Accelerator Physics
Particle Acceleration

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Lecture 6

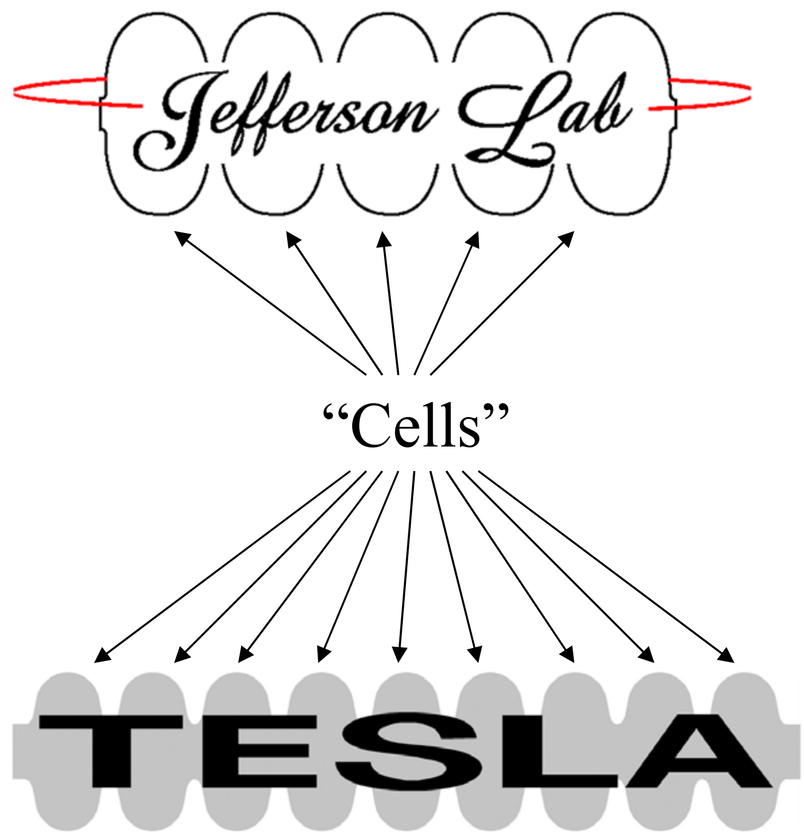
RF Acceleration

- Characterizing Superconducting RF (SRF) Accelerating Structures
  - Terminology
  - Energy Gain, R/Q, Q_0, Q_L and Q_{ext}
- RF Equations and Control
  - Coupling Ports
  - Beam Loading
- RF Focusing
- Betatron Damping and Anti-damping

Terminology

1 CEBAF Cavity

5 Cell Cavity



1 DESY Cavity

9 Cell Cavity

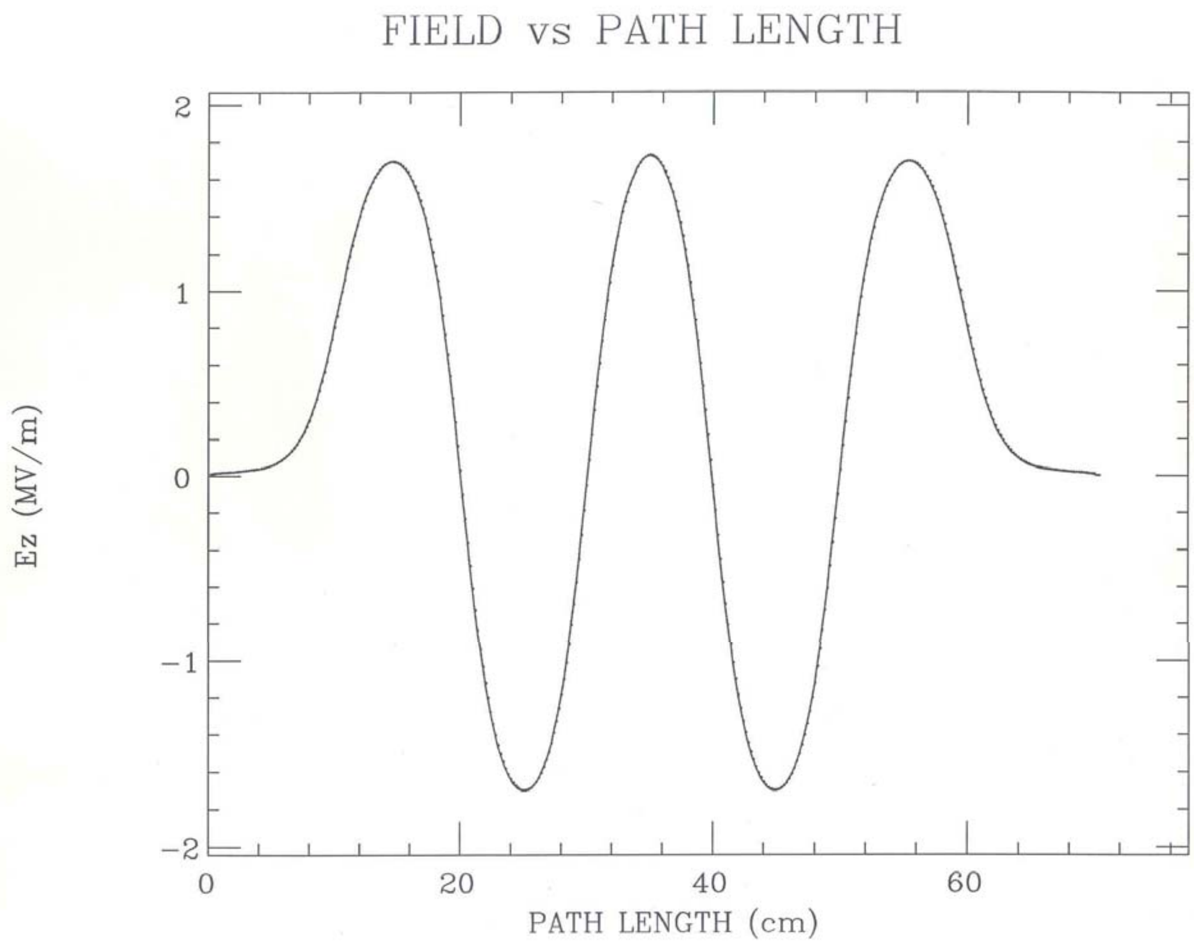
Modern Jefferson Lab Cavities (1.497 GHz) are optimized around a 7 cell design

Typical cell longitudinal dimension: $\lambda_{RF}/2$

Phase shift between cells: π

Cavities usually have, in addition to the resonant structure in picture:

(1) At least 1 input coupler to feed RF into the structure
(2) Non-fundamental high order mode (HOM) damping
(3) Small output coupler for RF feedback control



FIELD vs PATH LENGTH
Ez (MV/m)
2
1
0
−1
−2
0
20
40
60
PATH LENGTH (cm)

Some Fundamental Cavity Parameters

- Energy Gain

$$\frac{d\left(\gamma mc^2\right)}{dt} = -e\vec{E}\left(\vec{x}\left(t\right),t\right)\cdot\vec{\mathrm{v}}$$

- For standing wave RF fields and velocity of light particles

$$\vec{E}\left(\vec{x},t\right) = \vec{E}\left(\vec{x}\right)\cos\left(\omega_{RF}t+\delta\right) \rightarrow \Delta\left(\gamma mc^2\right) \approx -e\int_{-\infty}^{\infty} E_z\left(0,0,z\right)\cos\left(z/\lambda_{RF}+\delta\right)dz$$

$$= \frac{e\tilde{E}_z\left(2\pi/\lambda_{RF}\right)e^{-i\delta}+\mathrm{c.c.}}{2} \qquad V_c \equiv \left|e\tilde{E}_z\left(2\pi/\lambda_{RF}\right)\right|$$

- Normalize by the cavity length L for gradient

$$\mathrm{E}_{\mathrm{acc}}\left(\mathrm{MV/m}\right) = \frac{V_c}{L}$$

Shunt Impedance *R/Q*

- Ratio between the square of the maximum voltage delivered by a cavity and the product of ω_{RF} and the energy stored in a cavity

$$\frac{R}{Q} \equiv \frac{V_c^2}{\omega_{RF}\,(\text{stored energy})}$$

- Depends only on the cavity geometry, independent of frequency when uniformly scale structure in 3D
- Piel's rule: *R/Q* ~100 Ω/cell

CEBAF 5 Cell	**480 Ω**
CEBAF 7 Cell	**760 Ω**
DESY 9 Cell	**1051 Ω**

Unloaded Quality Factor

- As is usual in damped harmonic motion define a quality factor by

$$Q \equiv \frac{2\pi\left(\text{energy stored in oscillation}\right)}{\text{energy dissipated in 1 cycle}}$$

- Unloaded Quality Factor Q_0 of a cavity

$$Q_0 \equiv \frac{\omega_{RF}\left(\text{stored energy}\right)}{\text{heating power in walls}}$$

- Quantifies heat flow directly into cavity walls from AC resistance of superconductor, and wall heating from other sources.

Loaded Quality Factor

- When add the *input* coupling port, must account for the energy loss through the port on the oscillation

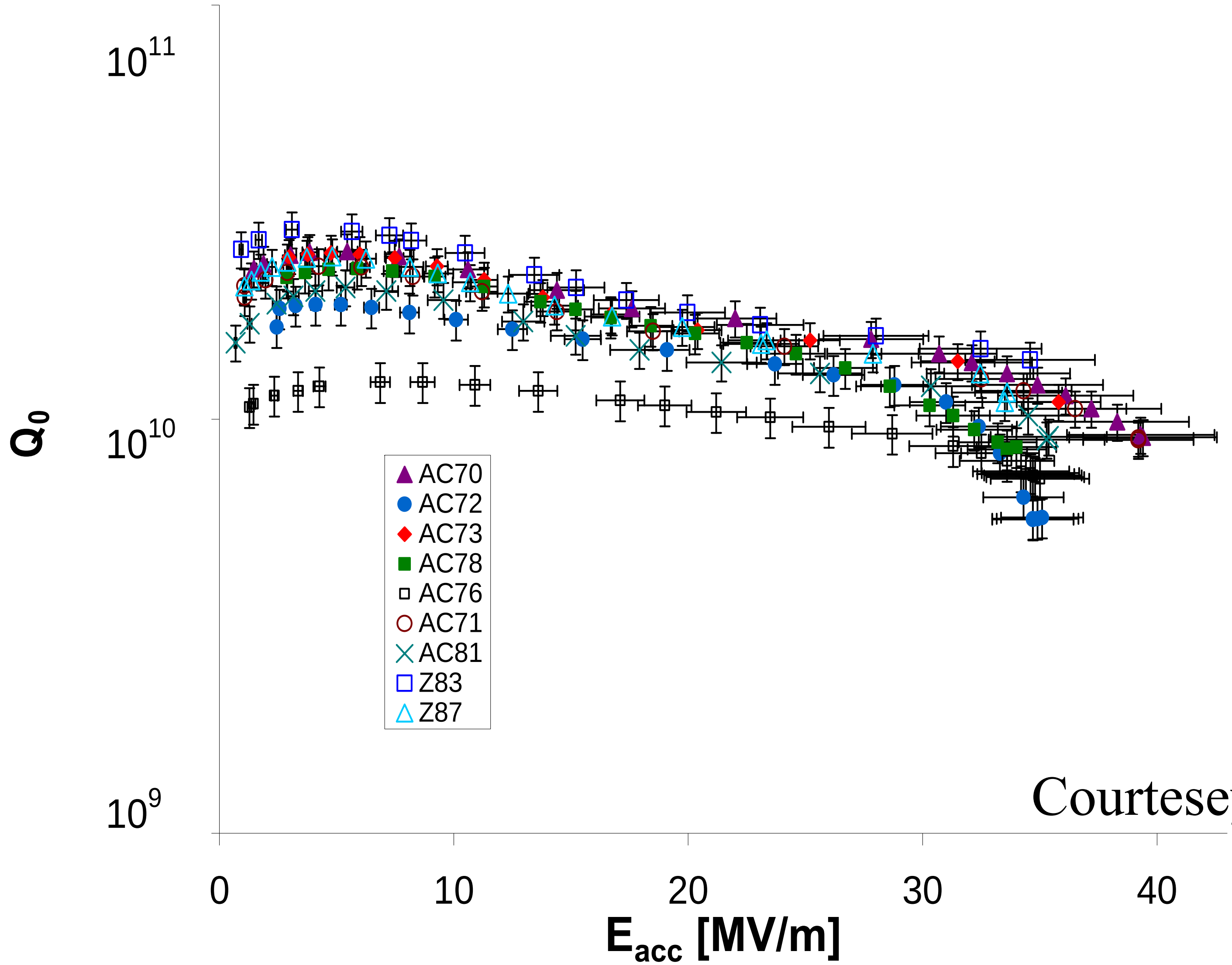
$$\frac{1}{Q_{tot}} \equiv \frac{1}{Q_L} = \frac{\text{total power lost}}{\omega_{RF}\left(\text{stored energy}\right)} = \frac{1}{Q_{ext}} + \frac{1}{Q_0}$$

- Coupling Factor

$$\beta \equiv \frac{Q_0}{Q_{ext}} \gg 1 \quad \text{for present day SRF cavities,} \qquad Q_L = \frac{Q_0}{1+\beta}$$

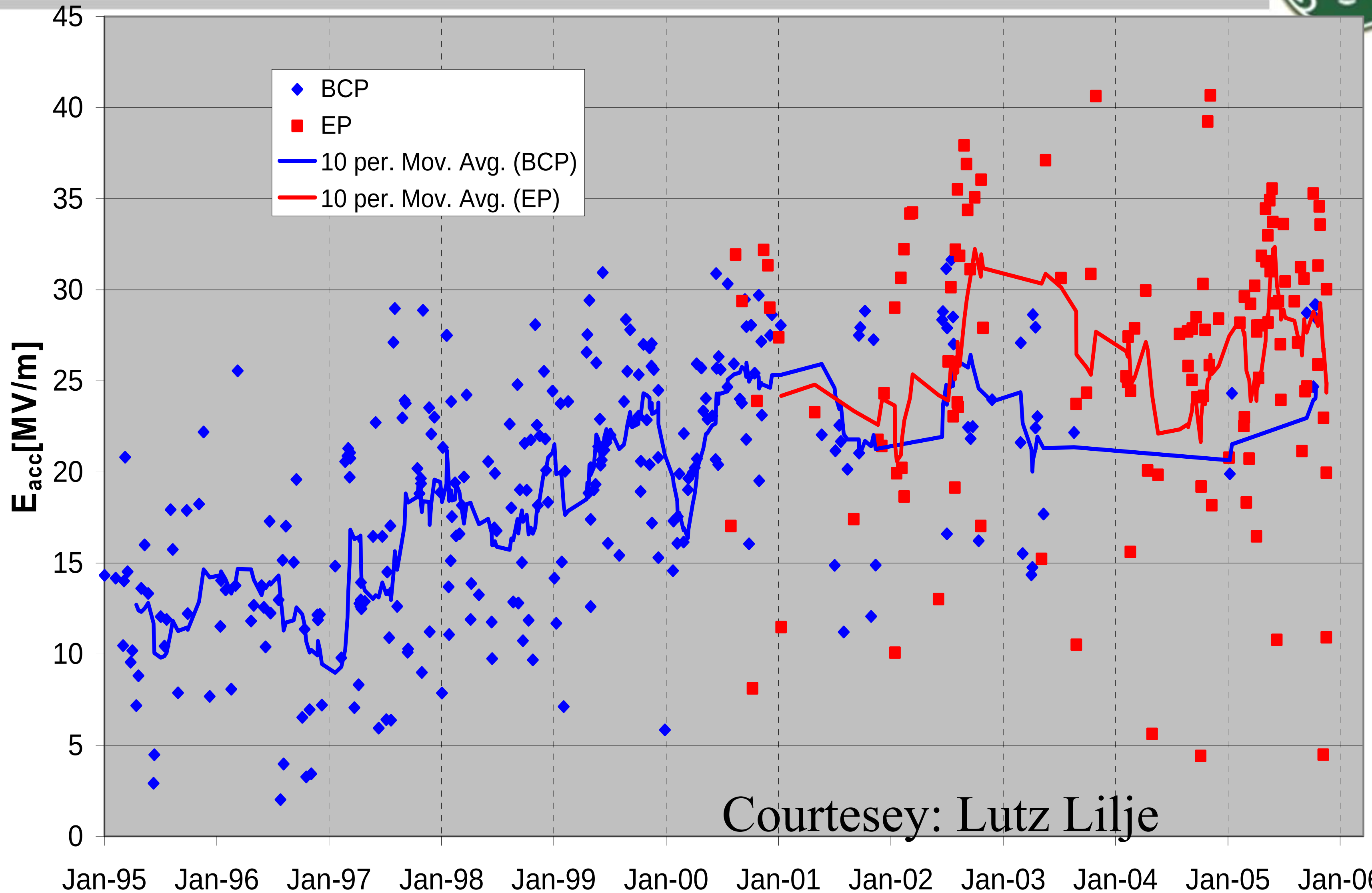
- It’s the loaded quality factor that gives the effective resonance width that the RF system, and its controls, seen from the superconducting cavity
- Chosen to minimize operating RF power: current matching (CEBAF, FEL), rf control performance and microphonics (SNS, ERLs)

Q_0 vs. Gradient for Several 1300 MHz Cavities



Courtesey: Lutz Lilje

E_{acc} vs. time


BCP
EP
10 per. Mov. Avg. (BCP)
10 per. Mov. Avg. (EP)
E_{acc}[MV/m]
0
5
10
15
20
25
30
35
40
45
Jan-95
Jan-96
Jan-97
Jan-98
Jan-99
Jan-00
Jan-01
Jan-02
Jan-03
Jan-04
Jan-05
Jan-06
Courtesey: Lutz Lilje

RF Cavity Equations

- Introduction
- Cavity Fundamental Parameters
- RF Cavity as a Parallel LCR Circuit
- Coupling of Cavity to an rf Generator
- Equivalent Circuit for a Cavity with Beam Loading
  - On Crest and on Resonance Operation
  - Off Crest and off Resonance Operation
    - Optimum Tuning
    - Optimum Coupling
- RF cavity with Beam and Microphonics
- Q_{ext} Optimization under Beam Loading and Microphonics
- RF Modeling
- Conclusions

Introduction

- Goal: Ability to predict rf cavity's steady-state response and develop a differential equation for the transient response

- We will construct an equivalent circuit and analyze it

- We will write the quantities that characterize an rf cavity and relate them to the circuit parameters, for

  a) a cavity

  b) a cavity coupled to an rf generator

  c) a cavity with beam

RF Cavity Fundamental Quantities

- Quality Factor Q_0:

$$Q_0 \equiv \frac{\omega_0 W}{P_{diss}} = \frac{\text{Energy stored in cavity}}{\text{Energy dissipated in cavity walls per radian}}$$

- Shunt impedance R_a:

$$R_a \equiv \frac{V_a^2}{P_{diss}} \quad \text{in ohms per cell}$$

(accelerator definition); V_a = accelerating voltage

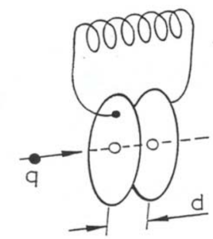
- Note: Voltages and currents will be represented as complex quantities, denoted by a tilde. For example:

$$V_c(t) = \mathrm{Re}\left\{\tilde{V}_c(t)e^{i\omega t}\right\} \qquad \tilde{V}_c(t) = V_c e^{i\phi(t)}$$

where $V_c = \left|\tilde{V}_c\right|$ is the magnitude of $\tilde{V}_c$ and ϕ is a slowly varying phase.

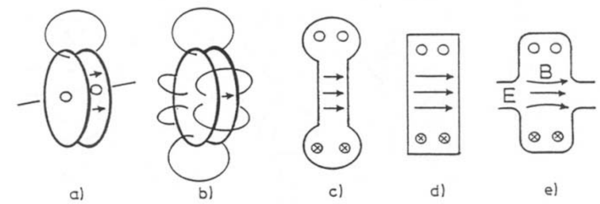
Equivalent Circuit for an rf Cavity

Simple LC circuit representing an accelerating resonator.



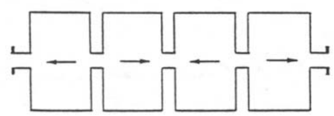
Simple lumped L-C circuit repesenting an accelerating resonator.
$\omega_o^2 = 1/LC$

Metamorphosis of the LC circuit into an accelerating cavity.

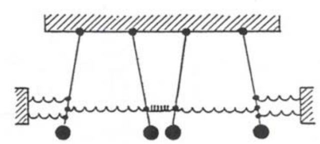


Metamorphosis of the L-C circuit of Fig. 1 into an accelerating cavity (after R.P.Feynman[33]). Fig. 5d shows the cylindrical "pillbox cavity" and Fig. 5e a slightly modified pillbox cavity with beam holes (typical β between 0.5 and 1.0). Fig. 5c resembles a low β version of the pillbox variety ($0.2<\beta<0.5$).

Chain of weakly coupled pillbox cavities representing an accelerating cavity.
Chain of coupled pendula as its mechanical analogue.



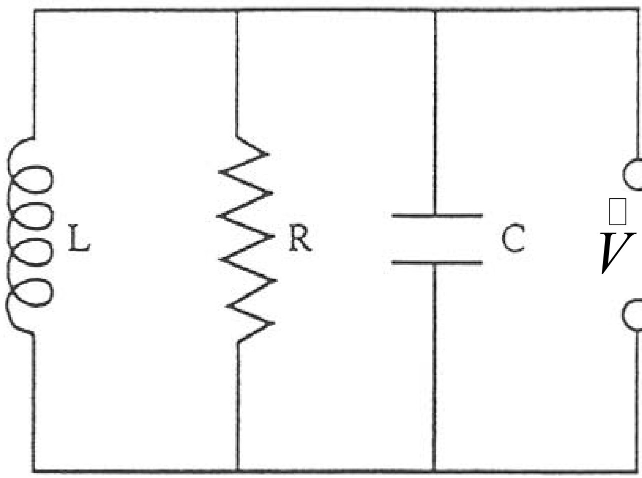
Chain of weakly-coupled pillbox cavities representing an accelerating module



Chain of coupled pendula as a mechanical analogue to Fig. 6a

Equivalent Circuit for an rf Cavity

- An rf cavity can be represented by a parallel LCR circuit:



$$\vec{V}_c(t) = \vec{v}_c\, e^{i\omega t}$$

- Impedance Z of the equivalent circuit: $\tilde{Z} = \left[\frac{1}{R} + \frac{1}{iL\omega} + iC\omega\right]^{-1}$

- Resonant frequency of the circuit: $\omega_0 = 1/\sqrt{LC}$

- Stored energy W: $W = \frac{1}{2} C V_c^2$

Equivalent Circuit for an rf Cavity

- Power dissipated in resistor R: $$P_{diss} = \frac{1}{2}\frac{V_c^2}{R}$$

$$R_a \equiv \frac{V_a^2}{P_{diss}} \qquad \therefore\; R_a = 2R$$

- From definition of shunt impedance

$$Q_0 \equiv \frac{\omega_0 W}{P_{diss}} = \omega_0 CR$$

- Quality factor of resonator:

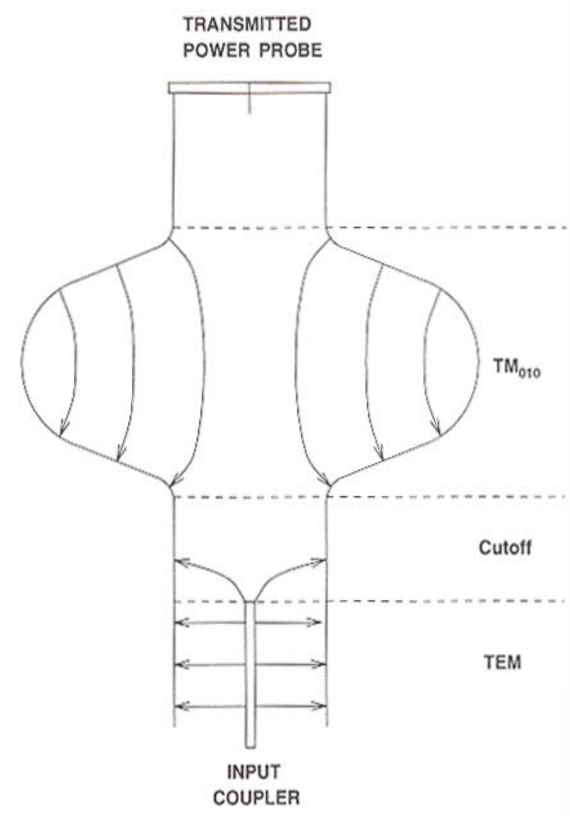
Wiedemann 16.13

- Note: $$\tilde{Z} = R\left[1 + iQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]^{-1}$$

For $\omega \approx \omega_0$, $$\tilde{Z} \approx R\left[1 + 2iQ_0\left(\frac{\omega - \omega_0}{\omega_0}\right)\right]^{-1}$$

Cavity with External Coupling

- Consider a cavity connected to an rf source
- A coaxial cable carries power from an rf source to the cavity
- The strength of the input coupler is adjusted by changing the penetration of the center conductor
- There is a fixed output coupler, the *transmitted power probe*, which picks up power transmitted through the cavity



Cavity with External Coupling (cont'd)

Consider the rf cavity after the rf is turned off.

Stored energy W satisfies the equation: $\frac{dW}{dt} = -P_{tot}$

Total power being lost, $P_{tot,}$ is: $P_{tot} = P_{diss} + P_e + P_t$

P_e is the power leaking back out the input coupler. P_t is the power coming out the transmitted power coupler. Typically P_t is very small $\Rightarrow P_{tot} \approx P_{diss} + P_e$

$$Q_0 \equiv \frac{\omega_0 W}{P_{diss}}$$

Recall

$$\frac{dW}{dt} = -\frac{\omega_0 W}{Q_L} \quad \Rightarrow \quad W = W_0 e^{-\frac{\omega_0 t}{Q_L}}$$

Similarly define a "loaded" quality factor Q_L: $Q_L \equiv \frac{\omega_0 W}{P_{tot}}$

Energy in the cavity decays exponentially with time constant:

$$\tau_L = \frac{Q_L}{\omega_0}$$

Cavity with External Coupling (cont'd)

Equation

$$\frac{P_{tot}}{\omega_0 W} = \frac{P_{diss} + P_e}{\omega_0 W}$$

suggests that we can assign a quality factor to each loss mechanism, such that

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_e}$$

where, by definition,

$$Q_e \equiv \frac{\omega_0 W}{P_e}$$

Typical values for CEBAF 7-cell cavities: $Q0=1\mathrm{x}10^{10}$, $Qe \approx QL=2\mathrm{x}10^{7}$.

Cavity with External Coupling (cont'd)

- Define "coupling parameter":

$$\beta \equiv \frac{Q_0}{Q_e}$$

therefore

$$\frac{1}{Q_L} = \frac{(1+\beta)}{Q_0}$$

Wiedemann 16.9

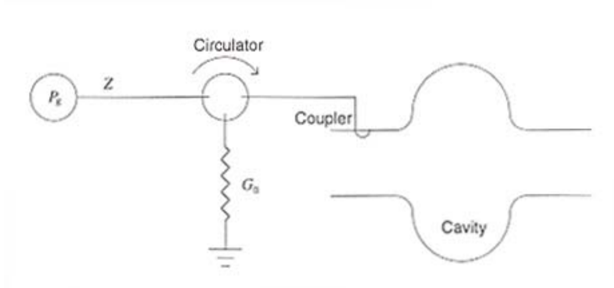
- β is equal to:

$$\beta = \frac{P_e}{P_{diss}}$$

It tells us how strongly the couplers interact with the cavity. Large β implies that the power leaking out of the coupler is large compared to the power dissipated in the cavity walls.

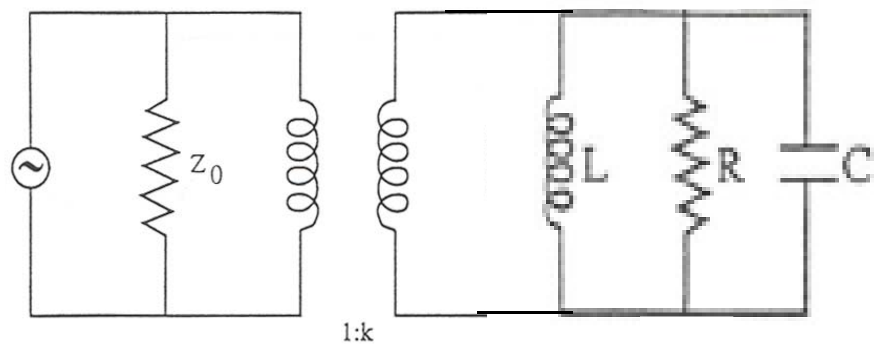
Cavity Coupled to an rf Source

- The system we want to model:



- Between the rf generator and the cavity is an isolator – a circulator connected to a load. Circulator ensures that signals coming from the cavity are terminated in a matched load.
- Equivalent circuit:

$$\tilde{I}_k(t) = \tilde{i}_k e^{i\omega t}$$



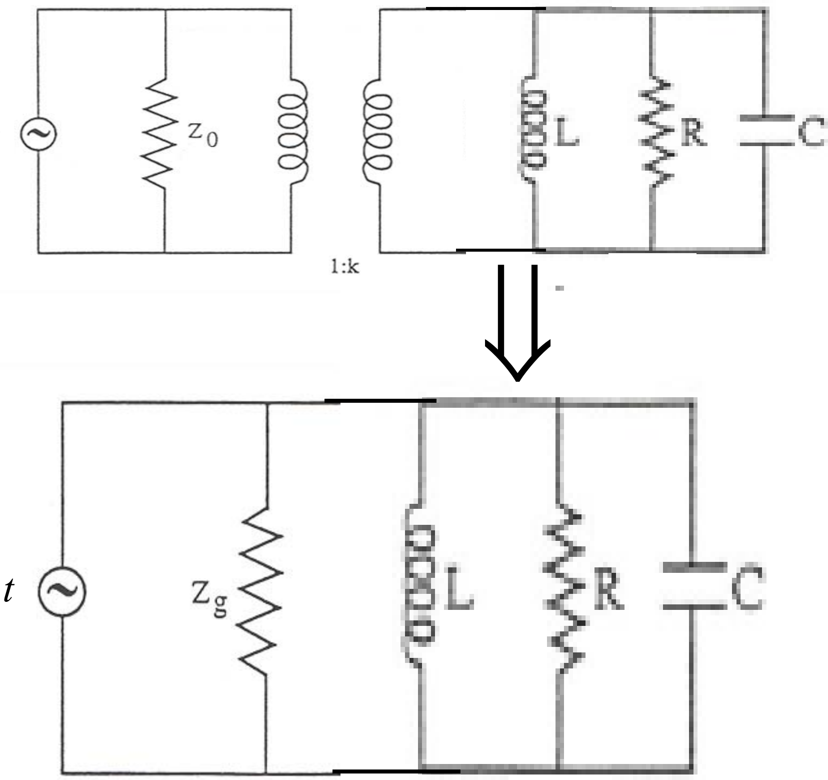
RF Generator + Circulator Coupler

Cavity

- Coupling is represented by an ideal transformer of turn ratio 1:k

Cavity Coupled to an rf Source

$I_k(t) = i_k e^{i\omega t}$



$I_g(t) = i_g e^{i\omega t}$

$$I_g = \frac{I_k}{k}$$

$$Z_g = k^2 Z_0$$

Wiedemann Fig. 16.1

By definition,

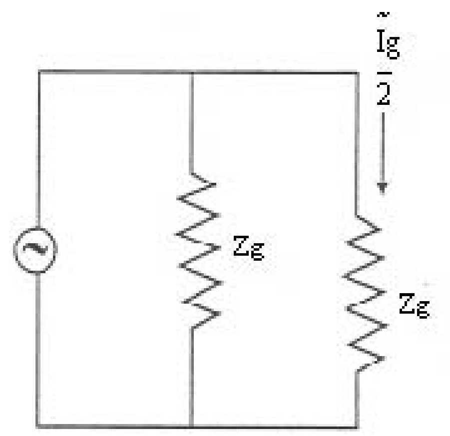
$$\beta \equiv \frac{R}{Z_g} = \frac{R}{k^2 Z_0} \quad \therefore \quad Z_g = \frac{R}{\beta}$$

Wiedemann 16.1

Generator Power

- When the cavity is matched to the input circuit, the power dissipation in the cavity is maximized.

$$\tilde{I}_g(t) = \tilde{i}_g e^{i\omega t}$$



$$P_{diss}^{\max} = \frac{1}{2} Z_g \left(\frac{I_g}{2} \right)^2 \quad \text{or} \quad P_{diss}^{\max} = \frac{1}{16\beta} R_a I_g^2 \equiv P_g$$

Wiedemann 16.6

- We define the available generator power P_g at a given generator current $\tilde{I}_g$ to be equal to P_{diss}^{max}.

Some Useful Expressions

- We derive expressions for W, P_{diss}, P_{refl}, in terms of cavity parameters

$$\frac{W}{P_g} = \frac{\frac{Q_0}{\omega_0} P_{diss}}{\frac{1}{16\beta} R_a I_g^2} = \frac{\frac{Q_0}{\omega_0}\frac{V_c^2}{R_a}}{\frac{1}{16\beta} R_a I_g^2} = \frac{16\beta}{R_a^2}\frac{Q_0}{\omega_0}\frac{V_c^2}{I_g^2}$$

$$V_c = I_g Z_{TOT}$$

$$Z_{TOT} = \left[\frac{1}{Z_g} + \frac{1}{Z}\right]^{-1}$$

$$Z_{TOT} = \frac{R_a}{2}\left[(1+\beta) + iQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]^{-1}$$

$$\therefore \quad W = 4\beta \frac{Q_0}{\omega_0} \frac{1}{(1+\beta)^2 + Q_0^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} P_g$$

$$\text{For} \quad \omega \approx \omega_0 \Rightarrow$$

$$W \approx \frac{4\beta}{(1+\beta)^2}\frac{Q_0}{\omega_0}\frac{1}{1 + \left[2\frac{Q_0}{(1+\beta)}\frac{\omega - \omega_0}{\omega_0}\right]^2} P_g$$

Some Useful Expressions (cont'd)

$$W \approx \frac{4\beta}{(1+\beta)^2}\frac{Q_0}{\omega_0}\frac{1}{1+\left[2\frac{Q_0}{(1+\beta)}\frac{\omega-\omega_0}{\omega_0}\right]^2}P_g$$

- Define "Tuning angle" Ψ:

$$\tan\Psi \equiv -Q_L\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right) \approx -2Q_L\frac{\omega-\omega_0}{\omega_0} \quad \text{for} \;\; \omega \approx \omega_0$$

$$\therefore$$

$$W = \frac{4\beta}{(1+\beta)^2}\frac{\mathrm{Q}_0}{\omega_0}\frac{1}{1+\tan^2\Psi}\mathrm{P}_\mathrm{g}$$

Wiedemann 16.12

- Recall:

$$P_{diss} = \frac{\omega_0 W}{Q_0}$$

$$\therefore$$

$$P_{diss} = \frac{4\beta}{(1+\beta)^2}\frac{1}{1+\tan^2\Psi}P_g$$

Some Useful Expressions (cont'd)

- Optimal coupling: W/Pg maximum or $Pdiss = Pg$

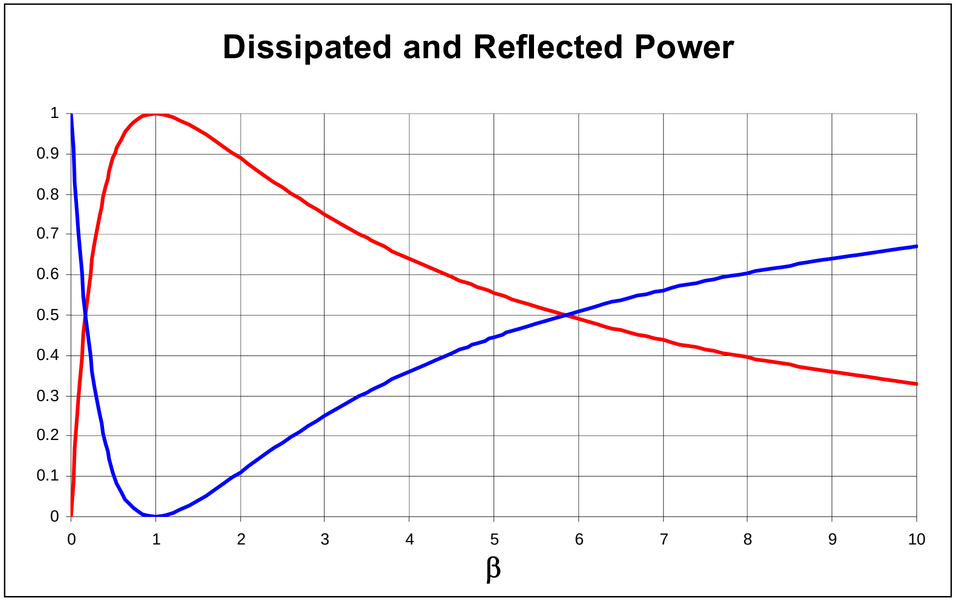
  which implies $\Delta\omega = 0,\ \beta = 1$

  this is the case of critical coupling

- Reflected power is calculated from energy conservation:

$$P_{refl} = P_g - P_{diss}$$

$$P_{refl} = P_g\left[1 - \frac{4\beta}{(1+\beta)^2}\frac{1}{1+\tan^2\Psi}\right]$$



- On resonance:

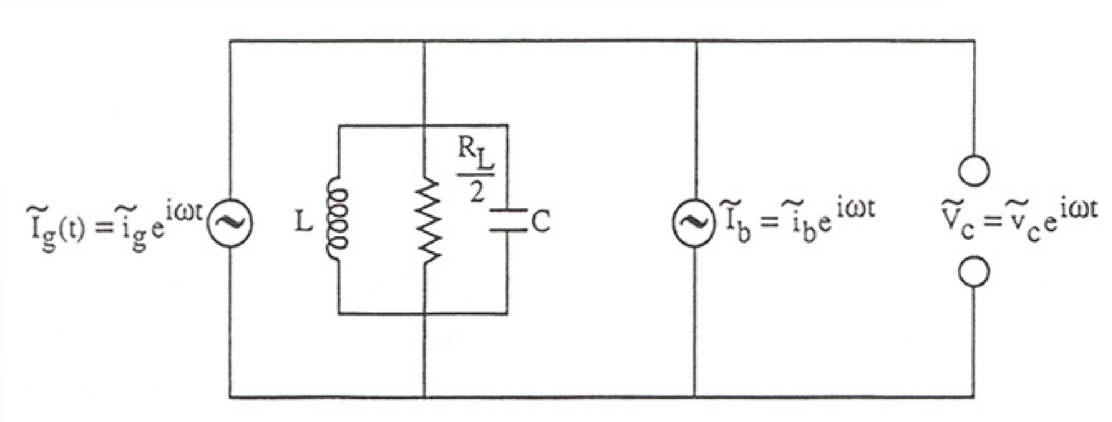
$$W = \frac{4\beta}{(1+\beta)^2}\frac{Q_0}{\omega_0}P_g$$

$$P_{diss} = \frac{4\beta}{(1+\beta)^2}P_g$$

$$P_{refl} = \left(\frac{1-\beta}{1+\beta}\right)^2 P_g$$

Equivalent Circuit for a Cavity with Beam

- Beam in the rf cavity is represented by a current generator.
- Equivalent circuit:



$$i_C = C\frac{dv_C}{dt}, \quad i_R = \frac{v_C}{R_L/2}, \quad v_C = L\frac{di_L}{dt}$$

- Differential equation that describes the dynamics of the system:

- R_L is the loaded impedance defined as: $R_L = \dfrac{R_a}{(1+\beta)}$

Equivalent Circuit for a Cavity with Beam (cont'd)

- Kirchoff's law:

$$\tilde{i}_L + \tilde{i}_R + \tilde{i}_C = \tilde{i}_g - \tilde{i}_b$$

- Total current is a superposition of generator current and beam current and beam current opposes the generator current.

$$\frac{d^2\tilde{v}_c}{dt^2} + \frac{\omega_0}{Q_L}\frac{d\tilde{v}_c}{dt} + \omega_0^2\tilde{v}_c = \frac{\omega_0 R_L}{2Q_L}\frac{d}{dt}\left(\tilde{i}_g - \tilde{i}_b\right)$$

- Assume that $\tilde{v}_c, \tilde{i}_g, \tilde{i}_b$ have a fast (rf) time-varying component and a slow varying component:

$$\tilde{v}_c = \tilde{V}_c e^{i\omega t}$$

$$\tilde{i}_g = \tilde{I}_g e^{i\omega t}$$

$$\tilde{i}_b = \tilde{I}_b e^{i\omega t}$$

where ω is the generator angular frequency and $\tilde{V}_c, \tilde{I}_g, \tilde{I}_b$ are complex quantities.

Equivalent Circuit for a Cavity with Beam (cont'd)

- Neglecting terms of order $\frac{d^2\tilde{V}_c}{dt^2}, \frac{d\tilde{I}}{dt}, \frac{1}{Q_L}\frac{d\tilde{V}_c}{dt}$ we arrive at:

$$\boxed{\frac{d\tilde{V}_c}{dt} + \frac{\omega_0}{2Q_L}(1 - i\tan\Psi)\tilde{V}_c = \frac{\omega_0 R_L}{4Q_L}(\tilde{I}_g - \tilde{I}_b)}$$

where Ψ is the tuning angle.

- For short bunches: $|\tilde{I}_b| \approx 2I_0$ where I_0 is the average beam current.

Wiedemann 16.19

Equivalent Circuit for a Cavity with Beam (cont'd)

$$\frac{d\tilde{V}_c}{dt}+\frac{\omega_0}{2Q_L}(1-i\tan\Psi)\tilde{V}_c=\frac{\omega_0 R_L}{4Q_L}(\tilde{I}_g-\tilde{I}_b)$$

- At steady-state: $\tilde{V}_c=\dfrac{R_L/2}{(1-i\tan\Psi)}\tilde{I}_g-\dfrac{R_L/2}{(1-i\tan\Psi)}\tilde{I}_b$

or $\tilde{V}_c=\dfrac{R_L}{2}\tilde{I}_g\cos\Psi e^{i\Psi}-\dfrac{R_L}{2}\tilde{I}_b\cos\Psi e^{i\Psi}$

or $\tilde{V}_c=\boxed{\tilde{V}_{gr}\cos\Psi e^{i\Psi}}+\boxed{\tilde{V}_{br}\cos\Psi e^{i\Psi}}$

or $\tilde{V}_c=\tilde{V}_g+\tilde{V}_b$

$\left\{\begin{aligned}\tilde{V}_{gr}&=\frac{R_L}{2}\tilde{I}_g\\ \tilde{V}_{br}&=-\frac{R_L}{2}\tilde{I}_b\end{aligned}\right\}$ are the generator and beam-loading voltages on resonance

and $\left\{\begin{matrix}\tilde{V}_g\\ \tilde{V}_b\end{matrix}\right\}$ are the generator and beam-loading voltages.

Equivalent Circuit for a Cavity with Beam (cont'd)

- Note that:

$$|\tilde{V}_{gr}| = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L} \approx 2\sqrt{P_g R_L} \quad \text{for large } \beta$$

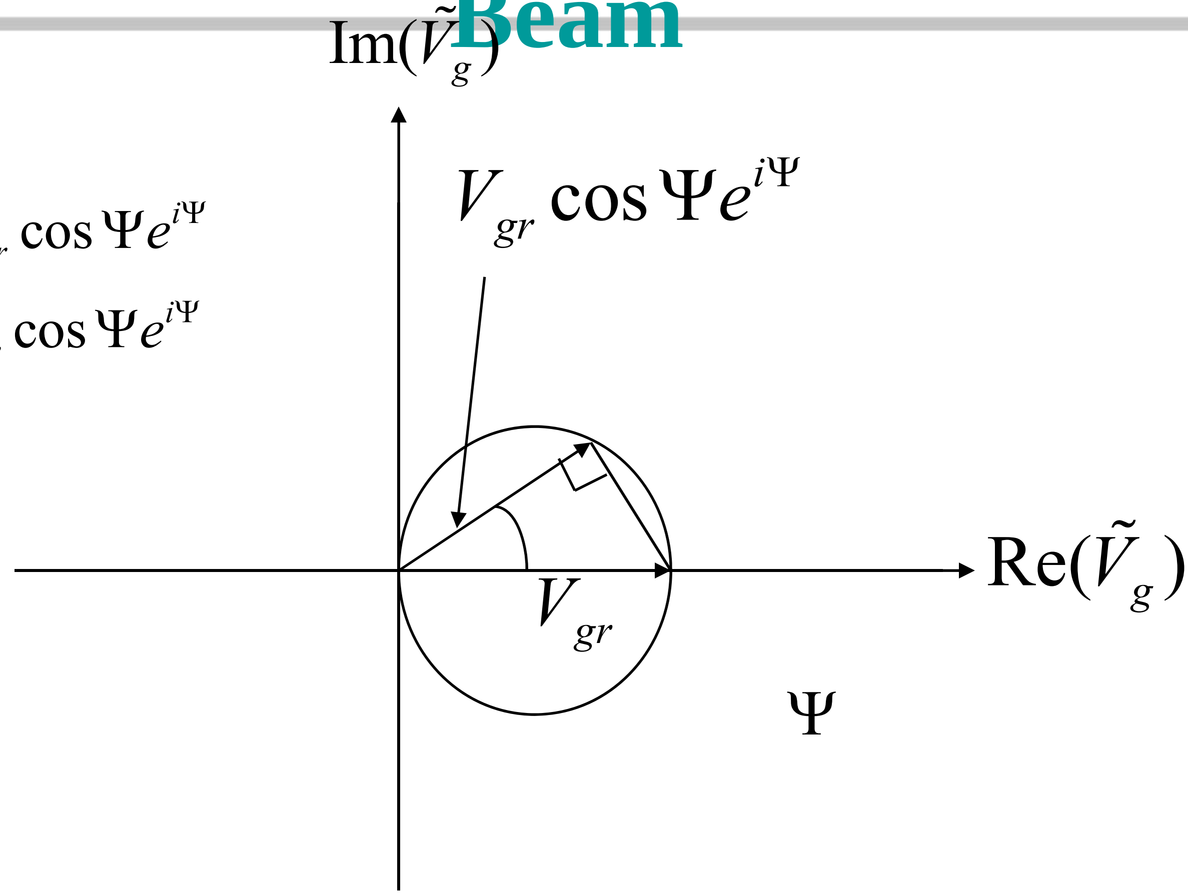
Wiedemann 16.16

$$|\tilde{V}_{br}| = R_L I_0$$

Wiedemann 16.20

Equivalent Circuit for a Cavity with Beam

$$\tilde{V}_g = \tilde{V}_{gr}\cos\Psi e^{i\Psi}$$
$$\tilde{V}_b = \tilde{V}_{br}\cos\Psi e^{i\Psi}$$



As Ψ increases the magnitude of both V_g and V_b decreases while their phases rotate by Ψ.

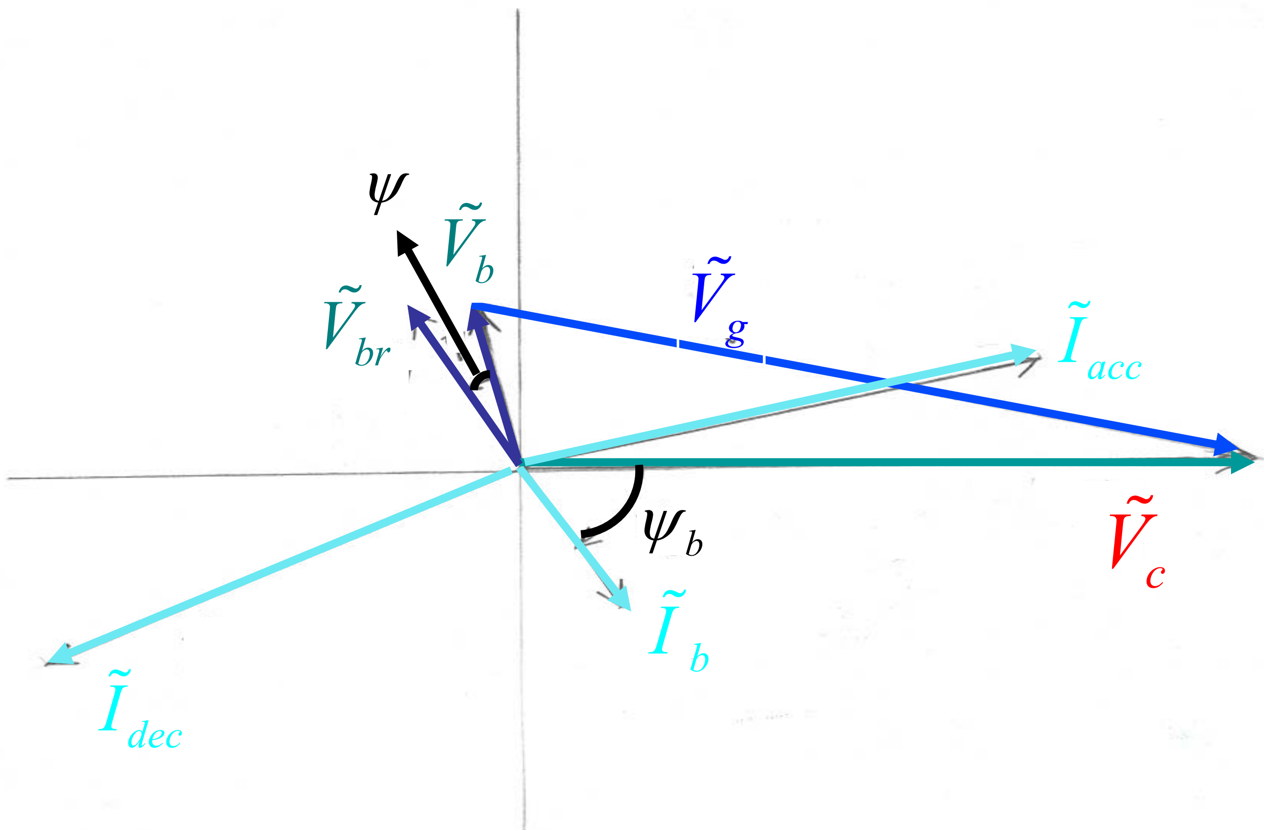
Equivalent Circuit for a Cavity with Beam (cont'd)

$$\tilde{V}_c = \tilde{V}_g + \tilde{V}_b$$

- Cavity voltage is the superposition of the generator and beam-loading voltage.

- This is the basis for the vector diagram analysis.

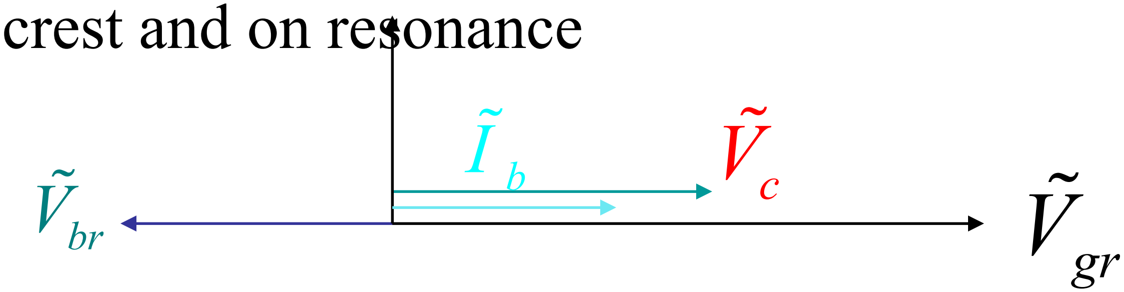
Example of a Phasor Diagram



Wiedemann
Fig. 16.3

On Crest and On Resonance Operation

- Typically linacs operate on resonance and on crest in order to receive maximum acceleration.
- On crest and on resonance



$$\Rightarrow \quad V_a = V_{gr} - V_{br}$$

where V_a is the accelerating voltage.

More Useful Equations

- We derive expressions for W, V_a, P_{diss}, P_{refl} in terms of β and the loading parameter K, defined by: $K = I_0/2\sqrt{R_a/P_g}$

From:

$$|\tilde{V}_{gr}| = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\sqrt{P_g R_L}$$

$$|\tilde{V}_{br}| = R_L I_0$$

$$V_a = V_{gr} - V_{br}$$

$$\Rightarrow$$

$$V_a = \sqrt{P_g R_a}\left\{\frac{2\sqrt{\beta}}{\sqrt{1+\beta}}\left(1 - \frac{K}{\sqrt{\beta}}\right)\right\}$$

$$W = \frac{4\beta}{(1+\beta)^2}\frac{Q_0}{\omega_0}\left(1 - \frac{K}{\sqrt{\beta}}\right)^2 P_g$$

$$P_{diss} = \frac{4\beta}{(1+\beta)^2}\left(1 - \frac{K}{\sqrt{\beta}}\right)^2 P_g$$

$$I_0 V_a = I_0\sqrt{R_a P_{diss}}$$

$$\eta \equiv \frac{I_0 V_a}{P_g} = \frac{2\sqrt{\beta}}{\sqrt{1+\beta}} 2K\left(1 - \frac{K}{\sqrt{\beta}}\right)$$

$$P_{refl} = P_g - P_{diss} - I_0 V_a \;\Rightarrow\; P_{refl} = \frac{\left[(\beta - 1) - 2K\sqrt{\beta}\right]^2}{(\beta+1)^2} P_g$$

More Useful Equations (cont'd)

- For β large,

$$P_g \approx \frac{1}{4R_L}(V_a + I_0 R_L)^2$$

$$P_{refl} \approx \frac{1}{4R_L}(V_a - I_0 R_L)^2$$

- For P_{refl}=0 (condition for matching) $\Rightarrow$

$$R_L = \frac{V_a^M}{I_0^M}$$

and

$$P_g \approx \frac{I_0^M V_a^M}{4}\left(\frac{V_a}{V_a^M} + \frac{I_0}{I_0^M}\right)^2$$

Example

- For V_a=20 MV/m, L=0.7 m, $Q_L=2\text{x}10^7$, $Q_0=1\text{x}10^{10}$:

Power	$I_0 = 0$	$I_0 = 100\ \mu A$	$I_0 = 1$ mA
P_g	3.65 kW	4.38 kW	14.033 kW
P_{diss}	29 W	29 W	29 W
I_0V_a	0 W	1.4 kW	14 kW
P_{refl}	3.62 kW	2.951 kW	~ 4.4 W

Off Crest and Off Resonance Operation

- Typically electron storage rings operate off crest in order to ensure stability against phase oscillations.
- As a consequence, the rf cavities must be detuned off resonance in order to minimize the reflected power and the required generator power.
- Longitudinal gymnastics may also impose off crest operation operation in recirculating linacs.
- We write the beam current and the cavity voltage as

$$\tilde{I}_b = 2I_0 e^{i\psi_b}$$

$$\tilde{V}_c = V_c e^{i\psi_c} \quad \text{and set } \psi_c = 0$$

- The generator power can then be expressed as:

$$\boxed{P_g = \frac{V_c^2}{R_L}\frac{(1+\beta)}{4\beta}\left\{\left[1+\frac{I_0 R_L}{V_c}\cos\psi_b\right]^2 + \left[\tan\Psi - \frac{I_0 R_L}{V_c}\sin\psi_b\right]^2\right\}}$$

Wiedemann 16.31

Off Crest and Off Resonance Operation

- Condition for optimum tuning:

$$\tan \Psi = \frac{I_0 R_L}{V_c} \sin \psi_b$$

- Condition for optimum coupling:

$$\beta_{\mathrm{opt}} = 1 + \frac{I_0 R_a}{V_c} \cos \psi_b$$

- Minimum generator power:

$$P_{g,\min} = \frac{V_c^2 \beta_{\mathrm{opt}}}{R_a}$$

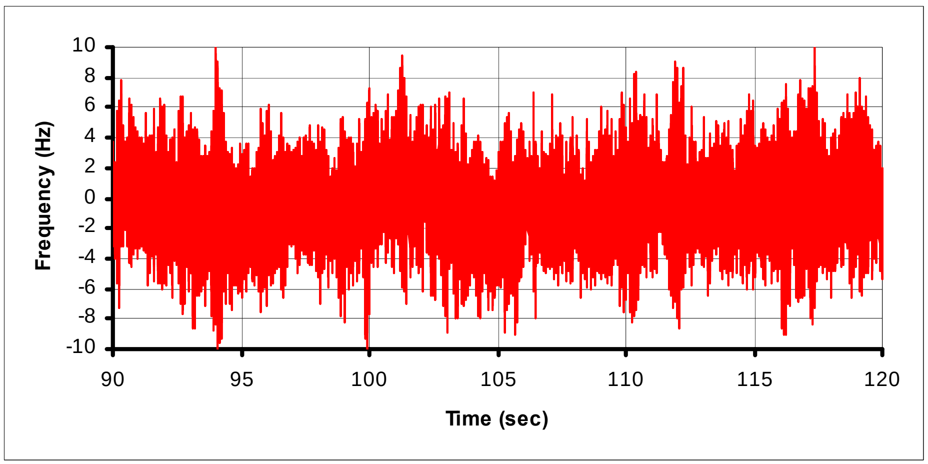
Wiedemann
16.36

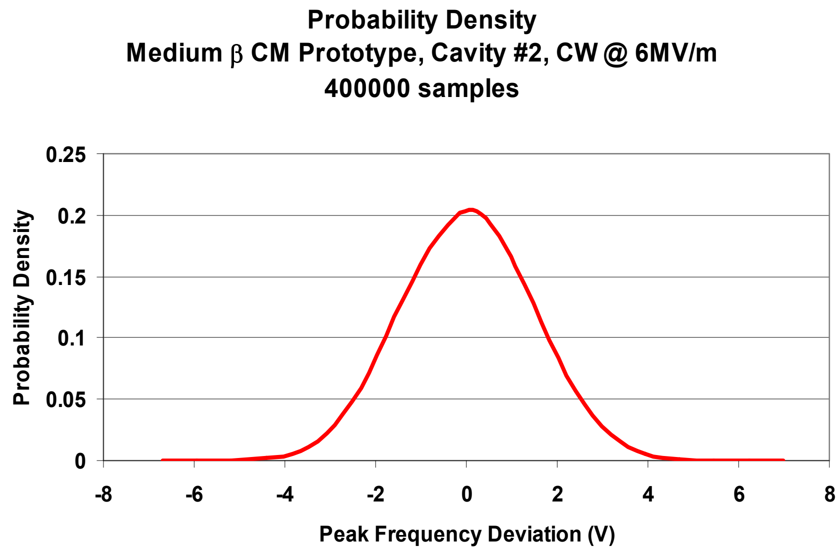
RF Cavity with Beam and Microphonics

The detuning is now: $\tan\Psi = -2Q_L \dfrac{\delta f_0 \pm \delta f_m}{f_0}$ $\qquad$ $\tan\psi_0 = -2Q_L \dfrac{\delta f_0}{f_0}$

where δf_0 is the static detuning (controllable)

and δf_m is the random dynamic detuning (uncontrollable)





Q_{ext} Optimization under Beam Loading and Microphonics

- Beam loading and microphonics require careful optimization of the external Q of cavities.
- Derive expressions for the optimum setting of cavity parameters when operating under

  a) heavy beam loading

  b) little or no beam loading, as is the case in energy recovery linac cavities

  and in the presence of microphonics.

Q_{ext} Optimization (cont'd)

$$P_g = \frac{V_c^2}{R_L}\frac{(1+\beta)}{4\beta}\left\{\left[1+\frac{I_{tot}R_L}{V_c}\cos\psi_{tot}\right]^2+\left[\tan\Psi-\frac{I_{tot}R_L}{V_c}\sin\psi_{tot}\right]^2\right\}$$

$$\tan\Psi = -2Q_L\frac{\delta f}{f_0}$$

where δf is the total amount of cavity detuning in Hz, including static detuning and microphonics.

- Optimizing the generator power with respect to coupling gives:

$$\beta_{opt} = \sqrt{(b+1)^2+\left[2Q_0\frac{\delta f}{f_0}+b\tan\psi_{tot}\right]^2}$$

$$\text{where}\quad b \equiv \frac{I_{tot}R_a}{V_c}\cos\psi_{tot}$$

where I_{tot} is the magnitude of the resultant beam current vector in the cavity and ψ_{tot} is the phase of the resultant beam vector with respect to the cavity voltage.

Q_{ext} Optimization (cont'd)

$$P_g = \frac{V_c^2}{R_L}\frac{(1+\beta)}{4\beta}\left\{\left[1+\frac{I_{tot}R_L}{V_c}\cos\psi_{tot}\right]^2+\left[\tan\Psi-\frac{I_{tot}R_L}{V_c}\sin\psi_{tot}\right]^2\right\}$$

where: $$\tan\Psi = -2Q_L\frac{\delta f_0+\delta f_m}{f_0}$$

- To minimize generator power with respect to tuning:

$$\delta f_0 = -\frac{f_0}{2Q_0}b\tan\Psi$$

$$\Rightarrow \quad P_g = \frac{V_c^2}{R_L}\frac{(1+\beta)}{4\beta}\left\{(1+b+\beta)^2+\left[2Q_0\frac{\delta f_m}{f_0}\right]^2\right\}$$

Q_{ext} Optimization (cont'd)

- Condition for optimum coupling:

$$\beta_{opt} = \sqrt{(b+1)^2 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}$$

and

$$P_g^{opt} = \frac{V_c^2}{2R_a}\left[b+1+\sqrt{(b+1)^2 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}\right]$$

- In the absence of beam (b=0):

$$\beta_{opt} = \sqrt{1 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}$$

and

$$P_g^{opt} = \frac{V_c^2}{2R_a}\left[1+\sqrt{1 + \left(2Q_0 \frac{\delta f_m}{f_0}\right)^2}\right]$$

Problem for the Reader

- Assuming no microphonics, plot β_{opt} and P^{g}_{opt} as function of b (beam loading), b=-5 to 5, and explain the results.

- How do the results change if microphonics is present?

Example

- ERL Injector and Linac:

  δf_m=25 Hz, Q_0=1x10^{10} , f_0=1300 MHz, I_0=100 mA, V_c=20 MV/m, L=1.04 m, R_a/Q_0=1036 ohms per cavity

- ERL linac: Resultant beam current, I_{tot} = 0 mA (energy recovery)

  and β_{opt}=385 $\Rightarrow$ Q_L=2.6x10^7 $\Rightarrow$ P_g = 4 kW per cavity.

- ERL Injector: I_0=100 mA and β_{opt}= 5x10^4 ! $\Rightarrow$ Q_L= 2x10^5 $\Rightarrow$ P_g = 2.08 MW per cavity!

  Note: $I_0 V_a$ = 2.08 MW $\Rightarrow$ optimization is entirely dominated by beam loading.

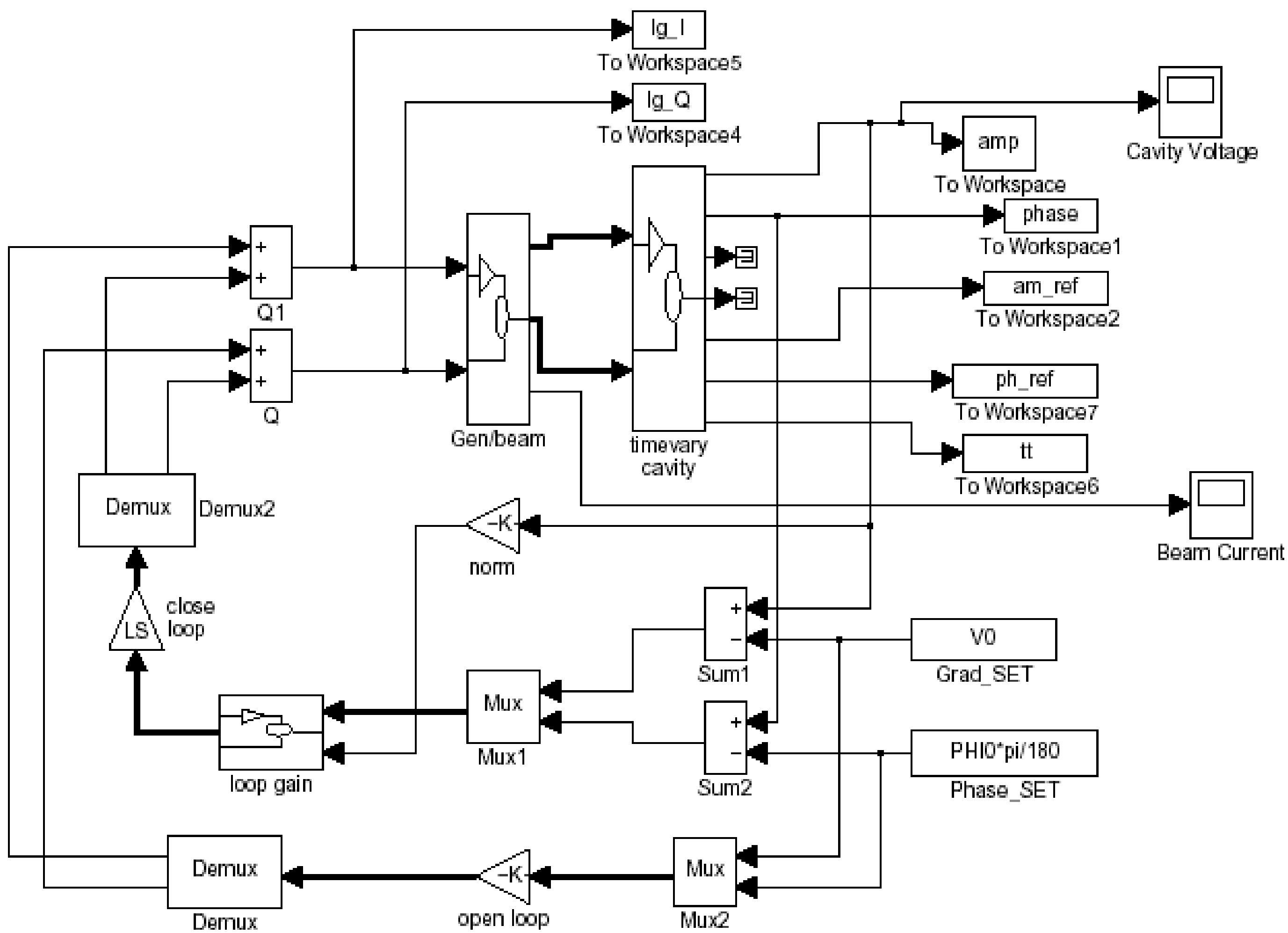
RF System Modeling

- To include amplitude and phase feedback, nonlinear effects from the klystron and be able to analyze transient response of the system, response to large parameter variations or beam current fluctuations

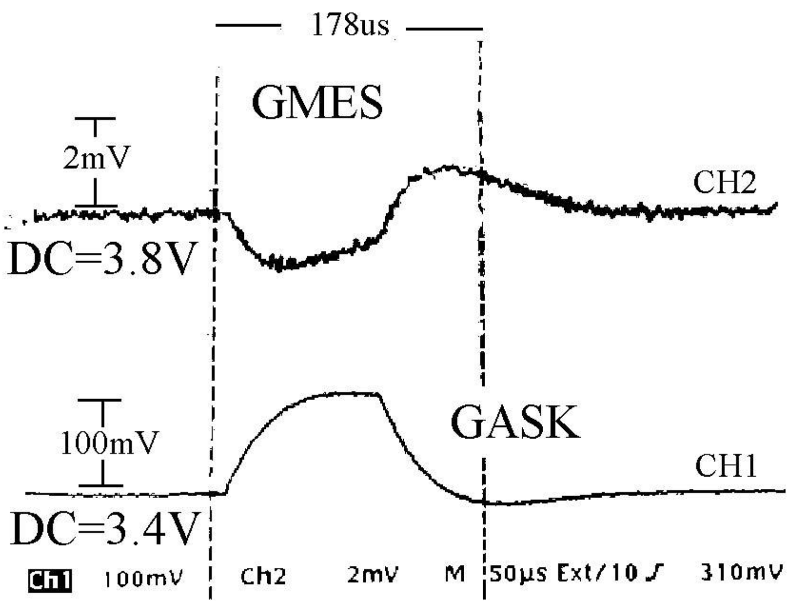
  - we developed a model of the cavity and low level controls using SIMULINK, a MATLAB-based program for simulating dynamic systems.

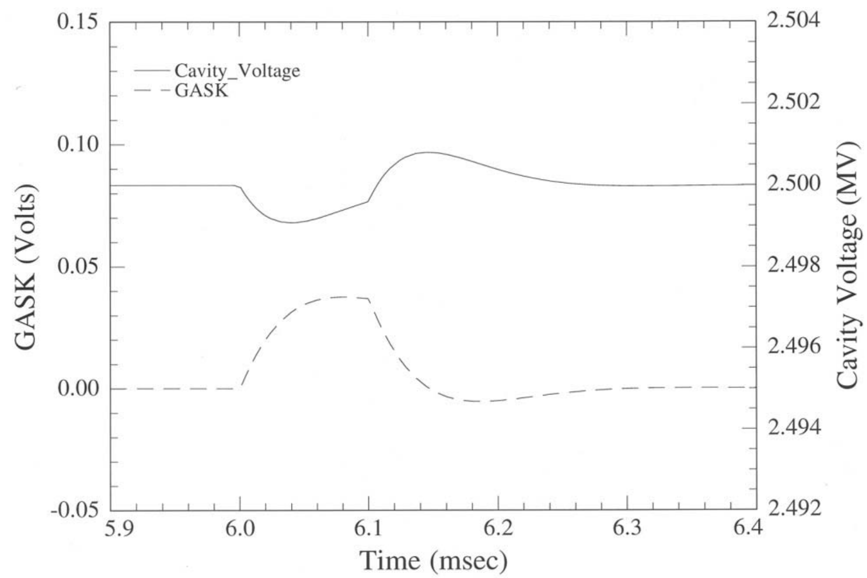
- Model describes the beam-cavity interaction, includes a realistic representation of low level controls, klystron characteristics, microphonic noise, Lorentz force detuning and coupling and excitation of mechanical resonances

RF System Model



RF Modeling: Simulations vs. Experimental Data





Measured and simulated cavity voltage and amplified gradient error signal (GASK) in one of CEBAF's cavities, when a 65 µA, 100 µsec beam pulse enters the cavity.

Conclusions

- We derived a differential equation that describes to a very good approximation the rf cavity and its interaction with beam.
- We derived useful relations among cavity's parameters and used phasor diagrams to analyze steady-state situations.
- We presented formula for the optimization of Q_{ext} under beam loading and microphonics.
- We showed an example of a Simulink model of the rf control system which can be useful when nonlinearities can not be ignored.

RF Focussing

In any RF cavity that accelerates longitudinally, because of Maxwell Equations there must be additional transverse electromagnetic fields. These fields will act to focus the beam and must be accounted properly in the beam optics, especially in the low energy regions of the accelerator. We will discuss this problem in greater depth in injector lectures. Let **A**(x,y,z) be the vector potential describing the longitudinal mode (Lorenz gauge)

$$\nabla \cdot \vec{A} = -\frac{1}{c}\frac{\partial \phi}{\partial t}$$

$$\nabla^2 \vec{A} = -\frac{\omega^2}{c^2}\vec{A} \qquad \nabla^2 \phi = -\frac{\omega^2}{c^2}\phi$$

For cylindrically symmetrical accelerating mode, functional form can only depend on r and z

$$A_z(r,z) = A_{z0}(z) + A_{z1}(z) r^2 + \ldots$$
$$\phi(r,z) = \phi_0(z) + \phi_1(z) r^2 + \ldots$$

Maxwell's Equations give recurrence formulas for succeeding approximations

$$(2n)^2 A_{zn} + \frac{d^2 A_{z,n-1}}{dz^2} = -\frac{\omega^2}{c^2} A_{z,n-1}$$
$$(2n)^2 \phi_n + \frac{d^2 \phi_{n-1}}{dz^2} = -\frac{\omega^2}{c^2} \phi_{n-1}$$

Gauge condition satisfied when

$$\frac{dA_{zn}}{dz} = -\frac{i\omega}{c}\phi_n$$

in the particular case $n = 0$

$$\frac{dA_{z0}}{dz} = -\frac{i\omega}{c}\phi_0$$

Electric field is

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \overleftarrow{A}}{\partial t}$$

And the potential and vector potential must satisfy

$$E_z(0,z) = -\frac{d\phi_0}{dz} - \frac{i\omega}{c} A_{z0}$$

$$\therefore \quad \frac{i\omega}{c} E_z(0,z) = \frac{d^2 A_{z0}}{dz^2} + \frac{\omega^2}{c^2} A_{z0} = -4 A_{z1}$$

So the magnetic field off axis may be expressed directly in terms of the electric field on axis

$$\therefore \quad B_\theta \approx -2rA_{z1} = \frac{i}{2}\frac{\omega r}{c} E_z(0,z)$$

And likewise for the radial electric field (see also $\nabla \cdot \vec{E} = 0$)

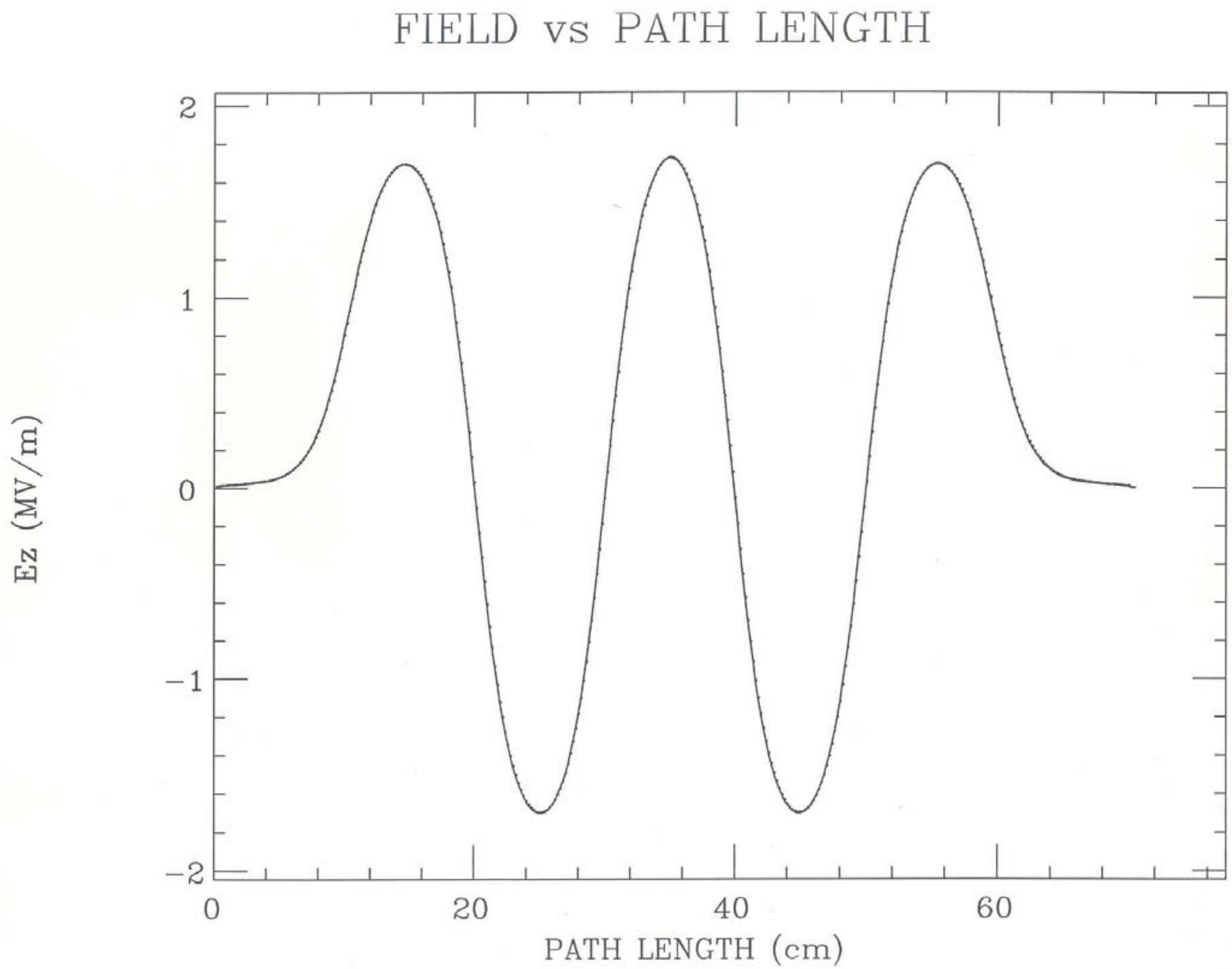
$$\therefore \quad E_r \approx -2r\phi_1(z) = -\frac{r}{2}\frac{dE_z(0,z)}{dz}$$

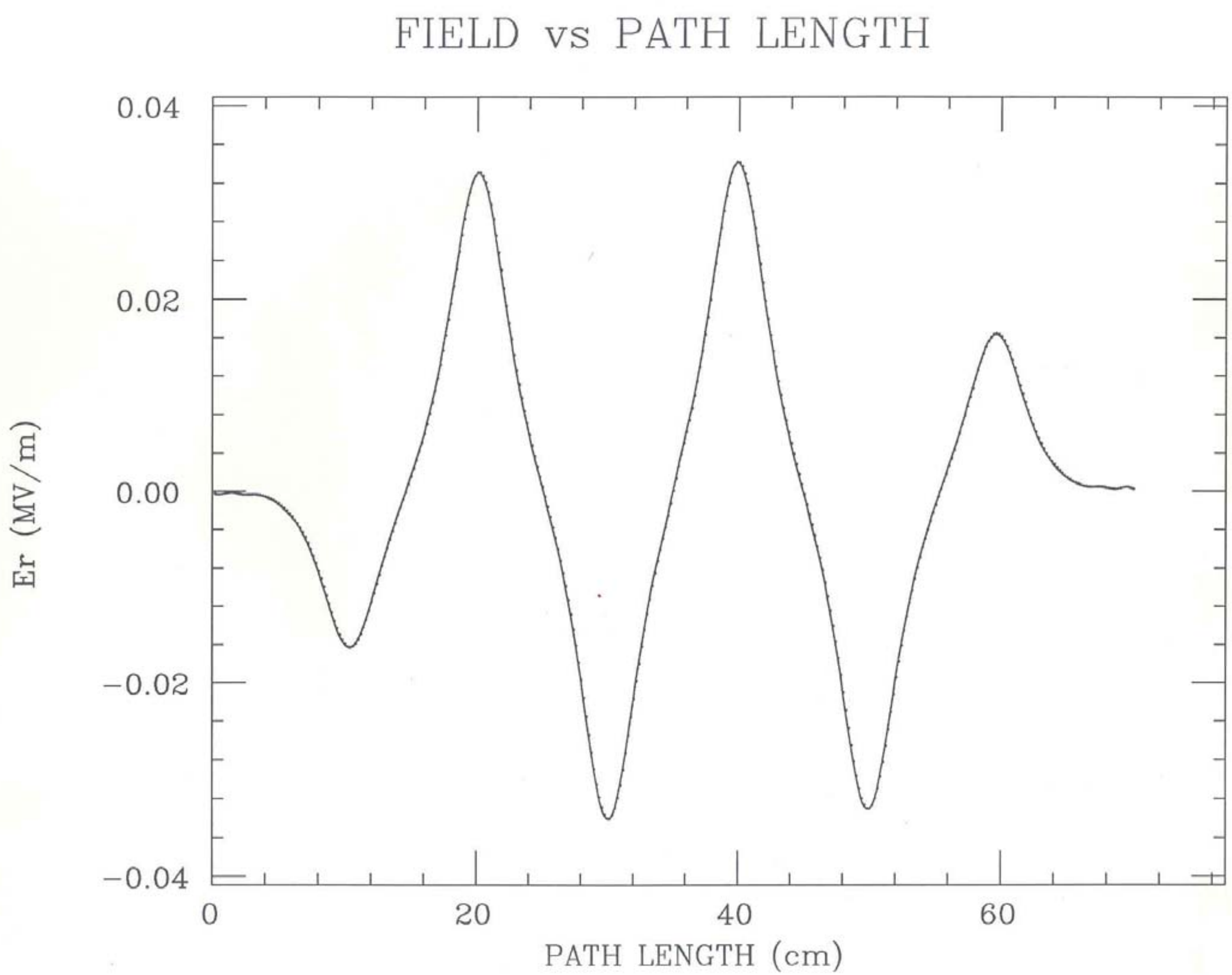
Explicitly, for the time dependence $\cos(\omega t + \delta)$

$$E_z(r,z,t) \approx E_z(0,z)\cos(\omega t + \delta)$$

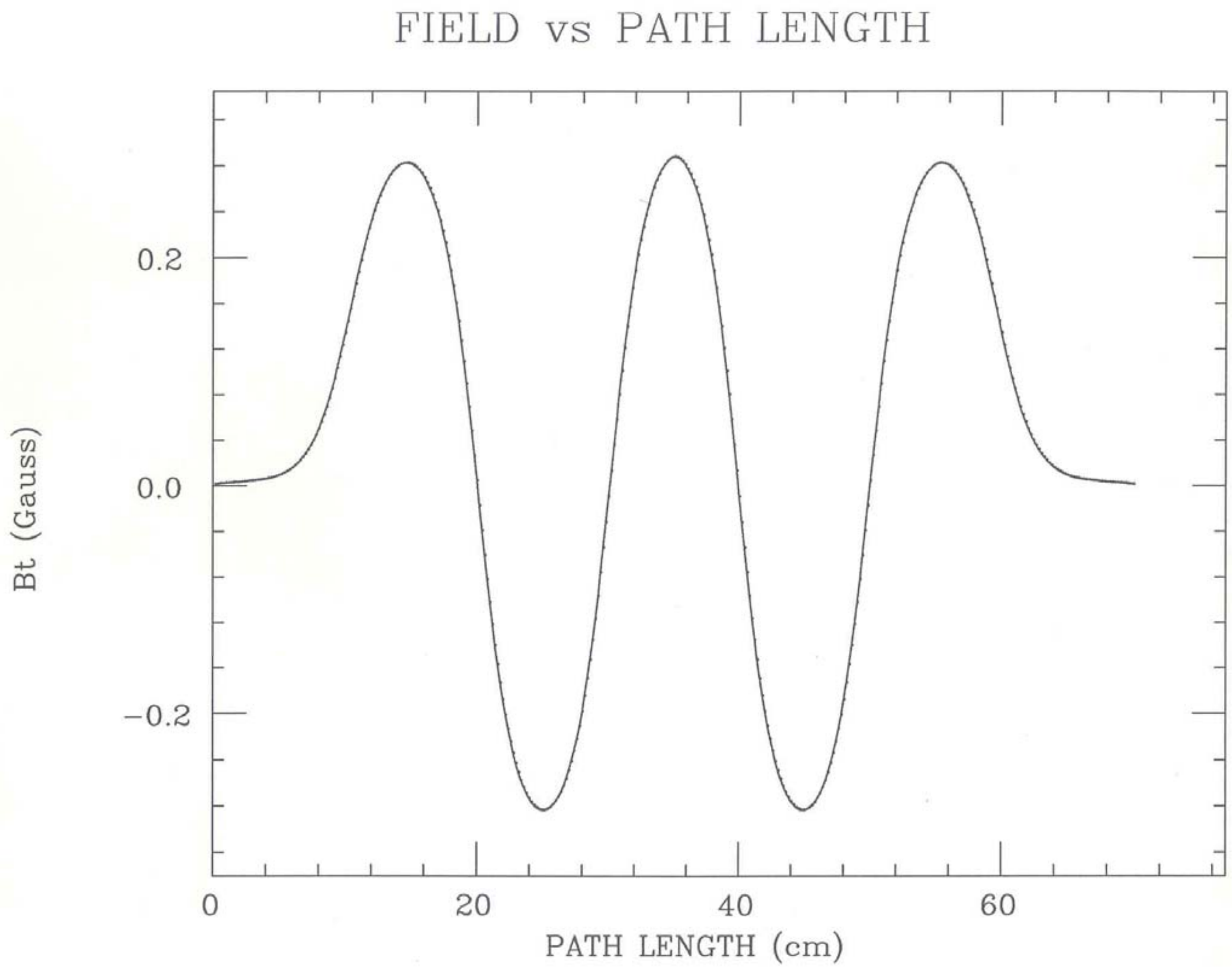
$$E_r(r,z,t) \approx -\frac{r}{2}\frac{dE_z(0,z)}{dz}\cos(\omega t + \delta)$$

$$B_\theta(r,z,t) \approx -\frac{\omega r}{2c}E_z(0,z)\sin(\omega t + \delta)$$


FIELD vs PATH LENGTH
Ez (MV/m)
PATH LENGTH (cm)
2
1
0
−1
−2
0
20
40
60



FIELD vs PATH LENGTH
0.04
0.02
0.00
−0.02
−0.04
Er (MV/m)
0
20
40
60
PATH LENGTH (cm)


FIELD vs PATH LENGTH
Bt (Gauss)
0.2
0.0
−0.2
0
20
40
60
PATH LENGTH (cm)

Motion of a particle in this EM field

$$\frac{d\left(\gamma m\vec{\mathrm{V}}\right)}{dt} = -e\left(\vec{E} + \frac{\vec{V}}{c}\times\vec{B}\right)$$

$$\gamma(z)\beta_x(z) = \gamma(-\infty)\beta_x(-\infty) + \int_{-\infty}^{z}\left[\begin{array}{c} -\dfrac{x(z')}{2}\dfrac{dG(z')}{dz'}\cos\left(\omega t(z')+\delta\right) \\ +\dfrac{\omega\beta_z(z')x(z')}{2c}G(z')\sin\left(\omega t(z')+\delta\right)\end{array}\right]\frac{dz'}{\beta_z(z')}$$

The normalized gradient is

$$G(z) = \frac{eE_z(z,0)}{mc^2}$$

and the other quantities are calculated with the integral equations

$$\gamma(z) = \gamma(-\infty) + \int_{-\infty}^{z} G(z')\cos(\omega t(z') + \delta)dz'$$

$$\gamma(z)\beta_z(z) = \gamma(-\infty)\beta_z(-\infty) + \int_{-\infty}^{z} \frac{G(z')}{\beta_z(z')}\cos(\omega t(z') + \delta)dz'$$

$$t(z) = \lim_{z_0 \to -\infty} \frac{z_0}{\beta_z(-\infty)c} + \int_{-\infty}^{z} \frac{dz'}{\beta_z(z')c}$$

These equations may be integrated numerically using the cylindrically symmetric CEBAF field model to form the Douglas model of the cavity focussing. In the high energy limit the expressions simplify.

$$x(z) = x(a) + \int_a^z \frac{\gamma(z')\beta_x(z')}{\gamma(z')\beta_z(z')} dz'$$

$$\approx x(a) + \frac{\beta_x(-\infty)}{\beta_z(-\infty)}(z-a) - \int_a^z \frac{x(z')}{2} \frac{G(z')}{\gamma(z')\beta_z^2(z')} \cos(\omega t(z') + \delta) dz'$$

Transfer Matrix

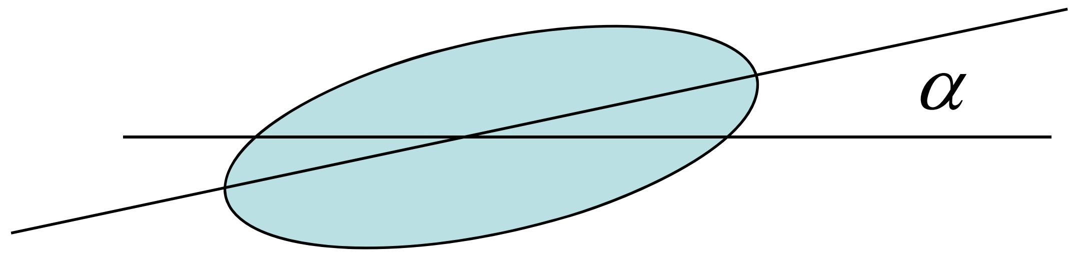
For position-momentum transfer matrix

$$T = \begin{pmatrix} 1 - \dfrac{E_G}{2E} & \dfrac{L}{\gamma} \\ -\dfrac{I}{4\gamma} & 1 + \dfrac{E_G}{2E} \end{pmatrix}$$

$$I = \cos^2(\delta) \int_{-\infty}^{\infty} G^2(z) \cos^2(\omega z / c) dz$$

$$+ \sin^2(\delta) \int_{-\infty}^{\infty} G^2(z) \sin^2(\omega z / c) dz$$

Kick Generated by mis-alignment



$$\Delta\gamma\beta = \frac{E_G\alpha}{2E}$$

Damping and Antidamping

By symmetry, if electron traverses the cavity exactly on axis, there is no transverse deflection of the particle, but there is an energy increase. By conservation of transverse momentum, there must be a decrease of the phase space area. For linacs NEVER use the word "adiabatic"

$$\frac{d\left(\gamma m \vec{\mathrm{V}}_{\text{transverse}}\right)}{dt} = 0$$

$$\gamma\left(z\right)\beta_x\left(z\right) = \gamma\left(-\infty\right)\beta_x\left(-\infty\right)$$

Conservation law applied to angles

$$\beta_x, \beta_y << \beta_z \approx 1$$

$$\theta_x = \beta_x / \beta_z \approx \beta_x \qquad \theta_y = \beta_y / \beta_z \approx \beta_y$$

$$\theta_x(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)}\theta_x(-\infty)$$

$$\theta_y(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)}\theta_y(-\infty)$$

Phase space area transformation

$$dx \wedge d\theta_x(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)} dx \wedge d\theta_x(-\infty)$$

$$dy \wedge d\theta_y(z) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)} dy \wedge d\theta_y(-\infty)$$

Therefore, if the beam is accelerating, the phase space area after the cavity is less than that before the cavity and if the beam is decelerating the phase space area is greater than the area before the cavity. The determinate of the transformation carrying the phase space through the cavity has determinate equal to

$$\mathrm{Det}\left(M_{cavity}\right) = \frac{\gamma(-\infty)\beta_z(-\infty)}{\gamma(z)\beta_z(z)}$$

By concatenation of the transfer matrices of all the accelerating or decelerating cavities in the recirculated linac, and by the fact that the determinate of the product of two matrices is the product of the determinates, the phase space area at each location in the linac is

$$dx \wedge d\theta_x(z) = \frac{\gamma(0)\beta_z(0)}{\gamma(z)\beta_z(z)} dx \wedge d\theta_x(0)$$

$$dy \wedge d\theta_y(z) = \frac{\gamma(0)\beta_z(0)}{\gamma(z)\beta_z(z)} dy \wedge d\theta_y(0)$$

Same type of argument shows that things like orbit fluctuations are damped/amplified by acceleration/deceleration.

Transfer Matrix Non-Unimodular

$$M_{tot} = M_1 \cdot M_2$$

$$P(M) \equiv \frac{M}{\det M}$$

$P(M)$ unimodular!

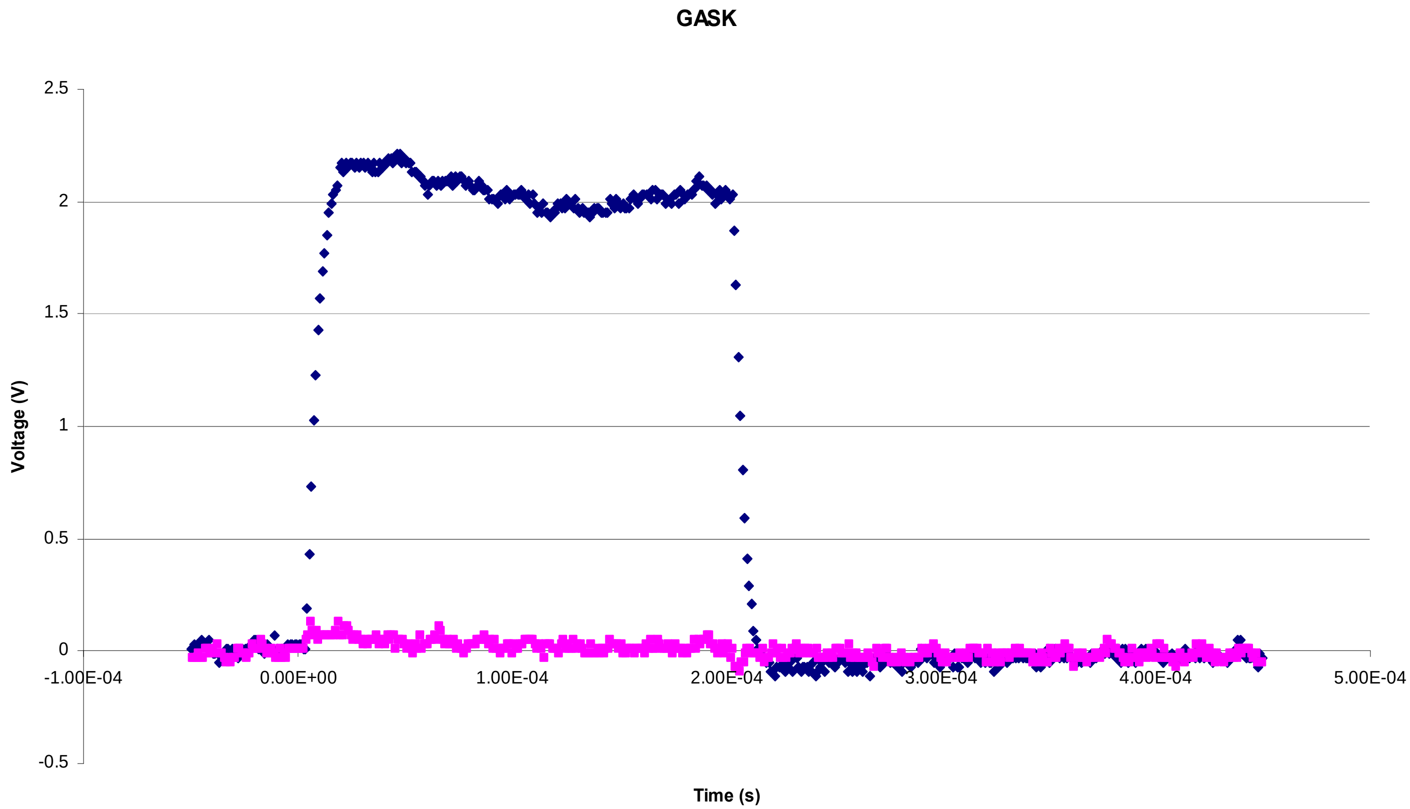
$$P(M_{tot}) = \frac{M_{tot}}{\det M_{tot}} = \frac{M_1}{\det M_1}\frac{M_2}{\det M_2} = P(M_1)\cdot P(M_2)$$

$\therefore$ can separately track the "unimodular part" (as before!)

and normalize by accumulated determinate

ENERGY RECOVERY WORKS

Gradient modulator drive signal in a linac cavity measured without energy recovery (signal level around 2 V) and with energy recovery (signal level around 0).



Courtesy: Lia Merminga