Electromagnetism I

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Lecture 22

Inductance

- Suppose have two separate wire loops

$$\boldsymbol{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\boldsymbol{l} \times \hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2}$$

- Flux through second loop

$$\Phi_2 = \int B_1 \cdot \hat{n} da_2$$

- The mutual inductance is

$$\Phi_2 = M_{21} I_1 = \left[\frac{\mu_0}{4\pi} \int \oint \frac{d\boldsymbol{l} \times \hat{\boldsymbol{\mathscr{r}}}}{|\boldsymbol{\mathscr{r}}|^2} \cdot \hat{n} da_2 \right] I_1$$

Symmetrical Version

- Transform expression using Stoke's Theorem

$$\Phi_2 = \int \boldsymbol{B}_1 \cdot \hat{n} da_2 = \int \nabla \times \boldsymbol{A}_1 \cdot \hat{n} da_2 = \oint \boldsymbol{A}_1 \cdot d\boldsymbol{l}_2$$

- Vector potential (Coulomb gauge)

$$\boldsymbol{A}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\boldsymbol{l}_1}{|\mathscr{r}|}$$

- Linking flux

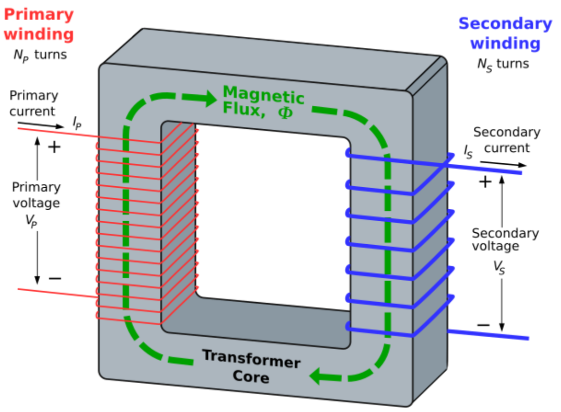
$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\boldsymbol{l}_1}{|\mathscr{r}|} \cdot d\boldsymbol{l}_2$$

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\boldsymbol{l}_1 \cdot d\boldsymbol{l}_2}{|\mathscr{r}|}$$

$$M_{21} = M_{12}$$

Example: Transformer

- Iron core linking different turns (ideally, no flux lost)



- Currents are sinusoids

$$\frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{n_1\dot{\Phi}_1}{n_2\dot{\Phi}_2} = \frac{n_1\dot{\Phi}_1}{n_2\dot{\Phi}_1} = \frac{n_1}{n_2}$$

- No power lost (high voltage, low current→low voltage high current)

$$\mathcal{E}_1 I_1 = \mathcal{E}_2 I_2 \to \frac{I_1}{I_2} = \frac{n_2}{n_1}$$

Self Inductance

- A single coil has a self inductance

$$\Phi_B = LI$$

- By Faraday's law, induces a back emf (essentially to create magnetic energy!)

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

- Work to create flux

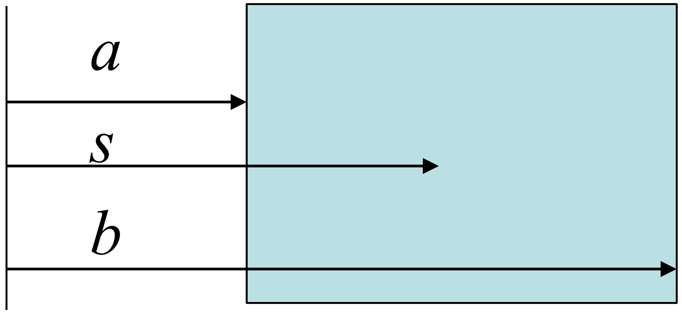
$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

$$W = \frac{1}{2}LI^2$$

Example 7.12

- Self inductance of toroidal coil

$$B = \frac{\mu_0 NI}{2\pi s}$$



- Flux through a single turn

$$\Phi_B = \int \boldsymbol{B} \cdot \hat{n} da = \frac{\mu_0 NI}{2\pi} h \int_a^b \frac{ds}{s} = \frac{\mu_0 NIh}{2\pi} \ln(b/a)$$

- Total flux linked is N times the flux per turn

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln(b/a)$$

Magnetic Energy (without materials)

- Flux through a magnetic coil loop is *LI*, but also

$$\Phi = \int \boldsymbol{B} \cdot \hat{n} da = \int \nabla \times \boldsymbol{A} \cdot \hat{n} da = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$$

$$LI = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$$

- Work to create field is

$$W = \frac{1}{2} L I^2 = \frac{1}{2} I \oint \boldsymbol{A} \cdot d\boldsymbol{l} = \frac{1}{2} \oint \boldsymbol{A} \cdot \boldsymbol{I} dl$$

- Generalization for current density

$$W = \frac{1}{2} \int \boldsymbol{A} \cdot \boldsymbol{J} dV$$

- Apply Ampere's Law

$$W = \frac{1}{2\mu_0} \int \boldsymbol{A} \cdot \nabla \times \boldsymbol{B} dV$$

- Vector identity

$$\nabla\cdot\left(\boldsymbol{A}\times\boldsymbol{B}\right)=\boldsymbol{B}\cdot\nabla\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\times\boldsymbol{B}$$

- So

$$\boldsymbol{A}\cdot\nabla\times\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{B}-\nabla\cdot\left(\boldsymbol{A}\times\boldsymbol{B}\right)$$

- Yielding

$$W=\frac{1}{2\mu_0}\left[\int\boldsymbol{B}\cdot\boldsymbol{B}dV-\int\nabla\cdot\left(\boldsymbol{A}\times\boldsymbol{B}\right)dV\right]$$

$$=\frac{1}{2\mu_0}\left[\int B^2dV-\int\left(\boldsymbol{A}\times\boldsymbol{B}\right)\cdot\hat{n}da\right]$$

- Boundary integral vanishes

$$W=\frac{1}{2\mu_0}\int B^2dV$$