Accelerator Physics
Statistical Effects

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Lecture 13

Solution by Characteristics

More subtle: a solution to the full Vlasov equation may be obtained from the distribution function at some the initial condition, provided the particle orbits may be found unambiguously from the initial conditions throughout phase space. Example: 1-D harmonic oscillator Hamiltonian.

$$\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} \cos\omega(t-t_0) & \sin\omega(t-t_0)/\omega \\ -\omega\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix} \begin{pmatrix} x(t_0) \\ x'(t_0) \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} \cos\omega(t-t_0) & -\sin\omega(t-t_0)/\omega \\ \omega\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$\psi(x,x';t=t_0) = f_0(x,x')$$

Let $\psi(x,x';t) = f_0\left(\cos\omega(t-t_0)x - \sin\omega(t-t_0)x'/\omega, \omega\sin\omega(t-t_0)x + \cos\omega(t-t_0)x'\right)$

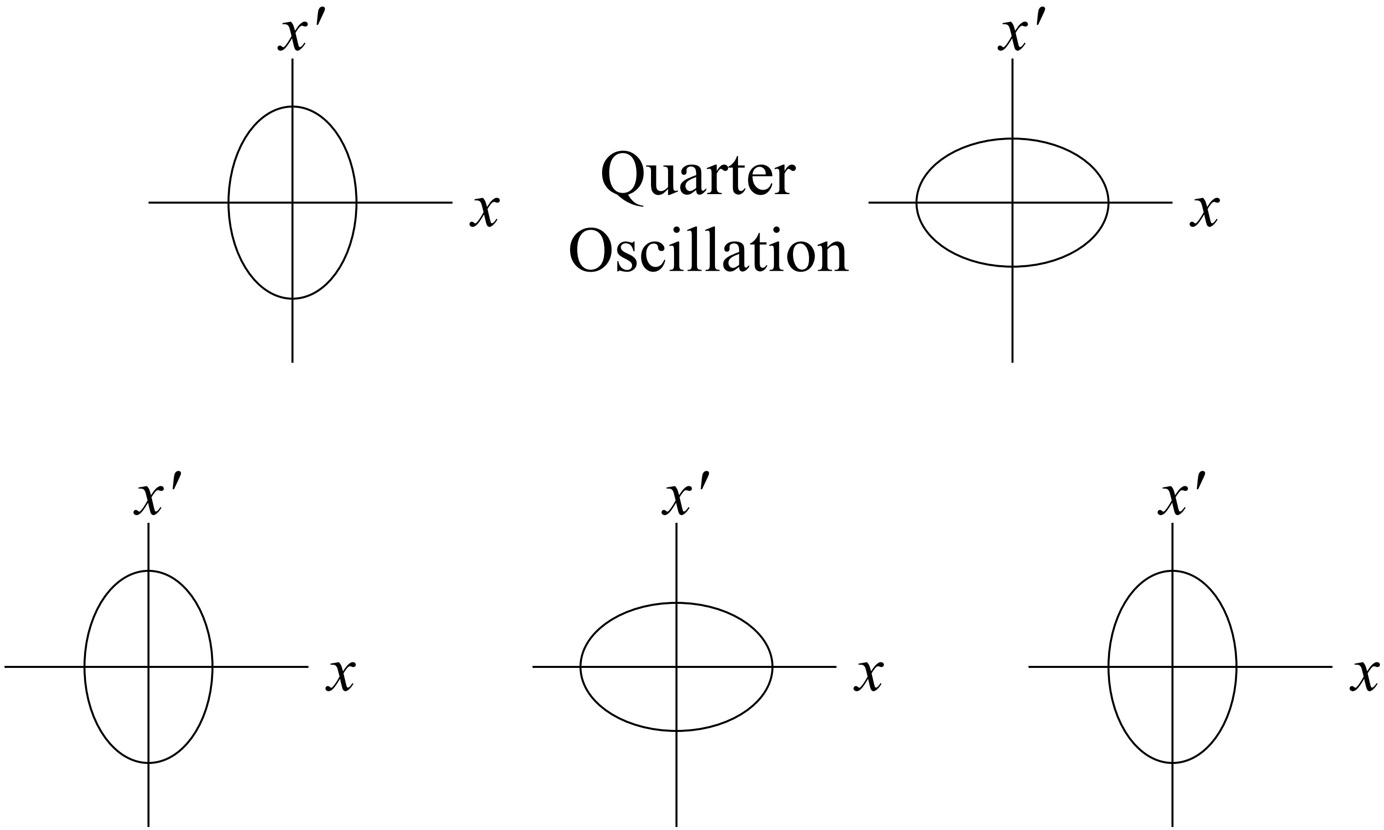
$$\frac{\partial\psi}{\partial t} = \frac{\partial f_0}{\partial x}\frac{dx(t;x,x')}{dt} + \frac{\partial f_0}{\partial x'}\frac{dx'(t;x,x')}{dt}$$

$$= \frac{\partial f}{\partial x}\left[-\omega\sin\omega(t-t_0)x - \cos\omega(t-t_0)x'\right] + \frac{\partial f}{\partial x'}\left[\omega^2\cos\omega(t-t_0)x - \omega\sin\omega(t-t_0)x'\right]$$

$$\frac{dx}{dt}\frac{\partial\psi}{\partial x} = x'\left[\frac{\partial f}{\partial x}\cos\omega(t-t_0) + \frac{\partial f}{\partial x'}\omega\sin\omega(t-t_0)\right]$$

$$\frac{dx'}{dt}\frac{\partial\psi}{\partial x'} = -\omega^2 x\left[-\frac{\partial f}{\partial x}\sin\omega(t-t_0)/\omega + \frac{\partial f}{\partial x'}\cos\omega(t-t_0)\right] \therefore \frac{d\psi}{dt} = 0$$

Breathing Mode



The particle envelope “breaths” at twice the revolution frequency!

Sacherer Theory

Assume beam is acted on by a linear focusing force plus additional linear or non-linear forces

$$x'' + k_x^2 x - F_x = 0$$

$$y'' + k_y^2 x - F_y = 0$$

For space charge example we'll see

$$F_{x(y)} = \frac{qE_{x(y)}\left(1-\beta^2\right)}{\gamma mc^2\beta^2} = \frac{qE_{x(y)}}{\gamma^3 mc^2\beta^2}$$

Now

$$\left\langle xx''\right\rangle + k_x^2\left\langle x^2\right\rangle - \left\langle F_x x\right\rangle = 0$$

$$\left\langle yy''\right\rangle + k_y^2\left\langle y^2\right\rangle - \left\langle F_y y\right\rangle = 0$$

Assume distributions zero-centered and let

$$\tilde{x}^2 = \left\langle x^2\right\rangle \quad \tilde{x}'^2 = \left\langle x'^2\right\rangle \qquad \tilde{y}^2 = \left\langle y^2\right\rangle \quad \tilde{y}'^2 = \left\langle y'^2\right\rangle$$

$$\left\langle x^2 \right\rangle' = 2\left\langle xx' \right\rangle = \tilde{x}^{2\prime} = 2\tilde{x}\tilde{x}'$$

$$\left\langle x^2 \right\rangle'' = \tilde{x}^{2\prime\prime} = \left(2\tilde{x}\tilde{x}'\right)' = 2\left(\tilde{x}\tilde{x}'' + \tilde{x}'^2\right)$$

Also

$$\left\langle xx' \right\rangle' = \left\langle x'^2 \right\rangle + \left\langle xx'' \right\rangle = \left\langle x'^2 \right\rangle - k_x^2 \left\langle x^2 \right\rangle + \left\langle F_x x \right\rangle$$

$$\frac{1}{2}\left\langle x^2 \right\rangle'' = \left\langle xx' \right\rangle' = \tilde{x}\tilde{x}'' + \tilde{x}'^2 = \left\langle x'^2 \right\rangle - k_x^2 \left\langle x^2 \right\rangle + \left\langle F_x x \right\rangle$$

$$\tilde{x}' = \left\langle xx' \right\rangle / \tilde{x} \rightarrow \tilde{x}\tilde{x}'' + \frac{\left\langle xx' \right\rangle^2 - \left\langle x'^2 \right\rangle \left\langle x^2 \right\rangle}{\tilde{x}^2} + k_x^2 \tilde{x}^2 - \left\langle F_x x \right\rangle = 0$$

$$\tilde{x}'' + \frac{\left\langle xx' \right\rangle^2 - \left\langle x'^2 \right\rangle \left\langle x^2 \right\rangle}{\tilde{x}^3} + k_x^2 \tilde{x} - \frac{\left\langle F_x x \right\rangle}{\tilde{x}} = 0$$

$$\tilde{x}'' - \frac{\varepsilon_{rms}^2}{\tilde{x}^3} + k_x^2 \tilde{x} - \frac{\left\langle F_x x \right\rangle}{\tilde{x}} = 0 \qquad \text{"Envelope" equation}$$

rms Emittance Conserved

$$\left(\left\langle x^2\right\rangle\left\langle x'^2\right\rangle-\left\langle xx'\right\rangle^2\right)'$$

$$=\left\langle x^2\right\rangle'\left\langle x'^2\right\rangle+\left\langle x^2\right\rangle\left\langle x'^2\right\rangle'-2\left\langle xx'\right\rangle\left\langle xx'\right\rangle'$$

$$=2\left\langle xx'\right\rangle\left\langle x'^2\right\rangle+2\left\langle x^2\right\rangle\left\langle x'x''\right\rangle-2\left\langle xx'\right\rangle\left(\left\langle x'^2\right\rangle-k_x^2\left\langle x^2\right\rangle+\left\langle F_x x\right\rangle\right)$$

$$=2\left\langle x^2\right\rangle\left(-k_x^2\left\langle x'x\right\rangle+\left\langle F_x x'\right\rangle\right)+2\left\langle xx'\right\rangle\left(k_x^2\left\langle x^2\right\rangle-\left\langle F_x x\right\rangle\right)$$

$$=2\left\langle x^2\right\rangle\left\langle F_x x'\right\rangle-2\left\langle xx'\right\rangle\left\langle F_x x\right\rangle$$

For linear forces derivative vanishes and *rms* emittance conserved. Emittance growth implies *non-linear forces*.

Space Charge and Collective Effects

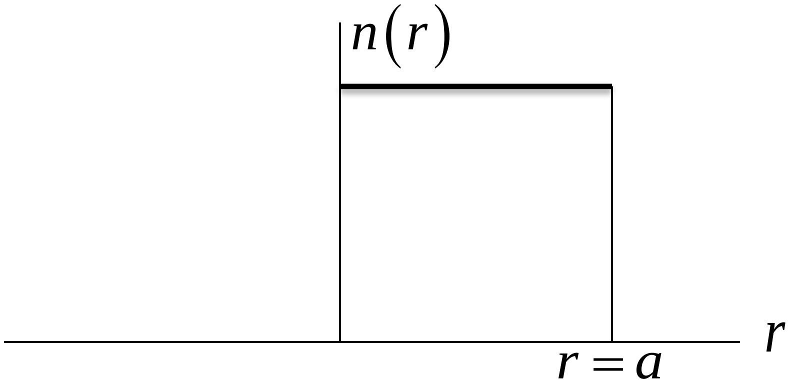
- Collective Effects
  - Brillouin Flow
  - Self-consistent Field
  - KV Equation
  - Bennet Pinch
  - Landau Damping

Simple Problem

- How to account for interactions between particles
- Approach 1: Coulomb sums
  - Use Coulomb's Law to calculate the interaction between each particle in beam
  - Unfavorable N^2 in calculation but perhaps most realistic
  - more and more realistic as computers get better
- Approach 2: Calculate EM field using ME
  - Need procedure to define charge and current densities
  - Track particles in resulting field

Uniform Beam Example

- Assume beam density is uniform and axi-symmetric going into magnetic field



Electric Field

$$\frac{1}{r}\frac{\partial}{\partial r} r E_r = \frac{qn}{\varepsilon_0} \rightarrow E_r = \frac{qn}{2\varepsilon_0} r$$

Self-Magnetic Field by Ampere's Law

$$2\pi r B_\theta = \mu_0 q n \beta c \pi r^2 \rightarrow B_\theta = \mu_0 \frac{qn\beta c}{2} r$$

Brillouin Flow

Total Collective Force on beam particle

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \doteq \left(\frac{q^2 n}{2\varepsilon_0}\left(1-\beta^2\right)\right) r$$

effective (de)focussing strength

$$k = \frac{\omega_p^2}{2\beta^2 c^2 \gamma^3}$$ where the non-relativistic "plasma frequency" is $\omega_p^2 = \frac{q^2 n}{\varepsilon_0 m}$

By previous work with solenoids in the rotating frame, can have equilibrium (force balance) when

$$\frac{\omega_p^2}{2\gamma} = \Omega_c^2$$ non-relativistic plasma and cyclotron frequencies $\omega_L = \frac{\Omega_c}{2\gamma}$

This state is known as Brillouin Flow and neglects beam temperature (fluid flow)

Comments

- Some authors, Reiser in particular, define a relativistic plasma frequency

$$\omega_p^2 = \frac{q^2 n}{\varepsilon_0 \gamma^3 m}$$

- Lawson's book has a nice discussion about why it is impossible to establish a relativistic Brillouin flow in a device where beam is extracted from a single cathode at an equipotential surface. In this case one needs to have either sheering of the rotation or non-uniform density in the self-consistent solution.

Vlasov-Poisson System

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \dot{q}\frac{\partial \psi}{\partial x} + \dot{p}\frac{\partial \psi}{\partial p} = 0$$

$$\dot{q} = p / m$$

$$\dot{p}_i = q\left(E_i + \left(\vec{v} \times \vec{B}\right)_i\right)$$

- Self-consistent Field

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 \phi = \frac{en}{\varepsilon_0}$$

$$n = \int \psi d^3 \dot{\vec{q}}$$

K-V Distribution

- Single value for the transverse Hamiltonian

$$\frac{1}{\varepsilon_x}\left(\frac{x^2+\left(\alpha_x x+\beta_x x'\right)^2}{\beta_x}\right)+\frac{1}{\varepsilon_y}\left(\frac{y^2+\left(\alpha_y y+\beta_y y'\right)^2}{\beta_y}\right)=C$$

$$\psi\left(x,x',y,y'\right)\propto\delta\left(C-1\right)$$

$$\rho\left(z\right)=\frac{I}{\pi\beta cXY}$$

$$K=\frac{I}{I_0}\frac{2}{\beta^3\gamma^3}\qquad I_0=\frac{4\pi\varepsilon_0 mc^3}{q}$$

$$E_x=\frac{I}{\pi\varepsilon_0\beta c}\frac{x}{Y\left(X+Y\right)}$$

K-V Envelope Equation

$$E_x = \frac{I}{\pi\varepsilon_0\beta c}\frac{x}{X\left(X+Y\right)}$$

$$E_y = \frac{I}{\pi\varepsilon_0\beta c}\frac{y}{Y\left(X+Y\right)}$$

$$x'' + k_x x - \frac{2K}{X\left(X+Y\right)}x = 0$$

$$y'' + k_y y - \frac{2K}{Y\left(X+Y\right)}x = 0$$

Envelope Equation

$$X'' + k_x X - \frac{2K}{\left(X+Y\right)} - \frac{\varepsilon_x^2}{X^3} = 0$$

$$Y'' + k_y Y - \frac{2K}{\left(X+Y\right)} - \frac{\varepsilon_y^2}{Y^3} = 0$$

Waterbag Distribution

- Lemons and Thode were first to point out SC field is solved as Bessel Functions for a certain equation of state. Later, others, including my advisor and I showed the equation of state was exact for the waterbag distribution.

$$H_T = \frac{p_z^2}{2m}\left(x'^2 + y'^2\right) + \frac{m\omega_0^2\left(x^2+y^2\right)}{2} + e\phi_{SC}$$

$$\psi = A\Theta\left(H_0 - H_T\right)$$

$$n(r) = \iint \psi\, dx'dy' = n_0\left(1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right) \qquad \phi_0 = H_0 / e$$

$$\sigma_v^2 = \frac{\iint \psi\, p_z^2\left(x'^2+y'^2\right)dx'dy'}{m^2\iint \psi\, dx'dy'} = \frac{H_0}{m}\left(1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right)$$

Integrals

$$\iint \psi\, dx'dy' = A\frac{2m}{p_z^2}\iint \Theta\left(1-\tilde{x}^2-\tilde{y}^2-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)d\tilde{x}d\tilde{y}$$

$$A\frac{2m}{p_z^2}\int_0^{\sqrt{1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}}}\int_0^{2\pi} r\,dr\,d\theta = A\frac{2m}{p_z^2}H_0\pi\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)$$

$$\sigma_v^2 = \frac{\iint \psi\, p_z^2\left(x'^2+y'^2\right)dx'dy'}{m^2\iint \psi\, dx'dy'} = \frac{2H_0A}{m}\frac{\frac{2m}{p_z^2}H_0\int_0^{\sqrt{1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}}}\int_0^{2\pi} r^3\,dr\,d\theta}{A\frac{2m}{p_z^2}H_0\pi\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)}$$

$$=\frac{H_0}{m}\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)^2\Bigg/\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)=\frac{H_0}{m}\left(1-\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}-\frac{\phi_{SC}}{\phi_0}\right)$$

Self-consistent potential solves

$$\nabla^2\phi_{SC} - \frac{\phi_{SC}}{\lambda_D^2} = -\frac{en_0}{\varepsilon_0}\left[1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0}\right]$$

$$\lambda_D = \frac{\sigma_v}{\omega_p} = \sqrt{\frac{\varepsilon_0 m}{e^2 n_0}\frac{H_0}{m}} = \sqrt{\frac{\varepsilon_0 H_0}{e^2 n_0}} \qquad \text{Debye Length}$$

Analytic solutions in terms of Modified Bessel Functions

$$e\phi(r) = -\frac{m\omega_0^2\left(x^2+y^2\right)}{2} + A\left(I_0\left(r/\lambda_D\right) - 1\right) + BK_0\left(r/\lambda_D\right)$$

$B = 0$ by boundary condition

A chosen so that solution without I_0 solution to inhomogeneous eqn.

Equation for Beam Radius

Now

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\frac{r^2}{2}=2$$

$$\therefore A=m\lambda_D^2\left(2\omega_0^2-\omega_p^2\right)$$

At $r=r_b$ the density vanishes

$$\omega_p^2=\left(2\omega_0^2-\omega_p^2\right)\left(I_0\left(r_b/\lambda_D\right)-1\right)$$

$$\frac{\omega_p^2}{2\omega_0^2-\omega_p^2}+1=\frac{2\omega_0^2}{2\omega_0^2-\omega_p^2}=I_0\left(r_b/\lambda_D\right) \qquad \omega_p^2<2\omega_0^2$$

$$n_b\left(r\right)=n_0\frac{I_0\left(r_b/\lambda_D\right)-I_0\left(r/\lambda_D\right)}{I_0\left(r_b/\lambda_D\right)-1}$$

In figure $\hat{n}_b=n_0$

Debye Length Picture*

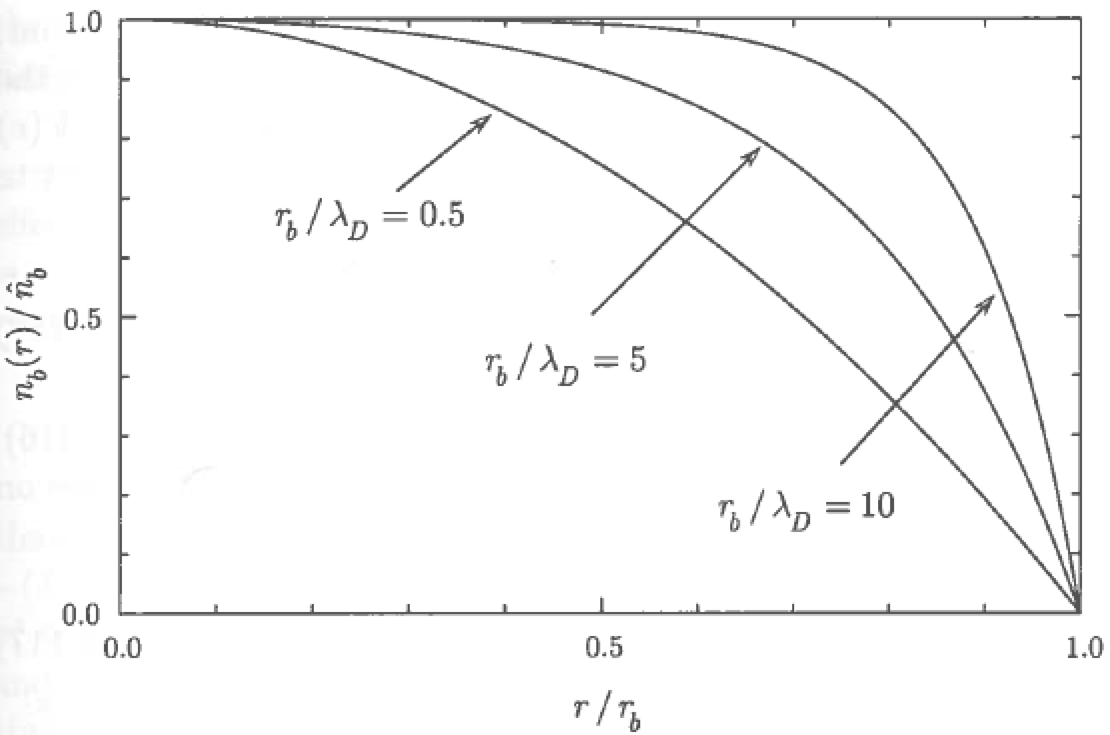


Figure 5.4. Plot of the normalized density profile $n_b(r)/\hat{n}_b$ versus r/r_b obtained from Eq. (5.115) for the choice of equilibrium distribution in Eq. (5.109). Here, the three cases correspond to the choices $r_b/\lambda_D = 0.5$, $r_b/\lambda_D = 5$ and $r_b/\lambda_D = 10$ [see Eq. (5.114)].

*Davidson and Qin