Electromagnetism I

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Lecture 16

Ampere's Law

- Analogue for magnetostatics of Gauss's Law for electrostatics

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

$$\int_S \nabla \times \boldsymbol{B} \cdot \hat{n} da = \mu_0 \int_S \boldsymbol{J} \cdot \hat{n} da$$

$$\int_{\partial S} \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{enclosed}$$

- The line integral for a closed loop, depends only on the total current passing though the loop
  - Electrostatics: Coulomb → Gauss's Law
  - Magnetostatics: Biot-Savart → Ampere's Law

Applications

- Magnetic field around a wire

$$\int_{Circle} \boldsymbol{B} \cdot d\boldsymbol{l} = 2\pi s B_\phi = \mu_0 I$$

$$B_\phi = \frac{\mu_0 I}{2\pi s}$$

- Magnetic field of an infinite surface current (in *x*-direction)

$$\int_{loop} \boldsymbol{B} \cdot d\boldsymbol{l} = 0 + Bl + 0 + Bl = \mu_0 K l$$

$$\boldsymbol{B} = \begin{cases} \hat{y}\mu_0 K/2 & z > 0 \\ -\hat{y}\mu_0 K/2 & z < 0 \end{cases}$$

The right hand rule implies the sign of ***B*** reversed above/below the sheet

Static Maxwell Equations

- Electrostatics

$$\nabla \cdot \boldsymbol{E} = \rho / \varepsilon_0 \qquad \text{(Gauss's Law)}$$

$$\nabla \times \boldsymbol{E} = 0 \qquad \text{(Scalar potential description)}$$

- Magnetostatics

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{(No magnetic poles, field lines closed)}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \qquad \text{(Ampere's Law)}$$

- Combined with Lorentz force

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$$

Magnetic Vector Potential

- Because $\boldsymbol{B}$ has no divergence, it can be described by a vector potential

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\nabla \times \boldsymbol{B} = \nabla \times \left(\nabla \times \boldsymbol{A}\right) = \nabla\left(\nabla \cdot \boldsymbol{A}\right) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

$$\boldsymbol{B} = \nabla \times \left(\boldsymbol{A} + \nabla \lambda\right)$$

- Vector potential is not be unique for a given $\boldsymbol{B}$. Any gradient can be added to it without changing the field. Use this flexibility to "fix the gauge". Most common choices

$\nabla \cdot \boldsymbol{A} = 0$ (Coulomb Gauge)

$\nabla \cdot \boldsymbol{A} - \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$ (Lorentz Gauge; Invariant to relativistic Lorentz transformations. Excellent for EM wave problems)

Coulomb Gauge

- Suppose $\boldsymbol{A}$ does NOT satisfy the Coulomb gauge condition

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\lambda$$

$$\nabla\cdot\boldsymbol{A}' = \nabla\cdot\boldsymbol{A} + \nabla^2\lambda$$

$$\lambda = \frac{1}{4\pi}\int\frac{\nabla\cdot\boldsymbol{A}}{|\boldsymbol{\mathscr{r}}|}dV'$$

$$\nabla\cdot\boldsymbol{A}' = \nabla\cdot\boldsymbol{A} - \nabla\cdot\boldsymbol{A} = 0$$

- $\boldsymbol{A}'$ has the same magnetic field, but DOES satisfy the Coulomb gauge. For reasonable boundary conditions on $\boldsymbol{A}'$, the equations for $\boldsymbol{A}'$ have unique solutions. For Coulomb gauge

$$\nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{J}$$

Cartesian Coordinate Solution

- Volume current density

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}'}{|\boldsymbol{\mathscr{r}}|} dV'$$

- Line current

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} I \int \frac{1}{|\boldsymbol{\mathscr{r}}|} d\boldsymbol{l}'$$

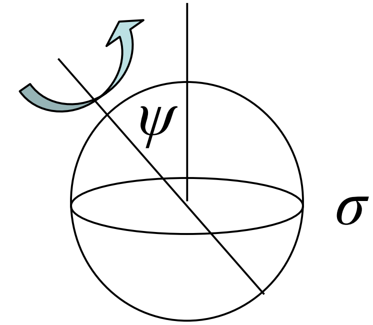
- Surface current

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{K}'}{|\boldsymbol{\mathscr{r}}|} da$$

Example 5.11

- Vector potential of a spinning charged spherical shell.

  Trick: align the *z*-axis with ***r***, ***ω*** in *x*-*z* plane



$$\boldsymbol{K}' = \sigma \boldsymbol{v}' = \sigma \boldsymbol{\omega} \times \boldsymbol{r}'$$

$$= \sigma\omega R \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\psi & 0 & \cos\psi \\ \sin\theta'\cos\phi' & \sin\theta'\sin\phi' & \cos\theta' \end{vmatrix}$$

$$= \sigma\omega R \begin{bmatrix} -\cos\psi\sin\theta'\sin\phi'\hat{x} + \left(\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta'\right)\hat{y} \\ \sin\psi\sin\theta'\sin\phi'\hat{z} \end{bmatrix}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{K}'}{|\boldsymbol{\mathscr{r}}|}da' = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{K}'}{\sqrt{R^2 + r^2 + 2Rr\cos\theta'}} R^2 d\cos\theta' d\phi'$$

$$= 0 + 0 - \frac{\mu_0}{2}\sigma\omega R^3 \hat{y}\int \frac{\sin\psi\cos\theta'}{\sqrt{R^2 + r^2 + 2Rr\cos\theta'}} d\cos\theta' + 0$$

$$\int_{-1}^{1}\frac{\cos\theta'}{\sqrt{R^2+r^2+2Rr\cos\theta'}}d\cos\theta' = -\frac{\left(R^2+r^2+Rr\cos\theta'\right)}{3R^2r^2}\sqrt{R^2+r^2-2Rr\cos\theta'}\Big|_{-1}^{1}$$

$$= -\frac{\left(R^2+r^2+Rr\right)}{3R^2r^2}\left|R-r\right| + \frac{\left(R^2+r^2-Rr\right)}{3R^2r^2}\left(R+r\right)$$

$$= \begin{cases} \dfrac{2r}{3R^2} & r<R \\ \dfrac{2R}{3r^2} & r>R \end{cases}$$

$$-\hat{y}\omega r\sin\psi = \boldsymbol{\omega}\times\boldsymbol{r}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}\boldsymbol{\omega}\times\boldsymbol{r} & r<R \\ \dfrac{\mu_0 R^4\sigma}{3r^3}\boldsymbol{\omega}\times\boldsymbol{r} & r>R \end{cases}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 R\sigma}{3}\nabla\times\left(\boldsymbol{\omega}\times\boldsymbol{r}\right) = \frac{\mu_0 R\sigma}{3}\left[-\left(\boldsymbol{\omega}\cdot\nabla\right)\boldsymbol{r} + \boldsymbol{\omega}\left(\nabla\cdot\boldsymbol{r}\right)\right]$$

$$= \frac{\left(-1+3\right)\mu_0 R\sigma}{3}\boldsymbol{\omega} = \frac{2\mu_0 R\sigma}{3}\boldsymbol{\omega} \qquad r<R$$

Boundary Conditions

- Use Ampere loop and Gaussian pillbox as before

$$\oint_{\partial S} \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{enclosed} \qquad \int \boldsymbol{B} \cdot \hat{n} da = \int \nabla \cdot \boldsymbol{B} dV = 0$$

$$B_+^{\|} l - B_-^{\|} l = \mu_0 K l \qquad B_+^{\perp} da - B_+^{\perp} da = 0$$

$$B_+^{\|} - B_-^{\|} = \mu_0 K \qquad B_+^{\perp} = B_+^{\perp}$$

$$\boldsymbol{B}_{above} - \boldsymbol{B}_{below} = \mu_0 \boldsymbol{K} \times \hat{n}$$

- For vector potential, $\boldsymbol{A}$ continuous guarantees normal magnetic field continuous

$$\frac{\partial A_{above}}{\partial n} - \frac{\partial A_{below}}{\partial n} = \mu_0 \boldsymbol{K}$$