Electromagnetism I

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Lecture 23

Electrodynamic Equations till now

- So far

$$\nabla \cdot \boldsymbol{E} = \rho / \varepsilon_0$$
$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B} / \partial t$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

- Take divergence of curl equations

$$\nabla \cdot \nabla \times \boldsymbol{E} = \nabla \cdot \left(-\partial \boldsymbol{B} / \partial t \right) = -\partial \left(\nabla \cdot \boldsymbol{B} \right) / \partial t$$
$$0 = 0 \qquad \text{OK!}$$
$$\nabla \cdot \nabla \times \boldsymbol{B} = \nabla \cdot \left(\mu_0 \boldsymbol{J} \right) = \mu_0 \nabla \cdot \boldsymbol{J}$$
$$0 = \nabla \cdot \boldsymbol{J} \quad \text{Inconsistent with } \nabla \cdot \boldsymbol{J} + \partial \rho / \partial t = 0!$$

Displacement Current (Density)

- Note

$$\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{E}\right)=\nabla\cdot\frac{\partial\boldsymbol{E}}{\partial t}=\frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial t}$$

- To fix, need to add something to the ***B*** curl equation

$$\nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$$

$$\nabla\cdot\nabla\times\boldsymbol{B}=\mu_0\nabla\cdot\boldsymbol{J}+\mu_0\varepsilon_0\frac{\partial\left(\nabla\cdot\boldsymbol{E}\right)}{\partial t}=0$$

- In the added term,

$$\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$$

was called by Maxwell the displacement current (density)

Maxwell's Electrodynamics

- Equations, now including all time dependences

$$\nabla \cdot \boldsymbol{E} = \rho / \varepsilon_0$$

$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B} / \partial t$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \partial \boldsymbol{E} / \partial t$$

- By Faraday's Law, a changing magnetic field generates and electric field. Now, due to the displacement current, *a changing electric field generates a magnetic field*. Next semester will show electromagnetic waves, including light, are possible solutions!

Magnetic Charges

- If there were magnetic charges, Maxwell's equations would have to be

$$\nabla \cdot \boldsymbol{E} = \rho_e / \varepsilon_0$$
$$\nabla \times \boldsymbol{E} = -\mu_0 \boldsymbol{J}_m - \partial \boldsymbol{B} / \partial t$$
$$\nabla \cdot \boldsymbol{B} = \mu_0 \rho_m$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}_e + \mu_0 \varepsilon_0 \partial \boldsymbol{E} / \partial t$$

- Continuity for electric and magnetic charges

$$\nabla \cdot \boldsymbol{J}_e = -\partial \rho_e / \partial t \qquad \nabla \cdot \boldsymbol{J}_m = -\partial \rho_m / \partial t$$

- Sign of $\boldsymbol{J}_m$ term must be negative to ensure continuity

Force Law (Problem 7.40)

- Magnetic analogue of "Coulomb's Law" must be

$$\boldsymbol{F} = \frac{\mu_0}{4\pi}\frac{q_{m1}q_{m2}}{|\boldsymbol{\mathscr{r}}|^2}\hat{\boldsymbol{\mathscr{r}}}$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{q_m}{|\boldsymbol{\mathscr{r}}|^2}\hat{\boldsymbol{\mathscr{r}}}$$

$$\nabla\cdot\boldsymbol{B} = \frac{\mu_0}{4\pi}q_m\nabla\cdot\frac{\hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} = \mu_0 q_m \delta^3(\boldsymbol{r}-\boldsymbol{r}')$$

- And force law is (− sign from relativity argument!)

$$\boldsymbol{F} = q_m\left(\boldsymbol{B} - \frac{\boldsymbol{v}\times\boldsymbol{E}}{c^2}\right)$$

Maxwell's Equations with Matter

- When allow time-dependent fields, need to allow polarization and magnetization (and the bound charges and currents) to be time dependent

$$\rho_b = -\nabla \cdot \boldsymbol{P}$$

$$\frac{\partial \rho_b}{\partial t} = -\nabla \cdot \boldsymbol{J}_p = -\nabla \cdot \frac{\partial \boldsymbol{P}}{\partial t}$$

$$\boldsymbol{J}_p = \frac{\partial \boldsymbol{P}}{\partial t}$$

- A time dependent magnetization ***M*** adds no new source term to Maxwell Equations. Still

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$$

Gauss's Law (with materials)

- As before

$$\rho = \rho_f + \rho_b$$
$$= \rho_f - \nabla \cdot \boldsymbol{P}$$

- Gauss's Law

$$\nabla \cdot \varepsilon_0 \boldsymbol{E} = \rho_f - \nabla \cdot \boldsymbol{P}$$

- In terms of the electric displacement ***D***

$$\nabla \cdot \boldsymbol{D} = \rho_f$$

- And for linear materials

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_0 \left(1 + \chi_e\right) \boldsymbol{E}$$

Ampere's Law (with materials)

- Now

$$\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b + \boldsymbol{J}_p = \boldsymbol{J}_f + \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}$$

- Ampere's Law is

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}_f + \mu_0 \nabla \times \boldsymbol{M} + \mu_0 \frac{\partial \boldsymbol{P}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

- In terms of ***H***

$$\nabla \times \left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} \right) = \nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t}$$

- Linear materials

$$\boldsymbol{H} + \chi_m \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} \rightarrow \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0 (1 + \chi_m)} = \frac{\boldsymbol{B}}{\mu}$$

Maxwell's Equations with Matter

- Full set

$$\nabla \cdot \boldsymbol{D} = \rho_f$$

$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B} / \partial t$$

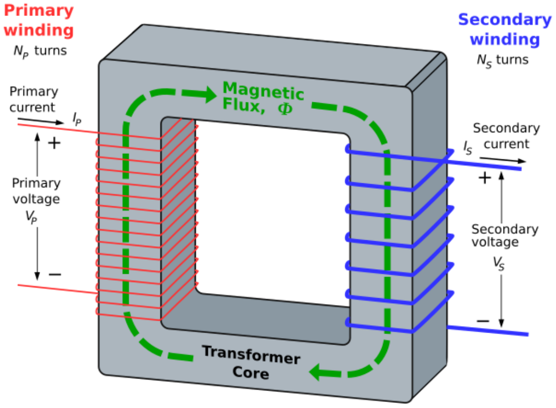
$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \partial \boldsymbol{D} / \partial t$$

- In general need a relationship (could be linear or non-linear) relating ***B*** and ***H*** and ***E*** and ***D*** (constitutive relations)
- Need to apply proper boundary conditions to make solutions!

Example: Transformer

- Iron core linking different turns (ideally, no flux lost)



- Currents are sinusoids

$$\frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{n_1\dot{\Phi}_1}{n_2\dot{\Phi}_2} = \frac{n_1\dot{\Phi}_1}{n_2\dot{\Phi}_1} = \frac{n_1}{n_2}$$

- No power lost (high voltage, low current→low voltage high current)

$$\mathcal{E}_1 I_1 = \mathcal{E}_2 I_2 \rightarrow \frac{I_1}{I_2} = \frac{n_2}{n_1}$$