\left(r/\lambda_D\right)-1\right) + BK_0\left(r/\lambda_D\right)$$

$B = 0$ by boundary condition

A chosen so that solution without I_0 solution to inhomogeneous eqn.

Equation for Beam Radius

Now

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\frac{r^2}{2}=2$$

$$\therefore A=m\lambda_D^2\left(2\omega_0^2-\omega_p^2\right)$$

At $r=r_b$ the density vanishes

$$\omega_p^2=\left(2\omega_0^2-\omega_p^2\right)\left(I_0\left(r_b/\lambda_D\right)-1\right)$$

$$\frac{\omega_p^2}{2\omega_0^2-\omega_p^2}+1=\frac{2\omega_0^2}{2\omega_0^2-\omega_p^2}=I_0\left(r_b/\lambda_D\right) \qquad \omega_p^2<2\omega_0^2$$

$$n_b\left(r\right)=n_0\frac{I_0\left(r_b/\lambda_D\right)-I_0\left(r/\lambda_D\right)}{I_0\left(r_b/\lambda_D\right)-1}$$

In figure $\hat{n}_b=n_0$

Debye Length Picture*

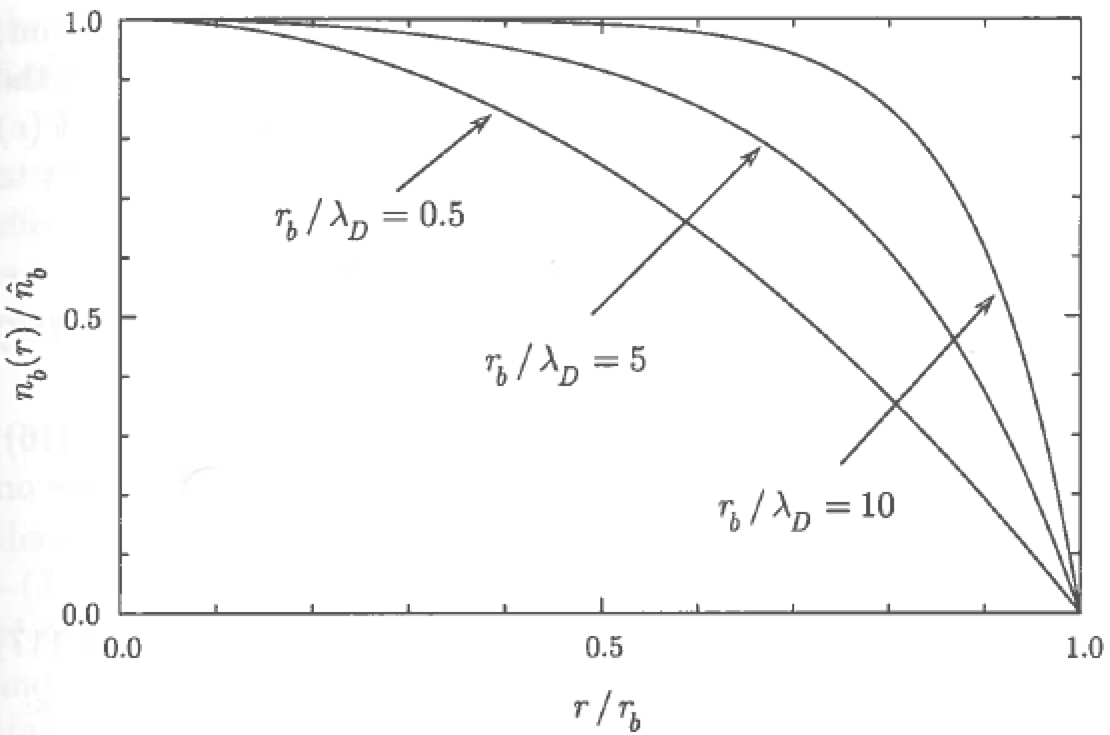


Figure 5.4. Plot of the normalized density profile $n_b(r)/\hat{n}_b$ versus r/r_b obtained from Eq. (5.115) for the choice of equilibrium distribution in Eq. (5.109). Here, the three cases correspond to the choices $r_b/\lambda_D = 0.5$, $r_b/\lambda_D = 5$ and $r_b/\lambda_D = 10$ [see Eq. (5.114)].

*Davidson and Qin

Collisionless (Landau) Damping

- Other important effect of thermal spreads in accelerator physics
- Longitudinal Plasma Oscillations (1 D)

$$\frac{\partial n}{\partial t} + \nabla \cdot v_z n = 0$$

$$\frac{d v_z}{dt} = \frac{-e}{m} E_z$$

$$\frac{\partial E_z}{\partial z} = \frac{-en}{\varepsilon_0}$$

Linearized

$$\frac{\partial \delta n}{\partial t} + n_0 \frac{\partial \delta v_z}{\partial z} = 0$$

$$\frac{\partial \delta v_z}{\partial t} = \frac{-e}{m} \delta E_z$$

$$\frac{\partial \delta E_z}{\partial z} = \frac{-e \delta n}{\varepsilon_0}$$

$$\frac{\partial^2 \delta n}{\partial t^2} = \frac{e n_0}{m} \frac{\partial \delta E_z}{\partial z} = -\frac{e^2 n_0}{\varepsilon_0 m} \delta n$$

$$\delta n \propto e^{\pm i \omega_p t} \qquad \omega_p = \sqrt{\frac{e^2 n_0}{\varepsilon_0 m}}$$

In fluid limit *plasma* oscillations are undamped

Vlasov Analysis of Problem

$$F_e\left(z,p_z,t\right)=\iint\Psi\left(z,\vec{p}\right)dp_x dp_y$$

$$\left\{\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}+e\frac{\partial\phi}{\partial z}\frac{\partial}{\partial p_z}\right\}F_e=0$$

$$\frac{\partial^2\phi}{\partial z^2}=\frac{e}{\varepsilon_0}\left(\int F_e dp_z-n_i\right)$$

0th order solution

$$F_e=F_0\left(p_z\right),\qquad \phi_0=0$$

linearized

$$\left(\frac{\partial}{\partial t}+v_z\frac{\partial}{\partial z}\right)\delta F_e=-e\frac{\partial F_0}{\partial p_z}\frac{\partial\delta\phi}{\partial z}$$

$$\frac{\partial^2\delta\phi}{\partial z^2}=\frac{e}{\varepsilon_0}\int\delta F_e dp_z$$

Initial Value Problem

- Laplace in t and Fourier in z

$$\hat{F}(\omega)=\int_0^\infty dt e^{i\omega t}F(t) \qquad \text{Im}\,\omega \text{ large enough to converge}$$

$$F(t)=\frac{1}{2\pi}\int_C d\omega e^{-i\omega t}\hat{F}(\omega)$$

$$\int_0^\infty dt e^{i\omega t}\frac{d}{dt}F(t)=-i\omega\hat{F}(\omega)-F(t=0)$$

$$\delta\phi(z,t)=\sum_{l=-\infty}^{\infty}\delta\hat{\phi}(l,t)e^{2\pi i l z/L}$$

$$\delta F_e(l,p_z,\omega)=\frac{i\delta F_e(l,p_z,t=0)}{\omega-v_z(2\pi l/L)}+\frac{e2\pi l\partial F_0/\partial p_z}{L(\omega-v_z(2\pi l/L))}\delta\hat{\phi}(l,\omega)$$

$$\delta\hat{\phi}(l,\omega)=-\left(\frac{L}{2\pi l}\right)^2\left[\frac{e^2}{\varepsilon_0}\int_{-\infty}^{\infty}dp_z\frac{2\pi l\partial F_0(p_z)/\partial p_z}{L(\omega-v_z(2\pi l/L))}\delta\hat{\phi}(l,\omega)+\frac{ei}{\varepsilon_0}\int_{-\infty}^{\infty}dp_z\frac{\delta F_e(l,p_z,t=0)}{\omega-v_z(2\pi l/L)}\right]$$

Dielectric function

- Landau (self-consistent) dielectric function

$$D(l,\omega)\delta\hat{\phi}(l,\omega) = N(l,\omega)$$

$$D(l,\omega) = 1 + \frac{e^2}{\varepsilon_0}\left(\frac{L}{2\pi l}\right)\int_{-\infty}^{\infty} dp_z \frac{\partial F_0(p_z)/\partial p_z}{\omega - v_z(2\pi l/L)}$$

- Solution for normal modes are

$$D(l,\omega) = 0$$

$$D(l,\omega) = 1 - \frac{e^2}{\varepsilon_0 m}\int_{-\infty}^{\infty} dp_z \frac{F_0(p_z)}{\left(\omega - v_z(2\pi l/L)\right)^2}$$

$$= 1 - \omega_p^2 \int_{-\infty}^{\infty} dp_z \frac{F_0(p_z)/n_0}{\left(\omega - v_z(2\pi l/L)\right)^2}$$

Collisionless Damping

- For Lorentzian distribution

$$\frac{F_0}{n_0} = \frac{\Delta}{\pi\left[p_z^2 + \Delta^2\right]}$$

$$1 = \omega_p^2 \int_{-\infty}^{\infty} dp_z \frac{1}{\left(\omega - p_z\left(2\pi l / Lm\right)\right)^2} \frac{\Delta}{\pi\left[p_z^2 + \Delta^2\right]}$$

$$= \frac{\omega_p^2}{\left(\omega + i\left(2\pi l / Lm\right)\Delta\right)^2}$$

- Landau damping rate

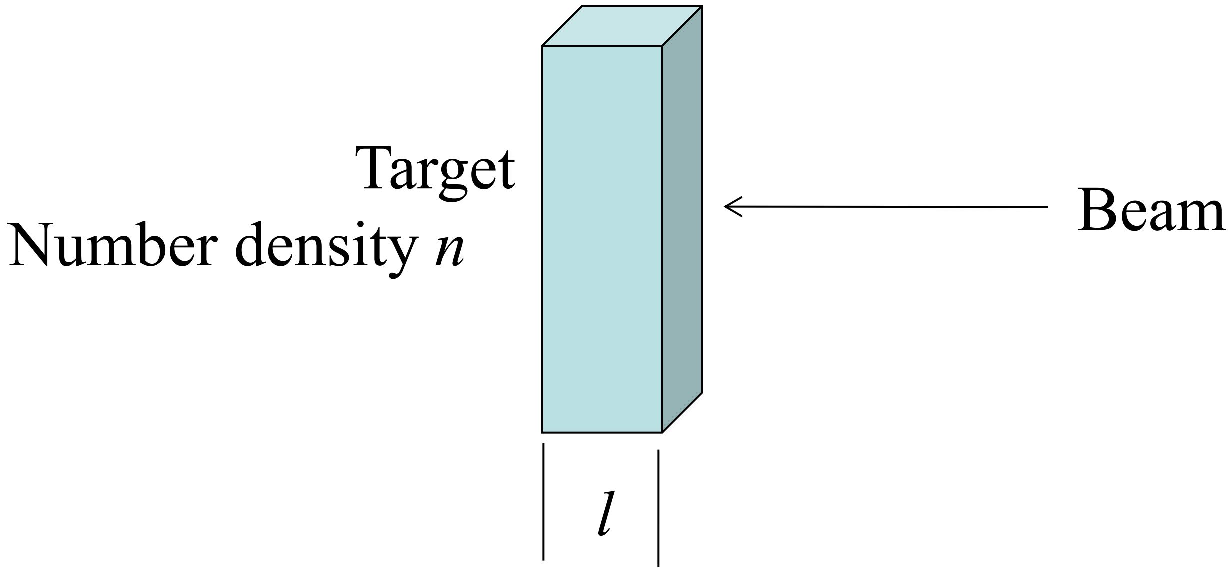
$$\omega = \pm\omega_p - i\frac{2\pi l}{L}\Delta$$

Luminosity and Beam-Beam Effect

- Luminosity Defined
- Beam-Beam Tune Shift
- Luminosity Tune-shift Relationship (Krafft-Ziemann Thm.)
- Beam-Beam Effect

Events per Beam Crossing

- In a nuclear physics experiment with a beam crossing through a thin fixed target

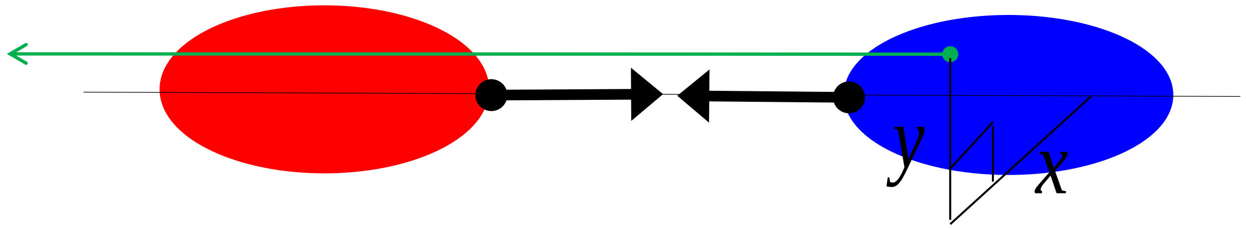


- Probability of single event, per beam particle passage is

$$P = n\sigma l$$

- σ is the "cross section" for the process (area units)

Collision Geometry



- Probability an event is generated by a single particle of Beam 1 crossing Beam 2 bunch with Gaussian density*

$$P = \sigma \frac{N_2 \exp\left(-x^2/2\sigma_{2x}^2\right)\exp\left(-y^2/2\sigma_{2y}^2\right)}{(2\pi)^{3/2}\sigma_{2x}\sigma_{2y}\sigma_{2z}} \int_{-\infty}^{\infty} \exp\left(-z^2/2\sigma_{2z}^2\right)dz$$

$$= \frac{N_2 \exp\left(-x^2/2\sigma_{2x}^2\right)\exp\left(-y^2/2\sigma_{2y}^2\right)}{2\pi\sigma_{2x}\sigma_{2y}}\sigma$$

* This expression still correct when relativity done properly

Collider Luminosity

- **Probability an event is generated by a Beam 1 bunch with Gaussian density crossing a Beam 2 bunch with Gaussian density**

$$P = \frac{N_1 N_2}{2\pi\sqrt{\sigma_{1x}^2 + \sigma_{2x}^2}\sqrt{\sigma_{1y}^2 + \sigma_{2y}^2}}\sigma$$

- **Event rate with equal transverse beam sizes**

$$\frac{dN}{dt} = \frac{f N_1 N_2}{4\pi\sigma_x\sigma_y}\sigma = \mathcal{L}\sigma$$

- **Luminosity**

$$\mathcal{L} = \frac{f N_1 N_2}{4\pi\sigma_x\sigma_y} \sim 10^{33}\,\mathrm{sec}^{-1}\mathrm{cm}^{-2},$$

$$\text{for } f = 100\ \text{MHz},\ N_1 = N_2 = 10^{10}, \sigma_x = \sigma_y = 10\ \text{microns}$$

Beam-Beam Tune Shift

- As we've seen previously, in a ring accelerator the number of transverse oscillations a particle makes in one circuit is called the "betatron tune" Q.
- Any deviation from the design values of the tune (in either the horizontal or vertical directions), is called a "tune shift". For long term stability of the beam in a ring accelerator, the tune must be highly controlled.

$$M_{tot} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} \cos\mu & \beta^* \sin\mu \\ -\sin\mu/\beta^* & \cos\mu \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu & \beta^* \sin\mu \\ -\cos\mu/f - \sin\mu/\beta^* & \cos\mu - \left(\beta^*/f\right)\sin\mu \end{pmatrix}$$

$$\cos(\mu+\Delta\mu)=\frac{\mathrm{Tr}\left(M_{tot}\right)}{2}=\cos\mu-\frac{\beta^*}{2f}\sin\mu$$

$$\xi=\Delta Q=\frac{\Delta\mu}{2\pi}=\frac{\beta^*}{4\pi f} \qquad \beta^* << f$$

Bessetti-Erskine Solution

ODU

- 2-D potential of Bi-Gaussian transverse distribution

$$\rho(x,y)=\frac{Q'}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2}\right)\exp\left(-\frac{y^2}{2\sigma_y^2}\right)$$

- Potential Theory gives solution to Poisson Equation

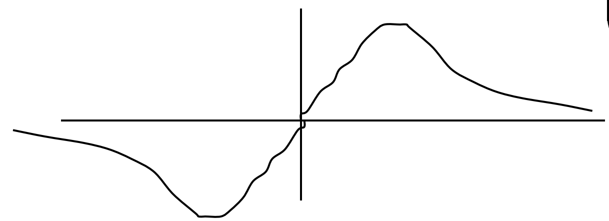
$$\nabla^2\phi=\frac{\rho(x,y)}{\varepsilon_0}$$

$$\phi(x,y)=\frac{Q'}{4\pi\varepsilon_0}\int_0^\infty\frac{\exp\left(-\frac{x^2}{2\sigma_x^2+q}\right)\exp\left(-\frac{y^2}{2\sigma_y^2+q}\right)}{\sqrt{2\sigma_x^2+q}\sqrt{2\sigma_y^2+q}}dq$$

- Bassetti and Erskine manipulate this to

$$E_x = \frac{Q'}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\,\mathrm{Im}\left[w\left(\frac{x+iy}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)-\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{x^2}{2\sigma_y^2}\right)w\left(\frac{x\left(\frac{\sigma_y}{\sigma_x}\right)+iy\left(\frac{\sigma_x}{\sigma_y}\right)}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)\right]$$

$$E_y = \frac{Q'}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\,\mathrm{Re}\left[w\left(\frac{x+iy}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)-\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{x^2}{2\sigma_y^2}\right)w\left(\frac{x\left(\frac{\sigma_y}{\sigma_x}\right)+iy\left(\frac{\sigma_x}{\sigma_y}\right)}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)\right]$$



$w(z)$ Complex error function

- We need 2-D linear field for small displacements

$$E_x(x,0) = -\frac{\partial\phi}{\partial x} \square \frac{Q'x}{2\pi\varepsilon_0}\int_0^\infty \frac{1}{\left(\sqrt{2\sigma_x^2+q}\right)^{3/2}\sqrt{2\sigma_y^2+q}}\,dq$$

- Can do the integral analytically

$$\int_0^\infty \frac{1}{\left(\sqrt{2\sigma_x^2+q}\right)^3\sqrt{2\sigma_y^2+q}}dq = \int_{\sigma_x^2+\sigma_y^2}^\infty \frac{\sigma_y^2-\sigma_x^2+q'}{\left(\sqrt{\sigma_x^2-\sigma_y^2+q'}\right)^3\left(\sqrt{\sigma_y^2-\sigma_x^2+q'}\right)^3}dq'$$

$$= \int_{\sigma_x^2+\sigma_y^2}^\infty \frac{\sigma_y^2-\sigma_x^2+q'}{\left(q'^2-\left(\sigma_y^2-\sigma_x^2\right)^2\right)^{3/2}}dq' = \left[-\frac{\left(\sigma_y^2-\sigma_x^2\right)q'}{\left(\sigma_y^2-\sigma_x^2\right)^2\left(q'^2-\left(\sigma_y^2-\sigma_x^2\right)^2\right)^{1/2}} - \frac{1}{\left(q'^2-\left(\sigma_y^2-\sigma_x^2\right)^2\right)^{1/2}}\right]\Bigg|_{\sigma_x^2+\sigma_y^2}^\infty$$

$$= -\frac{1}{\sigma_y^2-\sigma_x^2} + \frac{\sigma_y^2+\sigma_x^2}{\sigma_y^2-\sigma_x^2}\frac{1}{2\sigma_x\sigma_y} + \frac{1}{2\sigma_x\sigma_y} = \frac{-2\sigma_x\sigma_y+\sigma_y^2+\sigma_x^2+\sigma_y^2-\sigma_x^2}{\left(\sigma_y^2-\sigma_x^2\right)2\sigma_x\sigma_y} = \frac{1}{\left(\sigma_x+\sigma_y\right)\sigma_x}$$

- Similarly for the *y*-direction

$$E_y(0,y) = -\frac{\partial\phi}{\partial y} \square \frac{Q'y}{2\pi\varepsilon_0}\frac{1}{\sigma_y\left(\sigma_x+\sigma_y\right)}$$

Linear Beam-Beam Kick

- Linear kick received after interaction with bunch

$$\Delta\left(\gamma_1\beta_{1x}mc\right)=q_1\int_{-\infty}^{\infty}\left(\vec{E}_{2x}+\left(\vec{\mathrm{v}}\times\vec{B}\right)_{2x}\right)\left(\vec{x}_1\left(t\right),t\right)dt$$

by relativity, for oppositely moving beams

$$\Delta\gamma\beta_{1x}mc=q_1\left(1+\beta_{1z}\beta_{2z}\right)\int_{-\infty}^{\infty}\left(\vec{E}_{2x}\right)\left(\vec{x}_1\left(t\right),t\right)dt$$

Following linear Bassetti-Erskine model

$$E_{2x}\left(x,0,z,t\right)=\frac{q_2x}{2\pi\varepsilon_0}\frac{1}{\sigma_x\left(\sigma_x+\sigma_y\right)}\frac{1}{\sqrt{2\pi\sigma_2}}\exp\left(-\frac{\left(z-\beta_{2z}ct\right)^2}{2\sigma_z^2}\right)$$

q_1 moves with $\vec{x}\left(t\right)=\left(x,0,-\beta_{1z}ct\right)$

Linear Beam-Beam Tune Shift

$$\therefore \Delta\gamma\beta_{1x}mc = q_1 \frac{\left(1+\beta_{1z}\beta_{2z}\right)}{\beta_{1z}+\beta_{2z}} \frac{q_2 x}{2\pi\varepsilon_0 c} \frac{1}{\sigma_x\left(\sigma_x+\sigma_y\right)}$$

$$1/\,f = \frac{2N_2}{\gamma_1} \frac{\left(1+\beta_{1z}\beta_{2z}\right)}{\beta_{1z}+\beta_{2z}} \frac{r_1}{\sigma_x\left(\sigma_x+\sigma_y\right)} \qquad r_1 = \frac{e^2}{4\pi\varepsilon_0 m_1 c^2}$$

$$1/\,f \approx \frac{2N_2 r_1}{\gamma_1\sigma_x\left(\sigma_x+\sigma_y\right)}$$

Both beams relativistic

From linear Bassetti-Erskine model, and replacing the beam size

$$\xi_x^1 = \frac{N_2 r_1}{2\pi\gamma_1} \frac{1}{\varepsilon_x^1\left(1+\sigma_y/\sigma_x\right)} \qquad \xi_y^1 = \frac{N_2 r_1}{2\pi\gamma_1} \frac{1}{\varepsilon_y^i\left(1+\sigma_y/\sigma_x\right)\left(\sigma_x/\sigma_y\right)}$$

Argument entirely symmetric wrt choice of bunch 1 and 2

$$\xi_x^i = \frac{N_{\bar{i}} r_i}{2\pi\gamma_i} \frac{1}{\varepsilon_x^i\left(1+\sigma_y/\sigma_x\right)} \qquad \xi_y^i = \frac{N_{\bar{i}} r_i}{2\pi\gamma_i} \frac{1}{\varepsilon_y^i\left(1+\sigma_y/\sigma_x\right)\left(\sigma_x/\sigma_y\right)}$$

Luminosity Beam-Beam tune-shift relationship

- Express Luminosity in terms of the (larger!) vertical tune shift (*i* either 1 or 2)

$$\mathcal{L} = \frac{f N_i \xi_y^i \gamma_i}{2 r_i \beta_{iy}^*}\left(1+\sigma_y/\sigma_x\right) = \frac{I_i}{e}\frac{\xi_y^i \gamma_i}{2 r_i \beta_{iy}^*}\left(1+\sigma_y/\sigma_x\right)$$

- Necessary, but not sufficient, for self-consistent design
- Expressed in this way, and given a known limit to the beam-beam tune shift, the only variables to manipulate to increase luminosity are the stored current, the aspect ratio, and the β^* (beta function value at the interaction point)
- Applies to ERL-ring colliders, stored beam (ions) only

Luminosity-Deflection Theorem

- Luminosity-tune shift formula is linearized version of a much more general formula discovered by Krafft and generalized by V. Ziemann.

- Relates easy calculation (luminosity) to a hard calculation (beam-beam force), and contains all the standard results in beam-beam interaction theory.

- Based on the fact that the relativistic beam-beam force is almost entirely transverse, i. e., 2-D electrostatics applies.

2-D Electrostatics Theorem

$$\vec{E}(\vec{x}) = \frac{2Q'}{4\pi\varepsilon_0}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^2}$$

$$\vec{F}'_{21} = -\vec{F}'_{12} = \frac{1}{2\pi\varepsilon_0}\iint \rho_2(\vec{x}_2)\frac{\vec{x}_2-\vec{x}_1}{|\vec{x}_2-\vec{x}_1|}\rho_1(\vec{x}_1)\,d^2\vec{x}_1 d^2\vec{x}_2 \quad \text{1 on 2}$$

$$n_1(\vec{x}_1) = \rho_1(\vec{x}_1)/Q_1' \quad n_2(\vec{x}_2) = \rho_1\left(\vec{x}_2+\vec{b}\right)/Q_1' \text{ zero centerred}$$

$$Q_i' = \iint \rho_i(\vec{x})\,d^2\vec{x} \qquad \vec{b} = \iint \vec{x}\rho_2(\vec{x})\,d^2\vec{x}/Q_2'$$

$$\vec{F}'_{21} = -\vec{F}'_{12} = \frac{Q_1'Q_2'}{2\pi\varepsilon_0}\iint n_2(\vec{x}_2)\frac{\vec{x}_1+\vec{b}-\vec{x}_2}{|\vec{x}_1+\vec{b}-\vec{x}_2|^2}n_1(\vec{x}_1)\,d^2\vec{x}_1 d^2\vec{x}_2$$

$$\vec{\nabla}_{\vec{b}} \cdot \frac{\vec{x}_1 + \vec{b} - \vec{x}_2}{\left|\vec{x}_1 + \vec{b} - \vec{x}_2\right|^2} = 2\pi\delta\left(x_2 + b_x + x_1\right)\delta\left(y_2 + b_y + y_1\right)$$

$$\vec{\nabla}_{\vec{b}} \cdot \vec{F}'_{21} = \frac{1}{\varepsilon_0}\iint \rho_2\left(\vec{x} + \vec{b}\right)\rho_1\left(\vec{x}\right) d^2\vec{x}$$

Generalizes $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$ $\left(\text{take } \rho_2\left(\vec{x}\right) \propto \delta^2\left(\vec{x} + \vec{b}\right)\right)$

Transverse interaction in the beam-beam problem

$$\Delta p_1 = \frac{q_1 q_2}{2\pi\varepsilon_0 c}\frac{\vec{x}_1 - \vec{x}_2}{\left|\vec{x}_1 - \vec{x}_2\right|^2}$$

$$\vec{D}\left(\vec{b}\right) = \Delta\gamma_1\vec{\beta}_1 = -\Delta m_2\gamma_2\vec{\beta}_2 / m_1$$

$$= \frac{q_1 q_2}{m_1 c^2} \iint n_2\left(\vec{x}_2\right) \frac{\vec{x}_1 - \vec{x}_2 - \vec{b}}{\left|\vec{x}_1 - \vec{x}_2 - \vec{b}\right|^2} n_1\left(\vec{x}_1\right) d^2\vec{x}_1 d^2\vec{x}_2$$

$$\vec{\nabla}_{\vec{b}} \cdot \vec{D}\left(\vec{b}\right) = 4\pi N_2 r_e \iint n_2\left(\vec{x} - \vec{b}\right) n_1\left(\vec{x}\right) d^2\vec{x} \qquad r_e = \frac{e^2}{4\pi\varepsilon_0 mc^2}$$

$$L\left(\vec{b}\right) = N_1 N_2 \iint n_2\left(\vec{x} - \vec{b}\right) n_1\left(\vec{x}\right) d^2\vec{x}$$

$$L\left(\vec{b}\right) = \frac{N_1}{4\pi r_e} \vec{\nabla}_{\vec{b}} \cdot \vec{D}\left(\vec{b}\right)$$

$$L\left(\vec{b}\right) = -\frac{N_2}{4\pi r_e} \vec{\nabla}_{\vec{b}} \cdot \left(\Delta\gamma_2\vec{\beta}_2\right)$$

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \frac{\gamma_1}{2f} \begin{pmatrix} \sigma_y / \sigma_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

$$L = \frac{N_1 \gamma \xi}{2 r_e \beta^*} \left(1 + \sigma_y / \sigma_x \right) \qquad \text{as before}$$

Maximum when

$$\frac{\partial}{\partial b_x} \left[\frac{\partial D_x}{\partial b_x} \right] = 0, \qquad \frac{\partial}{\partial b_y} \left[\frac{\partial D}{\partial b_y} \right] = 0$$

Luminosity-Deflection Pairs

- Round Beam Fast Model

$$\vec{D}\left(\vec{b}\right)=\frac{2N_2 r_e \vec{b}}{\sigma^2+b^2} \qquad L\left(\vec{b}\right)=\frac{N_1 N_2 \sigma^2}{\pi\left(\sigma^2+b^2\right)^2}$$

- Gaussian Macroparticles

$$\vec{D}\left(\vec{b}\right)=\vec{D}_{Bassetti\_Erskine}\left(\vec{b};\sqrt{\sigma_{1x}^2+\sigma_{2x}^2};\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}\right)$$

$$L\left(\vec{b}\right)=\frac{N_1 N_2}{2\pi\sqrt{\sigma_{1x}^2+\sigma_{2x}^2}\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}}\exp\left(-\frac{b_x^2}{\sqrt{\sigma_{1x}^2+\sigma_{2x}^2}}\right)\exp\left(-\frac{b_y^2}{\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}}\right)$$

- Smith-Laslett Model

$$\vec{D}\left(\vec{b}\right)=\frac{2N_2 r_e \vec{b}}{\hat{b}^2 AB}\left\{\frac{\left(4\hat{b}^2+2\hat{b}^4\right)}{\left(4\hat{b}^2+\hat{b}^4\right)}-\frac{4\hat{b}^2}{\left(4\hat{b}^2+\hat{b}^4\right)^{3/2}}\left\{\sinh^{-1}\left[\frac{\hat{b}^3}{2}+\frac{3\hat{b}}{2}\right]+\sinh^{-1}\left[\frac{\hat{b}}{2}\right]\right\}\right\}$$

$$L\left(\vec{b}\right)=\frac{N_1 N_2}{\pi AB}\left\{\frac{\left(2\hat{b}^2-4\right)\hat{b}^2}{\left(4\hat{b}^2+\hat{b}^4\right)^2}-\frac{4\hat{b}^2\left(1+\hat{b}^2\right)}{\left(4\hat{b}^2+\hat{b}^4\right)^{5/2}}\left\{\sinh^{-1}\left[\frac{\hat{b}^3}{2}+\frac{3\hat{b}}{2}\right]+\sinh^{-1}\left[\frac{\hat{b}}{2}\right]\right\}\right\}$$

$$\hat{b}^2=\left(\frac{b_x}{A}\right)^2+\left(\frac{b_y}{B}\right)^2$$