Mid-term Exam Solution

Physics 425/525

Multiple Choice (circle or otherwise indicate the correct answer)

1. (5pts) Two 1 C positive charges are place 1 km apart in vacuum. The force on the charges is
    a. 9 Nt repulsive along the line between the charges
    b. 9 Nt attractive along the line between the charges
    c. 9000 Nt repulsive along the line between the charges
    d. 9000 Nt attractive along the line between the charges

Coulombs law gives the magnitude of the force as

$$\frac{1}{4\pi\cdot 8.85\times 10^{-12}}\frac{\mathrm{Nt\ m^2}}{\mathrm{C^2}}\frac{1\ \mathrm{C^2}}{10^6\ \mathrm{m^2}}=9000\ \mathrm{Nt}$$

Because the charges have the same sign, the force between them is repulsive, along the line between the charges. (c.) is the correct answer.

2. (5pts) What is $\nabla\cdot\hat{\mathscr{r}}$ for $\boldsymbol{r}-\boldsymbol{r}'\neq 0$?
    a. $1/|\mathscr{r}|^2$
    b. $2/|\mathscr{r}|$
    c. 0
    d. $-1/|\mathscr{r}|^3$

$$\nabla\cdot\left(\frac{(x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)$$

$$=\frac{3}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}-\frac{1}{2}\frac{2(x-x')^2+2(y-y')^2+2(z-z')^2}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{3/2}}$$

$$=\frac{2}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}=\frac{2}{|\mathscr{r}|}$$

(b.) is the correct answer.

3. (5pts) Which is NOT a property of a conductor
    a. The electric potential function varies inside the conductor
    b. Excess charge resides on the surface of a conductor
    c. The electric field vanishes inside the conductor
    d. Unbound electrons are free to move in a conductor

Because the electric field vanishes in a conductor, the potential is constant throughout the conductor. (a.) is NOT a property of a conductor.

4. (5pts) A conducting sphere of radius *R* is charged with a charge *Q*. The electric potential (referenced to 0 at ∞) is
   a. $Qr/\left(4\pi\varepsilon_0 R^2\right)\ \ r<R;\ \ QR/\left(4\pi\varepsilon_0 r^2\right)\ \ r>R$
   b. $Q/\left(4\pi\varepsilon_0 r\right)\ \ r<R;\ \ Q/\left(4\pi\varepsilon_0 R\right)\ \ r>R$
   c. $Qr^2/\left(4\pi\varepsilon_0 R^3\right)\ \ r<R;\ \ QR^2/\left(4\pi\varepsilon_0 r^3\right)\ \ r>R$
   d. $Q/\left(4\pi\varepsilon_0 R\right)\ \ r<R;\ \ Q/\left(4\pi\varepsilon_0 r\right)\ \ r>R$

Outside of any spherical surface containing the charge (for example by integrating the electric field found from Gauss's Law), the spherical distribution of charge has potential $Q/\left(4\pi\varepsilon_0 r\right)\ \ r>R$. On and inside the conductor, the electric potential is constant. (d.) is the correct answer.

5. (5pts) Suppose $\boldsymbol{E}=\left[E_0\sigma_s\left(1-\exp\left(-s^2/2\sigma_s^2\right)\right)/s\right]\hat{s}$ where *s* is the cylindrical radial coordinate. What is $\rho(s)$?
   a. $\left(\varepsilon_0 E_0/\sigma_s\right)^2\exp\left(-s^2/2\sigma_s^2\right)$
   b. $\left(\varepsilon_0 E_0/\sigma_s\right)\exp\left(-s^2/2\sigma_s^2\right)$
   c. $\left(\varepsilon_0 E_0/\sigma_s\right)\left(1-\exp\left(-s^2/2\sigma_s^2\right)\right)$
   d. $\left(\varepsilon_0 E_0 s/\sigma_2^2\right)\exp\left(-s^2/2\sigma_s^2\right)$

Using the formula for the divergence in cylindrical coordinates

$$\rho(s)=\varepsilon_0\nabla\cdot\boldsymbol{E}=\varepsilon_0\frac{E_0\sigma_s}{s}\frac{d}{ds}\left(1-\exp\left(-s^2/2\sigma_s^2\right)\right)=\varepsilon_0\frac{E_0\sigma_s}{s}\frac{2s}{2\sigma_s^2}\exp\left(-s^2/2\sigma_s^2\right)$$

(b.) is the correct answer

Problems

6. (20 pts) For each vector field, determine whether it can be described by a scalar potential, a vector potential, or whether it needs both for a complete description. Circle or otherwise indicate one answer for each field.
   a. $\boldsymbol{v}=x\hat{x}+y\hat{y}+z\hat{z}$ (scalar potential, vector potential, both needed)
   b. $\boldsymbol{v}=xy\hat{x}+yz\hat{y}+zx\hat{z}$ (scalar potential, vector potential, both needed)

c. $\boldsymbol{v} = (x\hat{x} + y\hat{y} + z\hat{z})(x^2 + y^2 + z^2)$ (scalar potential, vector potential, both needed)

d. $\boldsymbol{v} = yz\hat{x} - zx\hat{y} + xy\hat{z}$ (scalar potential, vector potential, both needed)

Extra credit: (10 pts) Record a scalar potential for those fields that can be so described

$\boldsymbol{v}_a = x\hat{x} + y\hat{y} + z\hat{z} = r\hat{r}$ $\quad \nabla \cdot \boldsymbol{v}_a = 3, \nabla \times \boldsymbol{v}_a = 0$ $\quad$ A potential function is $r^2/2$

Scalar potential

$\boldsymbol{v}_b = xy\hat{x} + yz\hat{y} + zx\hat{z}$ $\quad \nabla \cdot \boldsymbol{v}_b = x + y + z, \nabla \times \boldsymbol{v}_b = -y\hat{x} - z\hat{y} - x\hat{z}$

Both needed

$\boldsymbol{v}_c = (x\hat{x} + y\hat{y} + z\hat{z})(x^2 + y^2 + z^2) = r^3\hat{r}$ $\quad \nabla \cdot \boldsymbol{v}_c = 3r^2, \nabla \times \boldsymbol{v}_c = 0$ $\quad$ A potential function is $r^4/4$

Scalar potential

$\boldsymbol{v}_d = yz\hat{x} - zx\hat{y} + xy\hat{z}$ $\quad \nabla \cdot \boldsymbol{v}_d = 0, \nabla \times \boldsymbol{v}_d = 2x\hat{x} - 2z\hat{z}$

Vector potential

7. (25 pts) A conducting sphere of radius *R* is charged with a charge *Q*. It is surrounded by an UNCHARGED spherical conducting shell of inner radius *a* and outer radius *b*.
   a. What is the electric field in each of the regions $r < R$, $R < r < a$, $a < r < b$, and $r > b$?
   b. What is the value of the potential of the inner sphere assuming the electric potential function vanishes as $r \to \infty$?

a. By Gauss's Law, in both $R < r < a$ and $r > b$, the field is perfectly radial and

$$4\pi r^2 E_r = \frac{Q}{\varepsilon_0} \qquad E_r = \frac{Q}{4\pi\varepsilon_0 r^2}.$$

Inside conductor, that is for $r < R$ and $a < r < b$, $E_r = 0$.

b. Integrating the radial electric field to $r = b$,

$$\phi(r = b) = \phi(r = \infty) - \int_{\infty}^{b} \frac{Q}{4\pi\varepsilon_0 r^2} dr = 0 + \frac{Q}{4\pi\varepsilon_0 b}.$$

Integrating to $r = a$,

$$\phi(r = a) = \phi(r = b) - \int_{b}^{a} 0 dr = \frac{Q}{4\pi\varepsilon_0 b}.$$

Integrating to $r = R$,

$$\phi(r = R) = \phi(r = a) - \int_{a}^{R} \frac{Q}{4\pi\varepsilon_0 r^2} dr = \frac{Q}{4\pi\varepsilon_0 b} + \frac{Q}{4\pi\varepsilon_0 R} - \frac{Q}{4\pi\varepsilon_0 a}.$$

This is the potential throughout the sphere $r < R$.

8. (30 pts) Solve the potential for a conducting sphere in a uniform field by the method of images. Follow the steps indicated.
   a. Place a point charge of magnitude $-q$ at $z = a$ and a point charge of magnitude q at $z = -a$ on the *z*-axis. Show that near the origin, the electric field is uniform and

$$E_z \approx \frac{1}{4\pi\varepsilon_0}\frac{2q}{a^2}$$

   when $a \to \infty$.
   For the two point charges

$$\phi(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z+a)^2}} - \frac{1}{\sqrt{x^2+y^2+(z-a)^2}}\right].$$

   The electric field is

$$E_z = -\frac{\partial\phi}{\partial z} = -\left(-\frac{1}{2}\right)\frac{q}{4\pi\varepsilon_0}\left[\frac{2(z+a)}{\left(x^2+y^2+(z+a)^2\right)^{3/2}} - \frac{2(z-a)}{\left(x^2+y^2+(z-a)^2\right)^{3/2}}\right]$$

$$\approx \frac{q}{4\pi\varepsilon_0}\frac{2a}{a^3} \qquad a \to \infty$$

   b. Place a conducting sphere of radius R centered at the origin. By Example 3.2 which was used in a homework problem, what is the magnitude of two image charges and how should they be placed so that the potential from all 4 charges vanishes on the sphere?

$$q_3 = -q\frac{R}{a} \qquad z_3 = -\frac{R^2}{a}; \qquad q_4 = q\frac{R}{a} \qquad z_4 = \frac{R^2}{a}$$

   c. Show the dipole moment of the two image charges is

$$\mathbf{p} = \frac{2R^3}{a^2}q\hat{z} \to 4\pi\varepsilon_0 E_z R^3\hat{z},$$

   when $a \to \infty$ holding $2q/a^2$ constant.

$$\mathbf{p} = q_3\mathbf{z}_3 + q_4\mathbf{z}_4 = \left(-q\frac{R}{a}\right)\left(-\frac{R^2}{a}\hat{z}\right) + \left(q\frac{R}{a}\right)\left(\frac{R^2}{a}\hat{z}\right) = \frac{2qR^3}{a^2}\hat{z}$$

   Replacing $2q/a^2$ with the expression from (a.) gives the result. Because $a >> R$ in the limit, the result is exact.

   d. In this same limit, what is the dipole potential (in spherical coordinates) for $r > R$ for the two image charges?

   Now R^2/a is tiny, so the dipole potential gives an exact expression for the potential outside the sphere

$$\phi_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{|p|\cos\theta}{r^2} = \frac{E_z R^3 \cos\theta}{r^2}$$

e. By the uniqueness theorem, this same potential must govern the problem of a sphere in a uniform field. Find the total potential and compare to the answer for the potential of a sphere in a uniform field in the book or lectures.

   The potential for the uniform field produced by the first two charges is $-E_z z$. Thus

$$\phi_{tot} = -E_z z + \phi_{dipole} = -E_z\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

   This result is the same as in the book example and lectures.