1. (30 pts.) CERN’s Large Hadron Collider (LHC) accelerates protons to very high energy. In the design document the design energy is 7 TeV, or 7000 GeV.
   a. What is the relativistic $\gamma$ of the accelerated protons? What is their relativistic $\beta$?
   b. What is their magnetic rigidity?
   c. There are 1232 dipole magnets of length 14.3 m, bending in normal configuration. What is the required dipole magnetic field?
   d. What is the bend radius when the particles are in the dipoles? Is your result consistent with the total reported machine circumference of 26658.883 m? Explain.
   e. What is the revolution frequency?
   f. The RF frequency is reported to be 400.8 MHz. What is the harmonic number to four significant digits?
   g. The total energy gain per pass is 8 MeV. How long (seconds) does it take to accelerate from 450 GeV to 7 TeV?

You can find the answers to many of these questions in the design document https://edms.cern.ch/file/445830/5/Vol_1_Chapter_2.pdf

Of course, you will be graded on your approach to calculating the answers!

2. (20 pts.) In colliders, to increase the event rate, one would like the beam size very small at the collision point. This leads to the notion of a “low beta insertion”. A (grossly!) simplified model of such an insertion is a strong thin focusing lens of focal length $f$ followed by a drift. Suppose $s = 0$ at the location of the lens, $\beta(s = 0) = \beta_0$ is given, and the ellipse is upright (what does this imply about $\alpha(s = 0)$?).
   a. What is $\gamma(s = 0)$ for the ellipse?
   b. What is the transfer matrix to a location of distance $s$ downstream of the focusing lens?
   c. What is the inverse of the matrix?
   d. What is $\beta(s)$ (get this from the ellipse transformation formula)?
   e. Show $\beta(s)$ is an extremum at location

   $$s_{\text{extremum}} = \frac{\beta_0^2}{f(1 + \beta_0^2 / f^2)}.$$ 

   f. Is it a maximum or minimum?
   g. What is $\beta^* \equiv \beta(s_{\text{extremum}})$, the value at the location of the extremum?
   h. What is $\alpha(s_{\text{extremum}})$, calculated by the ellipse transformation formula?
   i. Extra Credit: Why does $\alpha(s_{\text{extremum}})$ have this value?
3. (20 pts.) Suppose a one period 2 by 2 transfer matrix has $|\text{Tr}(M)| > 2$.
   
   a. Find an expression for the two eigenvalues in terms of $\text{Tr}(M)$. Call them $\lambda_+$ and $\lambda_-$. 
   
   b. Are the two eigenvalues real, or do they have some imaginary components?
   
   c. Show $\lambda_+\lambda_- = 1$.
   
   d. By linear algebra, the matrix of the eigenvectors, $S$, diagonalizes the matrix $M$
      
      $$
      \begin{pmatrix}
      \lambda_+ & 0 \\
      0 & \lambda_-
      \end{pmatrix} = SMS^{-1}
      $$

      Give an expression for $M^n$ in terms of this diagonal matrix and $S$.
   
   e. Is $M$ bounded as $n$ increases?
   
   f. Comment on the stability or lack of stability of solutions given by repeated application of the transfer matrix $M$.
   
   g. Using $\det(AB) = \det(A)\det(B)$, the formula for the determinant of the inverse of a matrix, and part d, comment on the property of the transport matrix that ensures part c is true.

4. (15 pts.) Show that any canonical trajectory ($\hat{x}$) corresponding to a Hamiltonian motion ($\frac{d\hat{x}}{ds} = U\hat{H}\hat{x}$) evolves ($\hat{x} = \hat{M}(0, s)\hat{x}_0$) according to a symplectic matrix:

   $\hat{M}(0, s)^T U \hat{M}(0, s) = U$

   where $U$ is the unit symplectic matrix.

5. (15 pts.) Show the transfer matrix for a region with uniform defocusing $k^2$, $\frac{d^2x}{ds^2} - k^2x = 0$ is

   $$
   M_{s,s} = \begin{pmatrix}
   \cosh k(s' - s) & \sinh k(s' - s) / k \\
   k \sinh k(s' - s) & \cosh k(s' - s)
   \end{pmatrix}
   $$

   Therefore, show the transfer matrix for a quadrupole magnet of length $L$ in the defocusing direction is

   $$
   \begin{pmatrix}
   \cosh kL & \sinh kL / k \\
   k \sinh kL & \cosh kL
   \end{pmatrix}
   $$