Electromagnetism I

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Lecture 13

Boundary Conditions for the Potential

- When the relative permittivity of a dielectric is constant,

$$\rho_b = -\nabla\cdot\boldsymbol{P} = -\nabla\cdot\varepsilon_0\chi_e\boldsymbol{E} = -\nabla\cdot\varepsilon_0\chi_e\boldsymbol{D}/\varepsilon = -\frac{\chi_e}{(1+\chi_e)}\rho_f$$

- In particular, when $\rho_f = 0$ INSIDE a dielectric, ρ_b also vanishes and the electric potential solves Laplace's equation there!

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

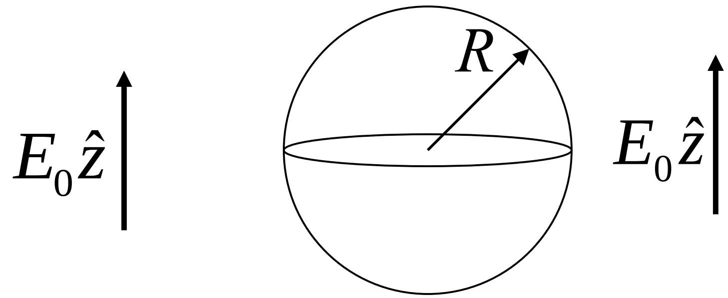
$$\varepsilon_{above}E^{\perp}_{above} - \varepsilon_{below}E^{\perp}_{below} = \sigma_f$$

$$\varepsilon_{above}\left.\frac{\partial\phi}{\partial n}\right|_{above} - \varepsilon_{below}\left.\frac{\partial\phi}{\partial n}\right|_{below} = -\sigma_f$$

$$E^{\parallel}_{above} = E^{\parallel}_{below} \quad \text{automatic if} \quad \phi_{above} = \phi_{below}$$

Dielectric Sphere in Uniform Field

- Put uncharged dielectric sphere in uniform field (field aligned with *z*-axis)
- Potential is



$$\phi(r,\theta)=\begin{cases}\displaystyle\sum_{l=1}^{\infty}A_l r^l P_l(\cos\theta) & r<R\\ \displaystyle -E_0 r P_1(\cos\theta)+\sum_{l=1}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) & r>R\end{cases}$$

- Boundary conditions

$$\sum_{l=1}^{\infty}A_l R^l P_l(\cos\theta)=-E_0 R P_1(\cos\theta)+\sum_{l=1}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta)$$

$$\varepsilon_0\left[-E_0 P_1(\cos\theta)-(l+1)\sum_{l=1}^{\infty}\frac{B_l}{R^{l+2}}P_l(\cos\theta)\right]-\varepsilon l\sum_{l=1}^{\infty}A_l R^{l-1}P_l(\cos\theta)=0$$

Orthogonality Relations

- Apply the orthogonality of the P_l

$$A_1 R = -E_0 R + \frac{B_1}{R^2} \qquad l = 1$$

$$A_l R^l = \frac{B_l}{R^{l+1}} \qquad l > 1$$

$$\varepsilon_0 \left[-E_0 - 2\frac{B_1}{R^3} \right] = \varepsilon A_1 \qquad l = 1$$

$$\varepsilon_0 \left[-(l+1)\frac{B_l}{R^{l+2}} \right] = \varepsilon l A_l R^{l-1} \qquad l > 1$$

- Results, including uniform field inside the sphere

$$A_1 = \frac{-3}{2+\varepsilon_r} \quad B_1 = \frac{\varepsilon_r - 1}{2+\varepsilon_r} E_0 R^3 \quad A_l = 0, B_l = 0 \qquad l > 1$$

$$E_z = \frac{3}{2+\varepsilon_r} E_0 \qquad r < R$$

- Conducting sphere limit: take ε_r to infinity

Capacitance

- Consider a parallel plate capacitor of area A filled with a material with permittivity ε. If a positive charge Q resides on the upper plate, Gauss's law gives (free charge only on plates)

$$D_z = -\sigma = -\frac{Q}{A}$$

$$E_z = \frac{D}{\varepsilon} = -\frac{Q}{\varepsilon_r \varepsilon_0 A} \rightarrow \Delta V = \frac{Qd}{\varepsilon_r \varepsilon_0 A}$$

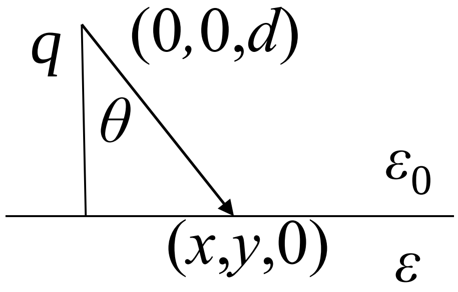
$$C = Q/V = \frac{\varepsilon_r \varepsilon_0 A}{d} = \varepsilon_r C_{vacuum}$$

- Capacitance increase by a factor ε_r . This result is pretty standard and provides a technical way to "beef up" capacitors

Force on a charge by a dielectric

- Only a bound surface charge on $z = 0$

$$\sigma_b = \boldsymbol{P} \cdot \hat{z} = \varepsilon_0 \chi_e \left(E_{z-}^p + E_{z-}^b \right)$$



- Coulomb field of charge is continuous across the surface and has

$$E_{z-}^p = -\frac{q}{4\pi\varepsilon_0} \frac{d}{\left(x^2 + y^2 + d^2\right)^{3/2}}$$

- Potential for bound surface charge continuous across the surface is

$$\phi_b(r,z) = \begin{cases} \int_0^\infty A(k) J_0(kr) \exp(-kz)\, dk & z > 0 \\ \int_0^\infty A(k) J_0(kr) \exp(kz)\, dk & z < 0 \end{cases}$$

- Gauss's law applied to the surface charge yields

$$E_{z+}^{b}-E_{z-}^{b}=\int_{0}^{\infty}A(k)kJ_{0}(kr)\,dk--\int_{0}^{\infty}A(k)kJ_{0}(kr)\,dk$$

$$=-2E_{z-}^{b}=\frac{\sigma_b}{\varepsilon_0}\rightarrow E_{z-}^{b}=-\frac{\sigma_b}{2\varepsilon_0}$$

- Solving for σ_b

$$\sigma_b=-\frac{1}{2\pi}\left(\frac{\chi_e}{2+\chi_e}\right)\left[\frac{qd}{\left(x^2+y^2+d^2\right)^{3/2}}\right]$$

- Note $\phi_b(r,z) = \phi_b(r,-z)$, so two *separate* image charges are needed to represent the bound potential

Total potential and force

- Images make bound potential to yield

$$\phi_{tot}(r,z)=\phi_{point}(r,z)+\phi_b(r,z)=$$

$$\begin{cases}\dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{q}{\sqrt{x^2+y^2+(z-d)^2}}-\dfrac{\chi_e}{2+\chi_e}\dfrac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] & z>0\\[2ex] \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{q}{\sqrt{x^2+y^2+(z-d)^2}}-\dfrac{\chi_e}{2+\chi_e}\dfrac{q}{\sqrt{x^2+y^2+(z-d)^2}}\right] & z<0\end{cases}$$

- Total force

$$\boldsymbol{F}=-\frac{1}{4\pi\varepsilon_0}\frac{\chi_e}{2+\chi_e}\frac{q^2}{(2d)^2}\hat{z}$$

Dielectric Energy

- Simple consideration of simple capacitor

$$W = \frac{1}{2} C V^2$$

- With dielectric inside

$$C = \varepsilon_r C_{vacuum}$$

- Without dielectric

$$W = \frac{\varepsilon_0}{2} \int \boldsymbol{E}^2 dV$$

- Suggests with dielectric

$$W = \frac{1}{2} \int \boldsymbol{D} \cdot \boldsymbol{E} dV$$