Final Examination Solution

Graduate Accelerator Physics

Dec. 12, 2017

1. (15 pts) The JLEIC's detector solenoid has a length of 6 m and a magnetic field of 1 T. What is the focal length of the solenoid for 100 GeV protons in m, assuming the lens is thin. Is the lens thin? (Extra Credit: 5 pts) What is the matched β (the value of the β leading to constant size throughout the solenoid)?

$$\omega_L = \frac{\Omega_c}{2} = \frac{eB}{2\gamma m} = \frac{1.602\times10^{-19}\mathrm{C}\cdot 1\ \mathrm{T}\cdot\left(2.998\times10^{8}\,\mathrm{m/sec}\right)^2}{2\cdot 100\times10^{9}\mathrm{eV}}$$

$$= 4.494\times10^{5}\mathrm{sec}^{-1} \quad 1\ \mathrm{T} = 1\ \frac{\mathrm{V\ sec}}{\mathrm{m}^2}$$

$$k = \frac{\omega_L}{c} = 1.5\times10^{-3}\mathrm{m}^{-1}$$

$$f = \frac{1}{k^2 L} = 74.2\ \mathrm{km}$$

One could also just use the formula from the end of Lecture 4

$$\frac{1}{f} = \frac{e^2\int_{-\infty}^{\infty} B_z^2 dz}{4\beta_z^2 c^2\gamma^2 m^2} \qquad \beta_z \approx 1$$

As mentioned in the lectures, solenoid focusing isn't very effective at relativistic energies. The lens IS very thin (6 m $\ll$ 74.2 km)!

When the beam is matched in the solenoid β is constant and therefore α must remain zero throughout the longitudinal length of the solenoid. For this to occur, by the evolution equation for α ,

$$\beta k^2 = \frac{1}{\beta} \rightarrow \beta = \frac{1}{k} = \frac{c}{\omega_L} = 667\ \mathrm{m}.$$

One could almost "guess" this was the correct answer for dimensional reasons.

2. (20 pts) In our coupled pendulum model ω_g is 20π sec^{-1} and ω_s is 0.5π sec^{-1}.
   a. What are the mode frequencies ω_+ and ω_- ?

   $$\omega_+ = \sqrt{\omega_g^2 + 2\omega_s^2} \doteq 20.0125\pi\ \mathrm{sec}^{-1}$$
   $$\omega_- = 20\pi\ \mathrm{sec}^{-1}$$

   b. The first ball is struck in the direction of the spring. How long does it take the resulting oscillation energy to transfer completely to the second ball?

   One quarter of an energy oscillation cycle with energy moving from ball 1 to ball 2 takes

$$\frac{1}{4}\frac{4\pi}{20.0125\pi-20\pi}\ \text{sec}=80\ \text{sec}$$

3. (25 pts) You are a young Geoff Krafft and just showed up at Jefferson Lab in 1986. The CEBAF/Cornell cavity had just been chosen for the CEBAF project. It has the following parameters: R/Q=480 Ω, f=1.5 GHz, $Q_0=2.4\times10^9$, Energy Gain=2.5 MV. The maximum project operating current has been specified as 800 μA. The cavity will be operated on crest without detuning and neglect microphonics.
   a. Show the optimal coupling factor, i.e., that which yields no reflected power at the maximum operating current, is $\beta_{\text{opt}}=369.64$.

$$\beta_{\text{opt}}=1+\frac{0.0008\ \text{A}\times480\ \Omega\times2.4\times10^9}{2.5\ \text{MV}}=1+368.64$$

   b. What is the consequent loaded Q (Q_L)?

$$Q_L=\frac{Q_0}{1+\beta_{\text{opt}}}=\frac{2.4\times10^9}{370.64}=6.49\times10^6$$

   c. With this coupling, what is the power required at zero current (no beam load)?

$$P_g=\frac{V_c^2}{(R/Q)Q_0}\frac{(1+\beta)^2}{4\beta}=504\ \text{W}$$

   d. With this coupling, what is the power required at 800 μA?

$$P_g=\frac{V_c^2}{(R/Q)Q_0}\frac{(1+\beta)^2}{4\beta}\left(1+\frac{368.64}{1+\beta}\right)^2=\frac{V_c^2}{(R/Q)Q_0}(1+368.64)=5.425\ \text{W}+2\ \text{kW}$$

   e. Compare with the beam power of 800 μA×2.5 MV = 2 kW. Does you result make sense? Yes, when there is no reflected power the generator must provide for the cavity losses and the beam load!
   f. (Extra Credit: 5 pts) Suppose the same cavity is used in a storage ring with $\psi_b=72^\circ$. What detuning ψ would lead to minimum incident power?

$$\tan\psi=\frac{I(R/Q)Q_L}{V_c}\sin72^\circ\rightarrow\Psi=43.5^\circ$$

4. (20 pts) Suppose a beam with a single particle distribution function

$$\rho(x,x')=\frac{1}{2\pi\varepsilon}\exp\left[-\left(\gamma x^2+2\alpha xx'+\beta x'^2\right)/2\varepsilon\right]$$

with $\gamma\beta-\alpha^2=1$, comes into a solid target with $\alpha=0$ at $z=0$.

a) What is the *rms* size in angle $\sigma_{x'}$ at $z=0$ in terms of β and ε?

When $\alpha=0$, $\gamma=1/\beta$. Because

$$\rho(x,x')=\frac{1}{\sqrt{2\pi}\sqrt{\varepsilon\beta}}\exp\left[-x^2/2\left(\sqrt{\varepsilon\beta}\right)^2\right]\frac{1}{\sqrt{2\pi}\sqrt{\varepsilon/\beta}}\exp\left[-x'^2/2\left(\sqrt{\varepsilon/\beta}\right)^2\right],$$

$\sigma_{x'} = \sqrt{\varepsilon / \beta}$.

b) Multiple scattering in the target causes the particle angles to be updated as

$$x'_{after} = x'_{before} + \Delta x'$$

where $\Delta x'$ is distributed according to the distribution function

$$\frac{1}{\sqrt{2\pi}\sigma'_r} \exp\left(-\Delta x'^2 / 2\sigma_r'^2\right).$$

Calculate the *rms* emittance and *rms* size in angle after the target in terms of β , ε , and σ'_r .

The distribution after the target is

$$\rho(x, x') = \frac{1}{\sqrt{2\pi}\sqrt{\varepsilon\beta}} \exp\left[-x^2 / 2\left(\sqrt{\varepsilon\beta}\right)^2\right]$$

$$\times \frac{1}{\sqrt{2\pi}\sqrt{\varepsilon / \beta}} \frac{1}{\sqrt{2\pi}\sigma'_r} \int_{-\infty}^{\infty} \exp\left[-\left(x' - \Delta x'\right)^2 / 2\left(\sqrt{\varepsilon / \beta}\right)^2\right] \exp\left[-\Delta x'^2 / 2\sigma_r'^2\right] d\Delta x'$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\varepsilon\beta}} \exp\left[-x^2 / 2\left(\sqrt{\varepsilon\beta}\right)^2\right] \frac{1}{\sqrt{2\pi}\sqrt{\varepsilon / \beta}} \frac{\exp\left[-x'^2 / 2\left(\sqrt{\varepsilon / \beta}\right)^2\right]}{\sqrt{2\pi}\sigma'_r}$$

$$\times \int_{-\infty}^{\infty} \exp\left[+2x'\Delta x' / 2\left(\sqrt{\varepsilon / \beta}\right)^2\right] \exp\left[-\Delta x'^2 / 2\left(\sqrt{\varepsilon / \beta}\right)^2\right] \exp\left[-\Delta x'^2 / 2\sigma_r'^2\right] d\Delta x'$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\varepsilon\beta}} \exp\left[-x^2 / 2\left(\sqrt{\varepsilon\beta}\right)^2\right] \frac{1}{\sqrt{2\pi}\sqrt{\varepsilon / \beta}} \frac{\exp\left[-x'^2 / 2\left(\sqrt{\varepsilon / \beta}\right)^2\right]}{\sqrt{2\pi}\sigma'_r}$$

$$\times \exp\left[+R^2 x'^2 \left(\left(\sqrt{\varepsilon / \beta}\right)^2 + \sigma_r'^2\right) / 2\left(\left(\sqrt{\varepsilon / \beta}\right)^2 \sigma_r'^2\right)\right]$$

$$\times \int_{-\infty}^{\infty} \exp\left[-\left(Rx' - \Delta x'\right)^2 \left(\left(\sqrt{\varepsilon / \beta}\right)^2 + \sigma_r'^2\right) / 2\left(\left(\sqrt{\varepsilon / \beta}\right)^2 \sigma_r'^2\right)\right] d\Delta x'$$

$$R = \frac{\sigma_r'^2}{\left(\sqrt{\varepsilon / \beta}\right)^2 + \sigma_r'^2}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\varepsilon\beta}} \exp\left[-x^2 / 2\left(\sqrt{\varepsilon\beta}\right)^2\right] \frac{1}{\sqrt{2\pi}\sqrt{\varepsilon / \beta}} \frac{\exp\left[-x'^2 / 2\left(\sqrt{\varepsilon / \beta}\right)^2\right]}{\sqrt{2\pi}\sigma'_r}$$

$$\times \exp\left[+\sigma_r'^2 x'^2 / 2\left(\left(\sqrt{\varepsilon / \beta}\right)^2 \left(\left(\sqrt{\varepsilon / \beta}\right)^2 + \sigma_r'^2\right)\right)\right] \sqrt{2\pi} \frac{\sqrt{\varepsilon / \beta}\,\sigma'_r}{\sqrt{\left(\sqrt{\varepsilon / \beta}\right)^2 + \sigma_r'^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\varepsilon\beta}} \exp\left[-x^2 / 2\left(\sqrt{\varepsilon\beta}\right)^2\right] \frac{1}{\sqrt{2\pi}\sqrt{\left(\sqrt{\varepsilon / \beta}\right)^2 + \sigma_r'^2}} \exp\left[-x'^2 / 2\left(\left(\sqrt{\varepsilon / \beta}\right)^2 + \sigma_r'^2\right)\right]$$

The rms angular size grows to

$$\sigma_{x'}^2 = \frac{\varepsilon}{\beta} + \sigma_r'^2$$

and the emittance grows to

$$\varepsilon' = \sigma_x \sigma_{x'} = \sqrt{\varepsilon\beta}\sqrt{\varepsilon / \beta + \sigma_r'^2} = \sqrt{\varepsilon^2 + \varepsilon\beta\sigma_r'^2} \ .$$

5. (20 pts) A soft X-ray source is designed to operate at 4 GeV. A 100 mA electron beam current passes through an undulator with $K = 1,\ N = 100$, and $\lambda_{ID} = 2.3\ \text{cm}$.
   a. What are the wavelength, frequency, and energy of the photons emitted in the forward direction?

$$\lambda = \frac{\lambda_{ID}}{2\gamma^2}\left(1 + \frac{K^2}{2}\right) = \frac{0.023\ \text{m}}{2(4000/0.511)^2}(1.5) = 0.2815\ \text{nm}$$

$$f = c/\lambda = \frac{2.998\times10^8\ \text{m/sec}}{0.2815\times10^{-9}\ \text{m}} = 1.065\times10^{18}\ \text{sec}^{-1}$$

$$E = hf = 6.626\times10^{-34}\ \text{J sec}\ 1.065\times10^{18}\ \text{sec}^{-1} = 7.056\times10^{-16}\ \text{J} = 4.405\ \text{keV}$$

   b. Given the expressions in the lectures for the total number and the average energy of photons emitted by a single electron, what is the total radiation power emitted by the electrons passing through the undulator?

$$P = \frac{I}{e} N_\gamma \langle E_\gamma \rangle = \frac{.1\ \text{A}}{1.602\times10^{-19}\ \text{C}} \frac{2\pi\times100\times1}{3\times137.04} \frac{7.056\times10^{-16}\ \text{J}}{2} = 337\ \text{W}$$

   Finally, although a good estimate for $K \ll 1$, this estimate entirely neglects harmonic emission and other phenomena that change the result at high K. It is much better to integrate Larmor's formula for better results as K approaches and exceeds 1.