Accelerator Physics
Statistical Effects

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Lecture 12

Statistical Treatments of Beams

- Distribution Functions Defined
  - Statistical Averaging
  - Examples
- Kinetic Equations
  - Liouville Theorem
  - Vlasov Theory
- Self-consistent Fields
- Collective Effects
  - KV Equation
  - Landau Damping
- Beam-Beam Effect

Beam *rms* Emittance

Treat the beam as a statistical ensemble as in Statistical Mechanics. Define the distribution of particles within the beam statistically. Define single particle distribution function

$$\psi\left(x, x'\right),$$

where $\psi(x,x')dxdx'$ is the number of particles in $[x,x+dx]$ and $[x',x'+dx']$, and statistical averaging as in Statistical Mechanics, e. g.

$$\langle q \rangle \equiv \int q\left(x, x'\right)\psi\left(x, x'\right)dxdx' / N$$

$$\langle q^2 \rangle \equiv \int q^2\left(x, x'\right)\psi\left(x, x'\right)dxdx' / N$$

$$\vdots$$

Closest *rms* Fit Ellipses

For zero-centered distributions, i.e., distributions that have zero average value for x and x'

$$\varepsilon_{rms} \equiv \sqrt{\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle x x' \right\rangle^2}$$

$$\beta = \frac{\left\langle x^2 \right\rangle}{\varepsilon_{rms}} = \frac{\sigma_x^2}{\varepsilon_{rms}}$$

$$\alpha = -\frac{\left\langle x x' \right\rangle}{\varepsilon_{rms}}$$

$$\gamma = \frac{\left\langle x'^2 \right\rangle}{\varepsilon_{rms}} = \frac{\sigma_{x'}^2}{\varepsilon_{rms}}$$

Case: Uniformly Filled Ellipse

$$\psi(x,x') = \frac{1}{\pi\varepsilon}\Theta\left(1 - \frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{\varepsilon}\right)$$

Θ here is the Heavyside step function, 1 for positive values of its argument and zero for negative values of its argument

$$\sigma_x^2 = \langle x^2 \rangle = \frac{\varepsilon\beta}{4}$$

$$\langle x x' \rangle = -\frac{\varepsilon\alpha}{4}$$

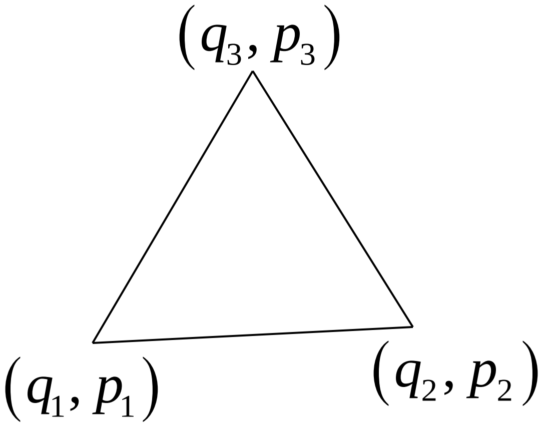
$$\sigma_{x'}^2 = \langle x'^2 \rangle = \frac{\varepsilon}{4\beta}\left(1+\alpha^2\right)$$

$$\therefore \varepsilon_{rms} = \frac{\varepsilon}{4}$$

Gaussian models (HW) are good, especially for lepton machines

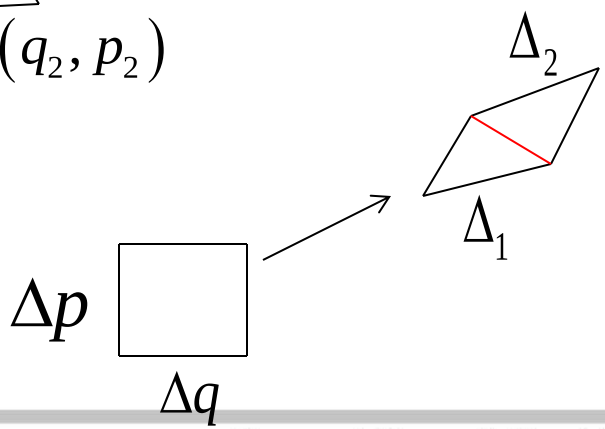
Dynamics? Start with Liouville Thm.

Generalization of the Area Theorem of Linear Optics. Simple Statement: For a dynamical system that may be described by a conserved energy function (Hamiltonian), the relevant phase space volume is conserved by the flow, *even if the forces are non-linear*. Start with some simple geometry!



$$\text{Area } \Delta = \frac{(q_2 - q_1)(p_3 - p_1)}{2} - \frac{(q_3 - q_1)(p_2 - p_1)}{2}$$

$$(\text{acute angle has line } 1-2 \text{ clockwise wrt line } 1-3)$$



In phase space
Area Before=Area After

Case: Uniformly Filled Ellipse

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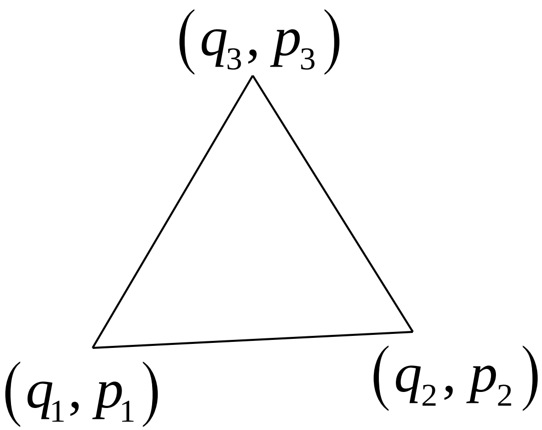

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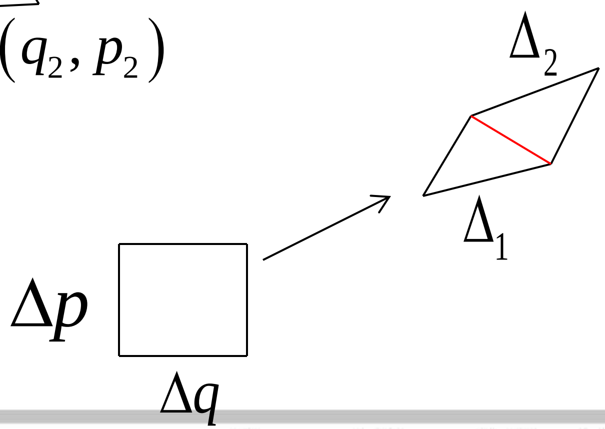

In phase space
Area Before=Area After

Liouville Theorem

$$(q_0, p_0) \rightarrow \left(q_0 + \frac{\partial H}{\partial p}(q_0, p_0)\Delta t + \cdots, p_0 - \frac{\partial H}{\partial q}(q_0, p_0)\Delta t + \cdots \right)$$

$$(q_0 + \Delta q, p_0) \rightarrow \left(q_0 + \Delta q + \frac{\partial H}{\partial p}(q_0 + \Delta q, p_0)\Delta t + \cdots, p_0 - \frac{\partial H}{\partial q}(q_0 + \Delta q, p_0)\Delta t + \cdots \right)$$

$$(q_0, p_0 + \Delta p) \rightarrow \left(q_0 + \frac{\partial H}{\partial p}(q_0, p_0 + \Delta p)\Delta t + \cdots, p_0 + \Delta p - \frac{\partial H}{\partial q}(q_0, p_0 + \Delta p)\Delta t + \cdots \right)$$

$$(q_0 + \Delta q, p_0 + \Delta p) \rightarrow \begin{pmatrix} q_0 + \Delta q + \frac{\partial H}{\partial p}(q_0 + \Delta q, p_0 + \Delta p)\Delta t + \cdots, \\ p_0 + \Delta p - \frac{\partial H}{\partial q}(q_0 + \Delta q, p_0 + \Delta p)\Delta t + \cdots \end{pmatrix}$$

$$\therefore \text{Area } \Delta_1 = \frac{1}{2}\det \begin{vmatrix} \Delta q + \left[\frac{\partial H}{\partial p}(q_0 + \Delta q, p_0) - \frac{\partial H}{\partial p}(q_0, p_0)\right]\Delta t & -\left[\frac{\partial H}{\partial q}(q_0 + \Delta q, p_0) - \frac{\partial H}{\partial q}(q_0, p_0)\right]\Delta t \\ \left[\frac{\partial H}{\partial p}(q_0, p_0 + \Delta p) - \frac{\partial H}{\partial p}(q_0, p_0)\right]\Delta t & \Delta p - \left[\frac{\partial H}{\partial q}(q_0, p_0 + \Delta p) - \frac{\partial H}{\partial q}(q_0, p_0)\right]\Delta t \end{vmatrix}$$

$$\underset{\Delta t \to 0}{\rightarrow} \frac{1}{2}(\Delta q \Delta p)\left[1 + \left\{\frac{\partial^2 H}{\partial q \partial p}(q_0, p_0) - \frac{\partial^2 H}{\partial p \partial q}(q_0, p_0)\right\}\Delta t\right] = \frac{(\Delta q \Delta p)}{2}$$

Likewise

$$\text{Area } \Delta_2 = \frac{1}{2}\det\begin{vmatrix} \Delta q + \left[\frac{\partial H}{\partial p}(q_0+\Delta q, p_0) - \frac{\partial H}{\partial p}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t & -\left[\frac{\partial H}{\partial q}(q_0+\Delta q, p_0) - \frac{\partial H}{\partial q}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t \\ \left[\frac{\partial H}{\partial p}(q_0, p_0+\Delta p) - \frac{\partial H}{\partial p}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t & \Delta p - \left[\frac{\partial H}{\partial q}(q_0, p_0+\Delta p) - \frac{\partial H}{\partial q}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t \end{vmatrix}$$

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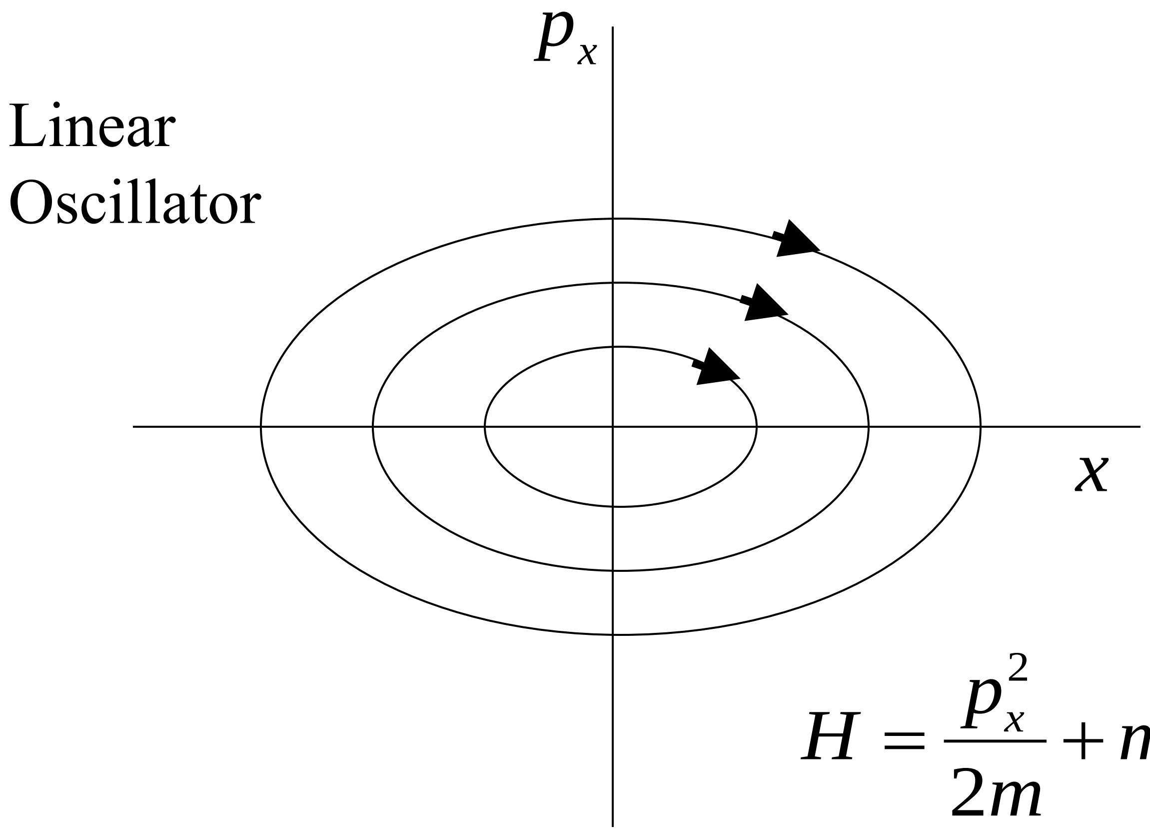
Because the starting point is arbitrary, phase space area is conserved at each location in phase space. In three dimensions, the full 6-D phase volume is conserved by essentially the same argument, as is the sum of the projected areas in each individual projected phase spaces (the so-called third Poincare and first Poincare invariants, respectively). Defeat it by adding non-Hamiltonian (dissipative!) terms later.

Phase Space

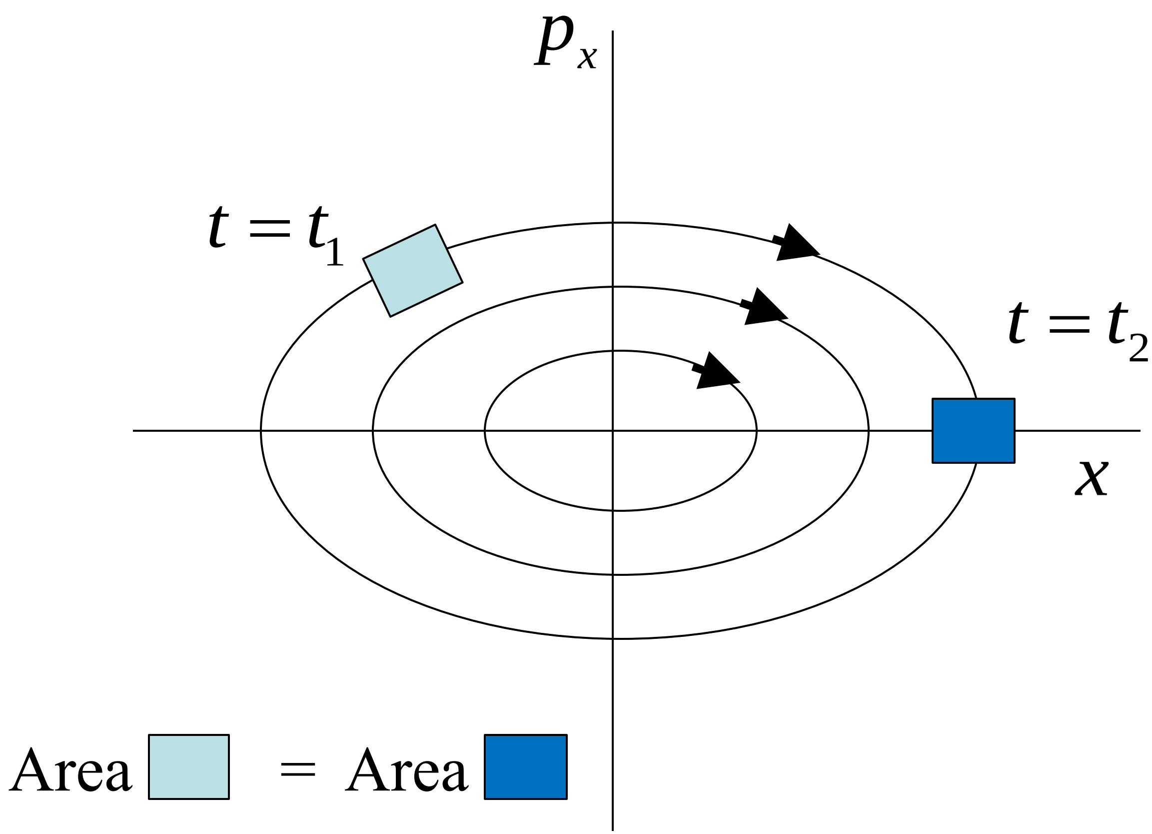
- Plot of dynamical system "state" with coordinate along abscissa and momentum along the ordinate



$$H = \frac{p_x^2}{2m} + m\omega^2\frac{x^2}{2}$$

Liouville Theorem

- Area in phase space is preserved when the dynamics is Hamiltonian



1D Proof

$$\frac{dV}{dt} = \lim_{\Delta t \to \infty} \frac{V(t+\Delta t) - V(t)}{\Delta t}$$

$$x(s,t+\Delta t) \doteq x(s) + \left.\frac{dx}{dt}\right|_t \Delta t + \cdots \doteq x(s) + \frac{\partial H}{\partial p_x}\big(x(s), p_x(s)\big)\Delta t + \cdots$$

$$p_x(s,t+\Delta t) \doteq p_x(s) + \left.\frac{dp_x}{dt}\right|_t \Delta t + \cdots \doteq p_x(s) - \frac{\partial H}{\partial x}\big(x(s), p_x(s)\big)\Delta t + \cdots$$

$$V(t) = \oint_{c(s,t)} p_x dx = \int_0^L p_x(s,t)\frac{dx}{ds}(s,t)\,ds$$

$$V(t+\Delta t) = \oint_{c(s,t+\Delta t)} p_x dx = \int_0^L p_x(s,t+\Delta t)\frac{dx}{ds}(s,t+\Delta t)\,ds$$

$$\doteq \int_0^L \left[p_x(s) + \frac{\partial H}{\partial x}\big(x(s), p_x(s)\big)\Delta t\right]\frac{d}{ds}\left[x(s) - \frac{\partial H}{\partial p_x}\big(x(s), p_x(s)\big)\Delta t\right]ds$$

$$\frac{dV}{dt} = \int_0^L \left[-p_x(s) \frac{d}{ds} \frac{\partial H}{\partial p_x}\big(x(s), p_x(s)\big) + \frac{\partial H}{\partial x}\big(x(s), p_x(s)\big) \frac{dx(s)}{ds} \right] ds$$

$$= \int_0^L \left[\frac{dp_x(s)}{ds} \frac{\partial H}{\partial p_x}\big(x(s), p_x(s)\big) + \frac{\partial H}{\partial x}\big(x(s), p_x(s)\big) \frac{dx(s)}{ds} \right] ds$$

(why is boundary term of integration by parts zero?)

$$= \oint_{c(s,t)} \left[\frac{\partial H}{\partial p_x} dp_x + \frac{\partial H}{\partial x} dx \right]$$

By Green's Thm.

=0 when the differential is an exact differential

i.e., $\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial p_x}\right) = \frac{\partial}{\partial p_x}\left(\frac{\partial H}{\partial x}\right)$, in other words always

(note the integrand above is really dH, so H is a "potential" for phase space!!!)

3D Poincare Invariants

- In a three dimensional Hamiltonian motion, the 6D phase space volume is conserved (also called Liouville's Thm.)

$$V_{6D} = \int_{V_6} dp_x dp_y dp_z dx dy dz$$

- Additionally, the sum of the projected volumes (Poincare invariants) are conserved

$$\int_{\text{proj}(V_2)} dp_x dx + \int_{\text{proj}(V_2)} dp_y dy + \int_{\text{proj}(V_2)} dp_z dz$$

$$\int_{\text{proj}(V_4)} dp_y dp_z dy dz + \int_{\text{proj}(V_4)} dp_z dp_x dz dx + \int_{\text{proj}(V_4)} dp_x dp_y dx dy$$

Emittance (phase space area) exchange based on this idea

- More complicated to prove, but are true because, as in 1D

$$\frac{\partial^2 H}{\partial q_i \partial p_i} = \frac{\partial^2 H}{\partial p_i \partial q_i}$$

γ is a loop in 6D phase space

$$\gamma(t) = (\vec{p}(s,t), \vec{q}(s,t))$$

$$\frac{d}{dt}\left[\oint_{\gamma(t)} \sum_{i=1}^{3} p_i dx_i\right] = \int_0^L \sum_{i=1}^{3}\left[-p_i(s)\frac{d}{ds}\frac{\partial H}{\partial p_i}(\vec{x}(s),\vec{p}(s)) + \frac{\partial H}{\partial x_i}(\vec{x}(s),\vec{p}(s))\frac{dx_i(s)}{ds}\right]$$

$$= \int_0^L \sum_{i=1}^{3}\left[\frac{dp_i(s)}{ds}\frac{\partial H}{\partial p_i}(\vec{x}(s),\vec{p}(s)) + \frac{\partial H}{\partial x_i}(\vec{x}(s),\vec{p}(s))\frac{dx_i(s)}{ds}\right] = \oint_{\gamma(t)} dH = 0$$

for any surface in 6D phase space V_2, with $\gamma = \partial V_2$

$$\oint_{\partial V_2} \sum_{i=1}^{3} p_i dx_i = \int_{V_2} \sum_{i=1}^{3} dp_i dx_i = \sum_{i=1}^{3} \int_{\mathrm{proj}(V_2)} dp_i dx_i$$

$$\left(\sum_{i=1}^{3} dp_i dx_i\right)^2 = dp_y dp_z dydz + dp_z dp_x dzdx + dp_x dp_y dxdy$$

$$\left(\sum_{i=1}^{3} dp_i dx_i\right)^3 = dp_x dp_y dp_z dxdydz$$

Vlasov Equation

By interpretation of ψ as the single particle distribution function, and because the individual particles in the distribution are assumed to not cross the boundaries of the phase space volumes (collisions neglected!), ψ must evolve so that

$$\frac{d\psi}{dt}=0 \qquad \text{as the distibution evolves}$$

$$\frac{d\psi}{dt}=\lim_{\delta t\to 0}\frac{\psi\left(t+\delta t,\vec{q}\left(t+\delta t\right),\vec{p}\left(t+\delta t\right)\right)-\psi\left(t,\vec{q}\left(t\right),\vec{p}\left(t\right)\right)}{\delta t}=0$$

where the equation for ANY (this is what makes

it hard to solve in general!) individual orbits

through phase space is given by $\vec{q}\left(t\right),\vec{p}\left(t\right)$

$$\therefore \frac{\partial\psi}{\partial t}+\frac{d\vec{q}}{dt}\frac{\partial\psi}{\partial\vec{q}}+\frac{d\vec{p}}{dt}\frac{\partial\psi}{\partial\vec{p}}=0$$

Conservation of Probability

$$N(t)=\int \psi(t;\vec{q},\vec{p})\,d^3\vec{x}\,d^3\vec{p}$$ is a conserved quantity

continuity equation for ψ is

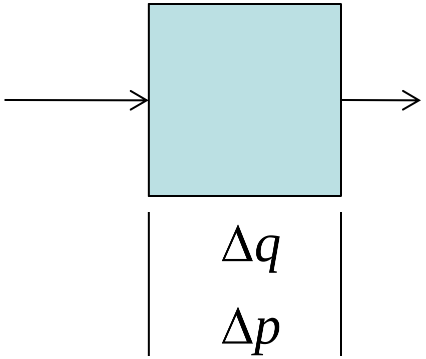
$$\frac{\partial\psi}{\partial t}+\nabla_{\vec{q}}\left(\dot{\vec{q}}\psi\right)+\nabla_{\vec{p}}\left(\dot{\vec{p}}\psi\right)=0$$

$$\therefore\ \frac{\partial\psi}{\partial t}+\frac{d\vec{q}}{dt}\frac{\partial\psi}{\partial\vec{q}}+\frac{d\vec{p}}{dt}\frac{\partial\psi}{\partial\vec{p}}+\psi\left[\nabla_{\vec{q}}\frac{\partial H}{\partial\vec{p}}-\nabla_{\vec{p}}\frac{\partial H}{\partial\vec{q}}\right]=0$$

for the Hamiltonian system

$$\therefore\ \frac{\partial\psi}{\partial t}+\frac{d\vec{q}}{dt}\frac{\partial\psi}{\partial\vec{q}}+\frac{d\vec{p}}{dt}\frac{\partial\psi}{\partial\vec{p}}=0$$

$$\psi\dot{q}(\cdots,q,\cdots)$$
$$\psi\dot{p}(\cdots,p,\cdots)$$



$$\psi\dot{q}(\cdots,q+\Delta q,\cdots)$$
$$\psi\dot{p}(\cdots,p+\Delta p,\cdots)$$

$$\Delta q$$
$$\Delta p$$

Jean's Theorem

The independent variable in the Vlasov equation is often changed to the variable *s*. In this case the Vlasov equation is

$$\frac{\partial \psi}{\partial s}+\frac{d\vec{q}}{ds}\frac{\partial \psi}{\partial \vec{q}}+\frac{d\vec{p}}{ds}\frac{\partial \psi}{\partial \vec{p}}=0$$

The equilibrium Vlasov problem, $\partial\psi/\partial t=0$, is solved by any function of the constants of the motion. This result is called Jean's theorem, and is the starting point for instability analysis as the "unperturbed problem".

If $\psi = f\left(A,B,C,\cdots\right),$ where $A,B,C,\cdots$ are constants of the motion

$$\frac{d\vec{x}}{dt}\frac{\partial \psi}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial \psi}{\partial \vec{p}}=\frac{\partial f}{\partial A}\left(\frac{d\vec{x}}{dt}\frac{\partial A}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial A}{\partial \vec{p}}\right)+\frac{\partial f}{\partial B}\left(\frac{d\vec{x}}{dt}\frac{\partial B}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial B}{\partial \vec{p}}\right)$$

$$+\frac{\partial f}{\partial C}\left(\frac{d\vec{x}}{dt}\frac{\partial C}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial C}{\partial \vec{p}}\right)+\cdots=\frac{\partial f}{\partial A}\frac{dA}{dt}+\frac{\partial f}{\partial B}\frac{dB}{dt}+\frac{\partial f}{\partial C}\frac{dC}{dt}+\cdots=0$$

Examples

1-D Harmonic oscillator Hamiltonian. Bi-Maxwellian distribution is a stationary distribution

$$\psi = \frac{1}{2\pi}\exp\left(-H/kT\right) = \frac{m\omega}{2\pi kT}\exp\left(-p_x^2/2mkT\right)\exp\left(-mx^2\omega^2/2kT\right),$$

As is any other function of the Hamiltonian. Contours of constant ψ line up with contours of constant H

2 D transverse Gaussians, including focusing structure in ring

$$\psi\left(s;x,x';y,y'\right) \propto \exp\left(-\left(\gamma_x\left(s\right)x^2 + 2\alpha_x\left(s\right)xx' + \beta_x\left(s\right)x'^2\right)/\varepsilon_x\right)$$
$$\times\exp\left(-\left(\gamma_y\left(s\right)y^2 + 2\alpha_y\left(s\right)yy' + \beta_y\left(s\right)y'^2\right)/\varepsilon_y\right)$$

Contours of constant ψ line up with contours of constant Courant-Snyder invariant. Stationary as particles move on ellipses!

Solution by Characteristics

More subtle: a solution to the full Vlasov equation may be obtained from the distribution function at some the initial condition, provided the particle orbits may be found unambiguously from the initial conditions throughout phase space. Example: 1-D harmonic oscillator Hamiltonian.

$$\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} \cos\omega(t-t_0) & \sin\omega(t-t_0)/\omega \\ -\omega\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix} \begin{pmatrix} x(t_0) \\ x'(t_0) \end{pmatrix} \to \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} \cos\omega(t-t_0) & -\sin\omega(t-t_0)/\omega \\ \omega\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$\psi(x,x';t=t_0) = f_0(x,x')$$

$$\text{Let } \psi(x,x';t) = f_0\left(\cos\omega(t-t_0)x - \sin\omega(t-t_0)x'/\omega, \omega\sin\omega(t-t_0)x + \cos\omega(t-t_0)x'\right)$$

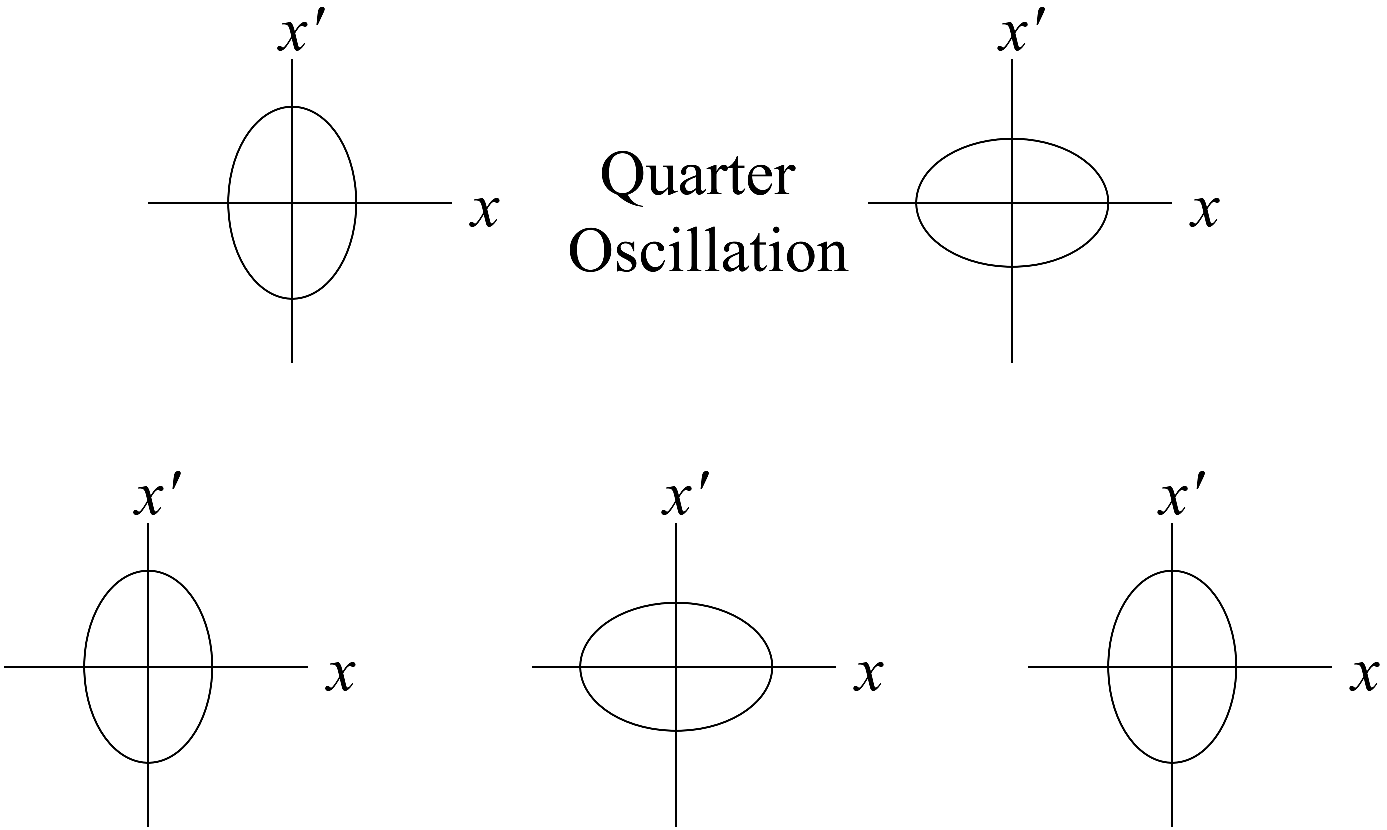
$$\frac{\partial\psi}{\partial t} = \frac{\partial f_0}{\partial x}\frac{dx(t;x,x')}{dt} + \frac{\partial f_0}{\partial x'}\frac{dx'(t;x,x')}{dt}$$

$$= \frac{\partial f}{\partial x}\left[-\omega\sin\omega(t-t_0)x - \cos\omega(t-t_0)x'\right] + \frac{\partial f}{\partial x'}\left[\omega^2\cos\omega(t-t_0)x - \omega\sin\omega(t-t_0)x'\right]$$

$$\frac{dx}{dt}\frac{\partial\psi}{\partial x} = x'\left[\frac{\partial f}{\partial x}\cos\omega(t-t_0) + \frac{\partial f}{\partial x'}\omega\sin\omega(t-t_0)\right]$$

$$\frac{dx'}{dt}\frac{\partial\psi}{\partial x'} = -\omega^2 x\left[-\frac{\partial f}{\partial x}\sin\omega(t-t_0)/\omega + \frac{\partial f}{\partial x'}\cos\omega(t-t_0)\right] \therefore \frac{d\psi}{dt} = 0$$

Breathing Mode



The particle envelope "breaths" at twice the revolution frequency!