Electromagnetism I

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Lecture 4

Three Dimensions

Define 3-D delta function by multiplication

$$\delta^3(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z) \qquad \boldsymbol{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

By repeated integrations

$$\iiint \delta^3(\boldsymbol{r})dxdydz = \iiint \delta(x)\delta(y)\delta(z)\,dxdydz$$
$$= 1\iint \delta(x)\delta(y)\,dxdy = 1\int \delta(x)\,dx = 1$$

Main properties

$$\iiint_{\substack{space\\containing\\ \boldsymbol{a}}} f(x,y,z)\delta^3(\boldsymbol{x}-\boldsymbol{a})\,dxdydz = f\left(a_x, a_y, a_z\right)$$

$$\nabla \cdot \frac{\boldsymbol{x}-\boldsymbol{a}}{\left|\boldsymbol{x}-\boldsymbol{a}\right|^3} = 4\pi\delta^3(\boldsymbol{x}-\boldsymbol{a}) \qquad \nabla^2\left(\frac{1}{\left|\boldsymbol{x}-\boldsymbol{a}\right|}\right) = -4\pi\delta^3(\boldsymbol{x}-\boldsymbol{a})$$

Helmholtz Theorem

Any vector field with appropriate boundary conditions (vanishing fast enough at infinity works) can be uniquely decomposed into a part with no curl (irrotational) and a part with no divergence (solenoidal). Conversely, if the divergence and curl of a vector field are given, along with boundary conditions, the vector field can be uniquely found.

In a single simply connected region of space, the irrotational part can be described by a scalar potential function ϕ and the solenoidal part by a vector potential function $\boldsymbol{A}$

$$\boldsymbol{v} = -\nabla\phi + \nabla\times\boldsymbol{A},$$

a deep result known as Poincare's lemma.

Finding the Potentials

The scalar potential is found simply by performing line integrals, as in mechanics potential energy functions for conservative fields

$$\phi(\boldsymbol{x}) = -\int_{\boldsymbol{a}}^{\boldsymbol{x}} \boldsymbol{v} \cdot d\boldsymbol{l}$$

(***a*** is a reference location where the potential vanishes).

Here is a method that allows you to find a vector potential for simple solenoidal vector fields (from a proof of Poincare's lemma)

$$V_x = \int_0^1 v_x(tx,ty,tz)\,t\,dt, V_y = \int_0^1 v_y(tx,ty,tz)\,t\,dt, V_z = \int_0^1 v_z(tx,ty,tz)\,t\,dt$$

$$A_x = V_y z - V_z y, A_y = V_z x - V_x y, A_z = V_x y - V_y x$$