Electromagnetism I

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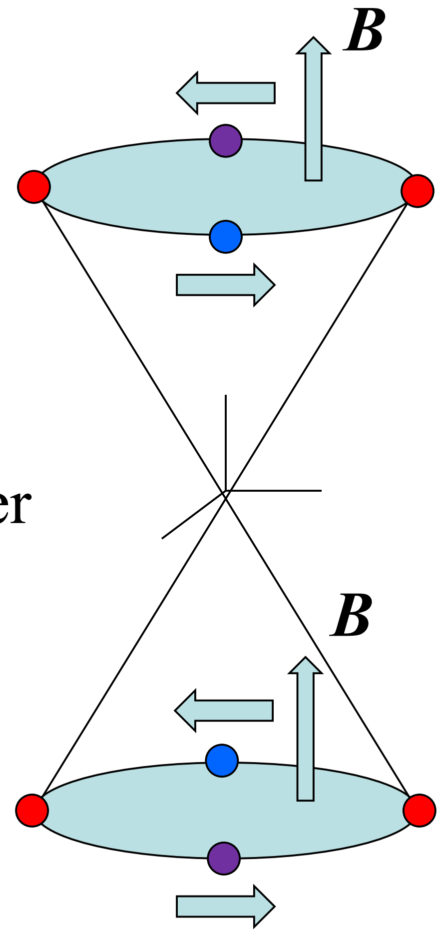
Lecture 15

(Pseudo)vector Magnetic Field



Sign of ***B*** does NOT change under parity

Circles!

- Simple harmonic motion in each degree of freedom

$$\frac{d}{dt}\left(v_x^2+v_y^2\right)=2\frac{qB}{m}\left(v_x v_y - v_y v_x\right)=0$$

$$\ddot{v}_x=\frac{qB}{m}\dot{v}_y=-\left(\frac{qB}{m}\right)^2 v_x$$

$$\ddot{v}_y=-\frac{qB}{m}\dot{v}_x=-\left(\frac{qB}{m}\right)^2 v_x$$

$$\dot{v}_z=0$$

- Cyclotron (angular) frequency

$$\omega_c=\frac{qB}{m} \qquad \left[\omega_c\right]=\frac{\text{C V sec}}{\text{kg m}^2}=\frac{1}{\text{sec}}$$

- Radius of orbit

$$\omega_c R=\sqrt{v_x^2+v_y^2}$$

Currents

- If have a wire, the current (I) is the amount of charge per unit time passing a location in the wire

$$I = \lambda \langle v \rangle \qquad [I] = \frac{\mathrm{C}}{\mathrm{sec}} = \mathrm{A}$$

- Force on a wire

$$d\boldsymbol{F}_{\mathrm{mag}} = dq \langle \boldsymbol{v} \rangle \times \boldsymbol{B}$$

$$= \lambda d\boldsymbol{l} \langle v \rangle \times \boldsymbol{B}$$

$$\boldsymbol{F}_{\mathrm{mag}} = \int I \left(d\boldsymbol{l} \times \boldsymbol{B} \right) = I \int \left(d\boldsymbol{l} \times \boldsymbol{B} \right)$$

depends only on current, not details of how current produced

Surface and Volume Currents

- When charge flows on a surface, need current per unit length *K*

$$\boldsymbol{K} = \frac{d\boldsymbol{I}}{dl_\perp} = \sigma \langle \boldsymbol{v} \rangle$$

$$\boldsymbol{F}_{\text{mag}} = \int (\boldsymbol{v} \times \boldsymbol{B}) \sigma \, da = \int (\boldsymbol{K} \times \boldsymbol{B}) \, da$$

- When charge flows in a volume, need current per unit area *J*

$$\boldsymbol{J} = \frac{d\boldsymbol{I}}{da_\perp} = \rho \langle \boldsymbol{v} \rangle$$

$$\boldsymbol{F}_{\text{mag}} = \int (\boldsymbol{v} \times \boldsymbol{B}) \rho \, dV = \int (\boldsymbol{J} \times \boldsymbol{B}) \, dV$$

Continuity Equation

- Charge flowing out of a volume V

$$\int_{\partial V} \boldsymbol{J} \cdot \hat{n} da = \int_V \nabla \cdot \boldsymbol{J} dV$$

- Change in charge within volume V

$$\frac{d}{dt} \int_V \rho dV = \int_V \frac{\partial \rho}{\partial t} dV$$

- When charge is conserved, outflow must equal the negative of the change in charge inside

$$\int_V \nabla \cdot \boldsymbol{J} dV = -\int_V \frac{\partial \rho}{\partial t} dV$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

Biot-Savart Law

- For steady currents

$$\nabla \cdot \boldsymbol{J} = 0$$

- Magnetic field for a steady line current

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\boldsymbol{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2}$$

- Permeability of free space

$$\mu_0 = 4\pi \times 10^{-7} \quad \frac{\mathrm{Nt}}{\mathrm{A}^2}$$

Long Straight Wire

- Can do integral analytically for long straight wire

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

has curl!

- Force between two wires

$$f = \frac{force}{length} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$$

- What is the line integral on a circle

$$\oint \boldsymbol{B}\cdot d\boldsymbol{l} = \frac{\mu_0 I}{2\pi s}2\pi s = \mu_0 I$$

Doesn’t depend on size of circle!

Divergence of $\boldsymbol{B}$

- For general volume current $\boldsymbol{J}$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}'\times\hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2}dV'$$

- What is the divergence?

$$\nabla\cdot\boldsymbol{B} = \frac{\mu_0}{4\pi}\int \nabla\cdot\left[\frac{\boldsymbol{J}'\times\hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2}\right]dV'$$

$$\nabla\cdot\left[\frac{\boldsymbol{J}'\times\hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2}\right] = \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2}\cdot\nabla\times\boldsymbol{J}' - \boldsymbol{J}'\cdot\nabla\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2} = 0+0$$

$$\nabla\cdot\boldsymbol{B} = 0$$

- Second Maxwell Equation (true including time dependence)

Curl of $\boldsymbol{B}$

- What is the curl?

$$\nabla \times \boldsymbol{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left[\frac{\boldsymbol{J}' \times \hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2} \right] dV'$$

$$\nabla \times \left[\frac{\boldsymbol{J}' \times \hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2} \right] = \boldsymbol{J}' \left[\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2} \right] - \left(\boldsymbol{J}' \cdot \nabla \right) \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2}$$

$$\left[\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{|\mathscr{r}|^2} \right] = 4\pi\delta^3(\mathscr{r})$$

$$\nabla \times \boldsymbol{B} = \mu_0 \int \boldsymbol{J}' \delta^3(\mathscr{r})\, dV' = \mu_0 \boldsymbol{J}$$

- Third Maxwell equation for magnetostatics

Red term vanishes

- Evaluate red term

$$-\left(\mathbf{J}'\cdot\nabla\right)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\left|\boldsymbol{\mathscr{r}}\right|^2}=\left(\mathbf{J}'\cdot\nabla'\right)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\left|\boldsymbol{\mathscr{r}}\right|^2}$$

$$\left(\mathbf{J}'\cdot\nabla'\right)\frac{x-x'}{\left|\boldsymbol{\mathscr{r}}\right|^3}=\mathbf{J}'\cdot\nabla'\frac{x-x'}{\left|\boldsymbol{\mathscr{r}}\right|^3}=\nabla'\cdot\left[\frac{x-x'}{\left|\boldsymbol{\mathscr{r}}\right|^3}\mathbf{J}'\right]-\frac{x-x'}{\left|\boldsymbol{\mathscr{r}}\right|^3}\nabla'\cdot\mathbf{J}'$$

$$\left[\frac{\mu_0}{4\pi}\int-\left(\mathbf{J}'\cdot\nabla\right)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\left|\boldsymbol{\mathscr{r}}\right|^2}dV'\right]_x=\frac{\mu_0}{4\pi}\int\nabla'\cdot\left[\frac{x-x'}{\left|\boldsymbol{\mathscr{r}}\right|^3}\mathbf{J}'\right]dV'$$

$$=\frac{\mu_0}{4\pi}\oint\frac{x-x'}{\left|\boldsymbol{\mathscr{r}}\right|^3}\mathbf{J}'\cdot\hat{n}da\rightarrow 0$$