Electromagnetism I

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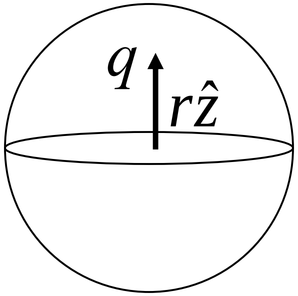
Lecture 12

Griffiths Problem 3.51

- Electric field from a point charge averaged over a sphere not singular



$$\boldsymbol{E}_{ave} = \frac{1}{4\pi R^3/3}\int_S \boldsymbol{E}(r')\,r'^2 dr' d\cos\theta\, d\phi$$

$$= \frac{1}{4\pi R^3/3}\frac{q}{4\pi\varepsilon_0}\int_S \frac{\boldsymbol{r}' - r\hat{z}}{\left(r'^2 + r^2 - 2r'r\cos\theta\right)^{3/2}} r'^2 dr' d\cos\theta\, d\phi$$

$$= \frac{1}{4\pi R^3/3}\frac{q2\pi\hat{z}}{4\pi\varepsilon_0}\frac{d}{dr}\left[\int_0^R\int_{-1}^{1}\frac{1}{\left(r'^2 + r^2 - 2r'r\cos\theta\right)^{1/2}} r'^2 dr' d\cos\theta\right]$$

$$= \frac{1}{4\pi R^3/3}\frac{q2\pi\hat{z}}{4\pi\varepsilon_0}\frac{d}{dr}\left[\int_0^R -\left.\frac{\left(r'^2 + r^2 - 2r'r\cos\theta\right)^{1/2}}{r'r}\right|_{-1}^{1} r'^2 dr'\right]$$

$$= \frac{1}{4\pi R^3/3}\frac{q4\pi\hat{z}}{4\pi\varepsilon_0}\frac{d}{dr}\left[\frac{1}{r}\int_0^r r'^2 dr' + \int_r^R r' dr'\right] = \frac{1}{4\pi R^3/3}\frac{q4\pi\hat{z}}{4\pi\varepsilon_0}\frac{d}{dr}\left[\frac{r^2}{3} + \frac{R^2}{2} - \frac{r^2}{2}\right]$$

$$= -\frac{qr\hat{z}}{4\pi\varepsilon_0 R^3}$$

- Next integrate over charge density creating the polarization

$$\boldsymbol{E}_{ave} = \int_S -\frac{\boldsymbol{r}\rho(\boldsymbol{r})}{4\pi\varepsilon_0 R^3} dV = -\frac{\boldsymbol{p}}{4\pi\varepsilon_0 R^3}$$

- Regardless of the details in ρ creating the polarization, for an averaging sphere small enough that $\boldsymbol{P}$ is constant, the average, or macroscopic field is

$$\boldsymbol{E}_{ave} = -\frac{1}{4\pi\varepsilon_0 R^3}\frac{4\pi R^3}{3}\boldsymbol{P} = -\frac{1}{3\varepsilon_0}\boldsymbol{P}$$

- So the potential for the macroscopic field indeed satisfies

$$\phi(\boldsymbol{r}) = \int \frac{1}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\boldsymbol{P}(\boldsymbol{r}')}{\mathscr{r}^2} dV'$$

Electric Displacement Vector

- The charges introduced onto conductors or fixed into dielectrics in an electrostatic system will be called free charges, and in general they will be described by a charge density ρ_f. In a given electrostatic problem, the charge density creating the *total* electric field $\boldsymbol{E}$ is

$$\rho = \rho_f + \rho_b$$

- By Gauss's Law

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} = \rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \boldsymbol{P}$$

- Combining the two divergence terms

$$\nabla \cdot \left(\varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \right) = \rho_f$$

yields the definition of the electric displacement

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \qquad \nabla \cdot \boldsymbol{D} = \rho_f \qquad \int_S \boldsymbol{D} \cdot \hat{n} da = Q_{f_{enc}}$$

Notes/Cautions

- $\boldsymbol{E}$ is the local value of the electric field including that field generated by any polarization
- Just the divergence may be insufficient for getting $\boldsymbol{D}$

$$\nabla \times \boldsymbol{D} = \nabla \times \boldsymbol{P}$$

- When $\boldsymbol{D}$ has curl, the simple Coulomb law solution isn't enough

$$\boldsymbol{D}(\boldsymbol{r}) \neq \frac{1}{4\pi} \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho_f(\boldsymbol{r}') dV'$$

- Calculating $\boldsymbol{D}$ may involve more than just Gauss's Law!
- $\boldsymbol{D}$ may not have a simple potential

Boundary Conditions

- From Gaussian pillbox argument applied to a surface charge

$$D_{above}^{\perp} - D_{below}^{\perp} = \sigma_f$$

- From the curl equation for ***D***, a Gaussian loop argument plus Stokes Theorem gives

$$D_{above}^{||} - D_{below}^{||} = P_{above}^{||} - P_{below}^{||}$$

- This relation consistent with what already know

$$E_{above}^{||} = E_{below}^{||}$$

- In many problems easiest to use the latter formulation

Linear Dielectrics

- For many materials stressed by small electric fields, our simple "atomic" model applies and the polarization is proportional to $\boldsymbol{E}$. Such materials are called linear dielectrics

$$\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}.$$

The dimensionless quantity χ_e is known as the electric susceptibility.

- In this case $\boldsymbol{D}$ is proportional to $\boldsymbol{E}$,

$$\boldsymbol{D} = \varepsilon_0 \left(1 + \chi_e\right) \boldsymbol{E} \equiv \varepsilon \boldsymbol{E}$$

and the proportionality constant ε is known as the material permittivity, the ratio $\varepsilon_r = \varepsilon/\varepsilon_0$ is known as the relative permittivity of the material. The relationship(s) $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ is/are known as the constitutive relation(s).

Boundary Conditions for the Potential

- When the relative permittivity of a dielectric is constant,

$$\rho_b = -\nabla\cdot\boldsymbol{P} = -\nabla\cdot\varepsilon_0\chi_e\boldsymbol{E} = -\nabla\cdot\varepsilon_0\chi_e\boldsymbol{D}/\varepsilon = -\frac{\chi_e}{\left(1+\chi_e\right)}\rho_f$$

- In particular, when $\rho_f = 0$ INSIDE a dielectric, ρ_b also vanishes and the electric potential solves Laplace's equation there!

$$D_{above}^{\perp} - D_{below}^{\perp} = \sigma_f$$

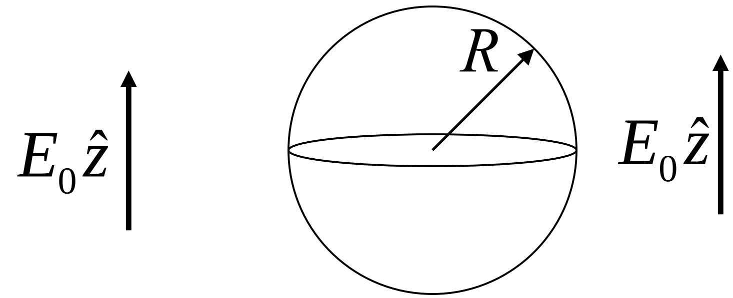
$$\varepsilon_{above}E_{above}^{\perp} - \varepsilon_{below}E_{below}^{\perp} = \sigma_f$$

$$\varepsilon_{above}\left.\frac{\partial\phi}{\partial n}\right|_{above} - \varepsilon_{below}\left.\frac{\partial\phi}{\partial n}\right|_{below} = -\sigma_f$$

$$E_{above}^{\parallel} = E_{below}^{\parallel} \quad \text{automatic if} \quad \phi_{above} = \phi_{below}$$

Dielectric Sphere in Uniform Field

- Put uncharged dielectric sphere in uniform field (field aligned with *z*-axis)
- Potential is



$$\phi(r,\theta)=\begin{cases}\displaystyle\sum_{l=1}^{\infty}A_l r^l P_l(\cos\theta) & r<R\\ \displaystyle -E_0 r P_1(\cos\theta)+\sum_{l=1}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) & r>R\end{cases}$$

- Boundary conditions

$$\sum_{l=1}^{\infty}A_l R^l P_l(\cos\theta)=-E_0 R P_1(\cos\theta)+\sum_{l=1}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta)$$

$$\varepsilon_0\left[-E_0 P_1(\cos\theta)-(l+1)\sum_{l=1}^{\infty}\frac{B_l}{R^{l+2}}P_l(\cos\theta)\right]-\varepsilon l\sum_{l=1}^{\infty}A_l R^{l-1}P_l(\cos\theta)=0$$