Electromagnetism I

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Lecture 14

Dielectric Energy

- Simple consideration of simple capacitor

$$W = \frac{1}{2} C V^2$$

- With dielectric inside

$$C = \varepsilon_r C_{vacuum}$$

- Without dielectric

$$W = \frac{\varepsilon_0}{2} \int \boldsymbol{E}^2 dV$$

- Suggests with dielectric

$$W = \frac{1}{2} \int \boldsymbol{D} \cdot \boldsymbol{E} dV$$

Atomic model again

- Assume the "atomic springs" have constant *k*

$$E_{spring} = \frac{1}{2}k\left(x_+ - x_-\right)^2 = \frac{1}{2}k\frac{q^2E^2}{k^2}$$

- In terms of individual atomic dipole moments

$$p = q\left(x_+ - x_-\right) = \frac{q^2E}{k}$$

- When *n* is the number of dipole moments per volume, the energy/volume in the springs is

$$\frac{U_{spring}}{Volume} = \frac{1}{2}(np)E = \frac{1}{2}P\times E$$

- The total energy in the springs generalizes to

$$U_{spring} = \frac{1}{2}\int \boldsymbol{P}\cdot\boldsymbol{E}dV$$

Total Energy

- Electromagnetic field energy plus spring energy for a linear dielectric is

$$W = U_{em} + U_{spring} = \frac{\varepsilon_0}{2}\int \boldsymbol{E}\cdot\boldsymbol{E}dV + \frac{1}{2}\int \varepsilon_0\chi_e \boldsymbol{E}\cdot\boldsymbol{E}dV = \frac{1}{2}\int \boldsymbol{D}\cdot\boldsymbol{E}dV$$

- Griffiths "add free charge" derivation. Start with 2.43 version of the work

$$W = \frac{1}{2}\int \rho\phi dV$$

where the potential ϕ is the total potential including any material polarization effects. Generally, for a given free charge distribution the total potential is proportional (but will have a different space distribution $\phi = \alpha(\boldsymbol{r})\rho_f$). Adding free charge $\Delta\rho_f$

$$\Delta W = \frac{1}{2}\int \alpha\left(\rho_f + \Delta\rho_f\right)^2 dV = \int\left(\Delta\rho_f\right)\phi dV = \int \nabla\cdot\left(\Delta\boldsymbol{D}\right)\phi dV$$

- "Integrating by parts"

$$\nabla\cdot\left(\phi\Delta\boldsymbol{D}\right) = \nabla\phi\cdot\Delta\boldsymbol{D} + \phi\nabla\cdot\left(\Delta\boldsymbol{D}\right)$$

$$\Delta W = \int\left[-\nabla\phi\cdot\Delta\boldsymbol{D} + \nabla\cdot\left(\phi\Delta\boldsymbol{D}\right)\right]dV$$

$$= \int \boldsymbol{E}\cdot\Delta\boldsymbol{D}dV + \int \phi\Delta\boldsymbol{D}\cdot\hat{n}da$$

Boundary integral vanishes at large radii

- For linear dielectric

$$\frac{1}{2}\Delta\left(\boldsymbol{D}\cdot\boldsymbol{E}\right) = \frac{1}{2}\Delta\left(\varepsilon\boldsymbol{E}\cdot\boldsymbol{E}\right) = \varepsilon\Delta\boldsymbol{E}\cdot\boldsymbol{E} = \Delta\boldsymbol{D}\cdot\boldsymbol{E}$$

$$W = \frac{1}{2}\int \boldsymbol{D}\cdot\boldsymbol{E}dV$$

Magnetic Fields

- Static magnetic fields (magnetostatics) produced by
  - Aligned atomic electron currents (permanent magnets)
  - Steady currents in wires (electromagnets)
- Unfortunate terminology problem
  - ***B*** is uniformly the notation for a (pseudo)vector field called by Griffiths the "magnetic field" and by other authors the "magnetic induction field". In this latter case, ***H***, which will be defined in Chapter 6, is called the "magnetic field". MKSA Units

$$[\boldsymbol{B}] = \text{Tesla (T)} = \frac{\text{Weber}}{\text{m}^2} = \frac{\text{V sec}}{\text{m}^2} = \frac{\text{Nt}}{\text{A m}}$$

$$[\boldsymbol{H}] = \frac{\text{A}}{\text{m}}$$

Lorentz Force

- Total electromagnetic force on a charged particle

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$$

- Force perpendicular to both velocity and magnetic field direction
- Magnetic fields can do no work on charged particles

$$\Delta T = \int \boldsymbol{F} \cdot d\boldsymbol{l}$$

$$\dot{T} = \boldsymbol{F} \cdot \boldsymbol{v}$$

- For magnetic forces

$$\dot{T} = q\left(\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \boldsymbol{v} = 0$$

Charged Particle Motion

- For a uniform magnetic field (say in *z*-direction)

$$\boldsymbol{F_B} = m\dot{\boldsymbol{v}} = q\boldsymbol{v}\times\boldsymbol{B} = q\begin{vmatrix}\hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B_z\end{vmatrix} = qB_z v_y\hat{x} - qB_z v_x\hat{y}$$

- Equations of motion

$$\dot{v}_x = \frac{qB_z}{m}v_y$$

$$\dot{v}_y = -\frac{qB_z}{m}v_x$$

$$\dot{v}_z = 0$$

Circles!

- Simple harmonic motion in each degree of freedom

$$\frac{d}{dt}\left(v_x^2+v_y^2\right)=2\frac{qB_z}{m}\left(v_x v_y - v_y v_x\right)=0$$

$$\ddot{v}_x=\frac{qB_z}{m}\dot{v}_y=-\left(\frac{qB_z}{m}\right)^2 v_x$$

$$\ddot{v}_y=-\frac{qB_z}{m}\dot{v}_x=-\left(\frac{qB_z}{m}\right)^2 v_x$$

$$\dot{v}_z=0$$

- Cyclotron (angular) frequency

$$\omega_c=\frac{qB_z}{m} \qquad \left[\omega_c\right]=\frac{\mathrm{C\ V\ sec}}{\mathrm{kg\ m^2}}=\frac{1}{\mathrm{sec}}$$

- Radius of orbit

$$\omega_c R=\sqrt{v_x^2+v_y^2}$$