PHYS 425: Electromagnetism I
Cheat Sheet

1. Chapter 1

$$\boldsymbol{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}$$

$$|\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$d\boldsymbol{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$\boldsymbol{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}' = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Fundamental Theorems (Gradient Theorem, Divergence Theorem, and Stokes Theorem)

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} \nabla f \cdot d\boldsymbol{l} = f(\boldsymbol{b}) - f(\boldsymbol{a})$$

$$\int_V (\nabla \cdot \boldsymbol{A}) dV = \oint_{S=\partial V} \boldsymbol{A} \cdot \hat{n} da$$

$$\int_S (\nabla \times \boldsymbol{A}) \cdot \hat{n} da = \oint_{L=\partial S} \boldsymbol{A} \cdot d\boldsymbol{l}$$

Delta-function formulas

$$\nabla \cdot \left(\frac{\boldsymbol{x} - \boldsymbol{a}}{|\boldsymbol{x} - \boldsymbol{a}|^3} \right) = 4\pi\delta^3(\boldsymbol{x} - \boldsymbol{a}) \leftrightarrow \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla^2 \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{a}|} \right) = -4\pi\delta^3(\boldsymbol{x} - \boldsymbol{a})$$

Helmholtz Theorem and Formulas for Potentials

Any vector field can be written as the sum of a Curl-free part and a Divergence-free part.

For a Curl-free vector field ***v***, there is a scalar potential ϕ

$$\boldsymbol{v} = -\nabla\phi$$

$$\phi(\boldsymbol{r}) = -\int_{\boldsymbol{a}}^{\boldsymbol{r}} \boldsymbol{v} \cdot d\boldsymbol{l}$$

For a Divergence-free vector field $\boldsymbol{v}$, there is a vector potential $\boldsymbol{A}$

$$\boldsymbol{v} = \nabla \times \boldsymbol{A}$$

$$V_x = \int_0^1 v_x(tx,ty,tz)\,tdt, V_y = \int_0^1 v_y(tx,ty,tz)\,tdt, V_z = \int_0^1 v_z(tx,ty,tz)\,tdt$$

$$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{V} \times \boldsymbol{r}$$

2. Chapter 2

Coulomb's Law and Integral form for $\boldsymbol{E}(\boldsymbol{r})$

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0} \frac{\boldsymbol{r} - \boldsymbol{r}_i'}{|\boldsymbol{r} - \boldsymbol{r}_i'|^3}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} (\boldsymbol{r} - \boldsymbol{r}')\, d\tau'$$

Guass's Law: Integral and Differential form

$$\Phi_{\boldsymbol{E}} = \oint_S \boldsymbol{E} \cdot \hat{n}\, da = \frac{Q_{inside}}{\varepsilon_0}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

Electric Potential

For electrostatics $\nabla \times \boldsymbol{E} = 0$ and there is a scalar potential function $\phi(\boldsymbol{r})$

$$\phi(\boldsymbol{r}) = -\int_{\boldsymbol{a}}^{\boldsymbol{r}} \boldsymbol{E} \cdot d\boldsymbol{l} \leftrightarrow \boldsymbol{E}(\boldsymbol{r}) = -\nabla \phi$$

For a point charge at $\boldsymbol{r}'$

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|}$$

Integral formula and Poisson's equation

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d\tau'$$

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

Electric field energy formulas:

$$U_e = \frac{\varepsilon_0}{2}\int \boldsymbol{E}\cdot\boldsymbol{E}dV$$
$$U_e = \frac{1}{2}\int \boldsymbol{D}\cdot\boldsymbol{E}dV$$
$$U_e = \frac{1}{2}\int \rho(\boldsymbol{r})V(\boldsymbol{r})dV$$

Two capacitance formulas

$$C = q/V$$
$$U_e = \frac{1}{2}CV^2$$

Capacitance of parallel plate capacitor of area *A* and with distance *d* between plates

$$C = \frac{\varepsilon A}{d} = \frac{\varepsilon_0(1+\chi_e)A}{d}$$

3. Chapter 3

In a charge free region the potential solves Laplace Equation:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Cartesian solutions in two variables (choose expansion functions based on boundary conditions)

$$\phi(x,y) = \sum_{n=1}^{\infty}\sin(n\pi x/a)\left[A_n\exp(n\pi y/a) + B_n\exp(-n\pi y/a)\right]$$

Axisymmetric spherical solutions

$$\phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\theta)$$

Normalization of Legendre polynomials

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = \int_{\pi}^{0} P_l(\cos\theta)P_{l'}(\cos\theta)\,d\cos\theta$$
$$= \begin{cases} 0 & l' \neq l \\ 2/(2l+1) & l' = l \end{cases}$$

Electric Dipole potential (*qd* is the electric dipole moment):

$$\phi(\boldsymbol{r}) \approx \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2}$$

Multipole expansion formula, moment formula

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{M_n}{r^{n+1}}$$

$$M_n = \int (r')^n \rho(\boldsymbol{r}') P_n(\cos\alpha)\, dV'$$

Dipole potential, dipole field

$$\boldsymbol{p} = \int \boldsymbol{r}'\rho(\boldsymbol{r}') dV'$$

$$\phi_1(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\boldsymbol{p}}{r^2}$$

$$\boldsymbol{E}_{dip} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3(\boldsymbol{p}\cdot\hat{r}) - \boldsymbol{p}\right)$$

4. Chapter 4

Polarization vector

$$\boldsymbol{P} \quad \equiv \text{"average dipole moment/volume"} \quad [\boldsymbol{P}] = \frac{Cm}{m^3} = \frac{C}{m^2}$$

Bound volume charge bound surface charge, and potential

$$\sigma_b = \boldsymbol{P}\cdot\hat{n}$$

$$\rho_b = -\nabla\cdot\boldsymbol{P}$$

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_{S=\partial V}\frac{\sigma_b}{\mathscr{r}}da' + \int_V\left(\frac{\rho_b}{\mathscr{r}}\right)dV'\right]$$

D Definition, Gauss's Law in differential and integral form (subscript *f* indicates the free charge only)

$$\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P}$$

$$\nabla\cdot\boldsymbol{D} = \rho_f$$

$$\int_S \boldsymbol{D}\cdot\hat{n}da = Q_{f_{enc}}$$

Susceptibility and permittivity definitions

$$\boldsymbol{P} = \varepsilon_0\chi_e\boldsymbol{E}$$

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} = \varepsilon_0(1+\chi_e)\boldsymbol{E}$$

5. Chapter 5

The magnetic field ***B*** is that which causes particles to move on circles, or causes forces between currents

$$[\boldsymbol{B}] = \text{Tesla (T)} = \frac{\text{Weber}}{\text{m}^2} = \frac{\text{V sec}}{\text{m}^2} = \frac{\text{Nt}}{\text{A m}}$$

Lorentz Force on a charged particle

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Magnetic force formulas for different currents

$$\boldsymbol{F}_{\text{mag}} = \int I(d\boldsymbol{l} \times \boldsymbol{B}) = I\int(d\boldsymbol{l} \times \boldsymbol{B})$$
$$\boldsymbol{F}_{\text{mag}} = \int(\boldsymbol{v} \times \boldsymbol{B})\sigma da = \int(\boldsymbol{K} \times \boldsymbol{B})da$$
$$\boldsymbol{F}_{\text{mag}} = \int(\boldsymbol{v} \times \boldsymbol{B})\rho dV = \int(\boldsymbol{J} \times \boldsymbol{B})dV$$

Continuity equation for electric current

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$$

Biot-Savart Law

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\boldsymbol{l}' \times \hat{\boldsymbol{\imath}}}{|\boldsymbol{\imath}|^2}$$

Divergence of ***B***

$$\nabla \cdot \boldsymbol{B} = 0$$

Because ***B*** divergence-free, must have a vector potential. Formula for vector potential in Coulomb gauge

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$
$$\nabla \cdot \boldsymbol{A} = 0 \qquad\qquad \text{(Coulomb Gauge)}$$
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}'}{|\boldsymbol{\imath}|} dV'$$

Curl of ***B*** (Ampere's Law), differential and integral versions

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$
$$\int_{\partial S} \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{enclosed}$$

For a single loop of current *I* and area *A*, the magnetic dipole moment is ($\hat{n}$ normal in right hand sense to current)

$$\boldsymbol{m} = IA\hat{n}$$

Magnetic dipole potential and magnetic field

$$\boldsymbol{A}_{dip}\left(\boldsymbol{r}\right)=\frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\hat{r}}{r^2}$$
$$\boldsymbol{B}_{dip}=\frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(\boldsymbol{m}\cdot\hat{r}\right)\hat{r}-\boldsymbol{m}\right]$$

6. Chapter 6

Mechanical potential of a magnetic dipole

$$U_{mech}=-\boldsymbol{m}\cdot\boldsymbol{B}$$

Magnetization vector

$$\boldsymbol{M}\ \equiv\ \text{"average magnetic dipole moment/volume"}$$
$$\left[\boldsymbol{M}\right]=\frac{\mathrm{C\ m^2}}{\mathrm{sec\,m^3}}=\frac{\mathrm{C}}{\mathrm{sec\,m}}=\frac{\mathrm{A}}{\mathrm{m}}$$

Induced field (formula for vector potential) from magnetization

$$\boldsymbol{J}_b=\nabla\times\boldsymbol{M}$$
$$\boldsymbol{K}_b=\boldsymbol{M}\times\hat{n}$$
$$\boldsymbol{A}_{induced}\left(\boldsymbol{r}\right)=\frac{\mu_0}{4\pi}\left[\int_V\frac{\boldsymbol{J}_b\left(\boldsymbol{r}'\right)}{\mathscr{r}}dV'+\int_{S=\partial V}\frac{\boldsymbol{K}_b\left(\boldsymbol{r}'\right)}{\mathscr{r}}da'\right]$$

H vector definition

$$\boldsymbol{H}=\frac{\boldsymbol{B}}{\mu_0}-\boldsymbol{M}$$

General Ampere’s Law differential and integral form (subscript *f* denotes free current/current density)

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}_f$$
$$\oint_{L=\partial S}\boldsymbol{H}\cdot d\boldsymbol{l}=I_{f\mathrm{enc}}$$

Divergence of ***H*** may not vanish

$$\nabla\cdot\boldsymbol{H}=-\nabla\cdot\boldsymbol{M}$$

Magnetic susceptibility and permeability

$$\boldsymbol{M}=\chi_m\boldsymbol{H}$$
$$\boldsymbol{B}=\mu_0\left(1+\chi_m\right)\boldsymbol{H}\equiv\mu\boldsymbol{H}$$

Static Boundary Condition Formulas

Electric

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

$$\text{or} \quad \varepsilon_{above} E^{\perp}_{above} - \varepsilon_{below} E^{\perp}_{below} = \sigma_f$$

$$\text{or} \quad \varepsilon_{above} \left.\frac{\partial \phi}{\partial n}\right|_{above} - \varepsilon_{below} \left.\frac{\partial \phi}{\partial n}\right|_{below} = -\sigma_f$$

$$E^{\parallel}_{above} = E^{\parallel}_{below} \quad \text{automatic if} \quad \phi_{above} = \phi_{below}$$

Magnetic

$$H^{\perp}_{above} - H^{\perp}_{below} = -\left(M^{\perp}_{above} - M^{\perp}_{below}\right)$$

$$\text{or} \qquad B^{\perp}_{above} = B^{\perp}_{below}$$

$$H^{\parallel}_{above} - H^{\parallel}_{below} = \boldsymbol{K}_f \times \hat{n}$$