Homework Problems II
Accelerator Physics

1. An electron goes through 10 revolutions in a betatron with $n = 0.6$. How many vertical oscillations does it perform in the small displacement approximation? How many radial oscillations does it perform? Because the time-scale is so short for 10 revolutions, you may assume that γ is constant.

2. Starting with the Lagrangian of a point particle with charge q and rest mass m in an electromagnetic field specified by the scalar potential Φ and the vector potential $\vec{A}$,

$$L = -mc^2\sqrt{1-\vec{\mathrm{v}}\cdot\vec{\mathrm{v}}/c^2} - q\Phi + q\vec{\mathrm{v}}\cdot\vec{A},$$

show the Euler-Langrange equations reduce to the well-known relativistic Lorentz Force Equation

$$\frac{d\left(\gamma m\vec{\mathrm{v}}\right)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}}\times\vec{B}\right),$$

where $\vec{E}$ and $\vec{B}$ are the electric field and magnetic field given by the usual relations between the fields and potentials

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$

and

$$\vec{B} = \vec{\nabla}\times\vec{A}.$$

From the relativistic Lorentz Force Equation derive

$$\vec{\mathrm{v}}\cdot\frac{d\left(\gamma m\vec{\mathrm{v}}\right)}{dt} = q\vec{\mathrm{v}}\cdot\vec{E}.$$

From the usual expression

$$\gamma = \frac{1}{\sqrt{1-\vec{\mathrm{v}}\cdot\vec{\mathrm{v}}/c^2}},$$

show

$$\frac{d\left(\gamma mc^2\right)}{dt} = q\vec{E}\cdot\vec{\mathrm{v}}.$$

Therefore, even at relativistic energies, magnetic fields cannot change the particle energy when radiation reaction is neglected. Verify that the relativistic force law is also written as $\frac{dp^\alpha}{d\tau} = qF^\alpha{}_\nu u^\nu$ (ν summation implied), where

$$F^{\alpha}{}_{\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix},$$

and yields the relativistic Lorentz force equation when evaluated on the space components $\alpha = 1,2,3$.

3. Repeat, using the relativistic equations of motion, the derivation in class of the cyclotron frequency. Show the relativistic cyclotron angular frequency is

$$\Omega_c = \frac{qB}{\gamma m}.$$

Show the radius of the cyclotron motion

$$r = \frac{\beta c}{qB / \gamma m}.$$