Accelerator Physics
Linear Optics

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Lecture 4

Linear Beam Optics Outline

- Particle Motion in the Linear Approximation
- Some Geometry of Ellipses
- Ellipse Dimensions in the β-function Description
- Area Theorem for Linear Transformations
- Phase Advance for a Unimodular Matrix
  - Formula for Phase Advance
  - Matrix Twiss Representation
  - Invariant Ellipses Generated by a Unimodular Linear Transformation
- Detailed Solution of Hill's Equation
  - General Formula for Phase Advance
  - Transfer Matrix in Terms of β-function
  - Periodic Solutions
- Non-periodic Solutions
  - Formulas for β-function and Phase Advance
- Beam Matching

Linear Particle Motion

Fundamental Notion: The *Design Orbit* is a path in an Earth-fixed reference frame, i.e., a differentiable mapping from [0,1] to points within the frame. As we shall see as we go on, it generally consists of *arcs of circles* and *straight lines*.

$$\sigma : [0,1] \to \mathrm{R}^3$$

$$\sigma \quad \to \quad \vec{X}(\sigma) = \left(X(\sigma), Y(\sigma), Z(\sigma) \right)$$

Fundamental Notion: *Path Length*

$$ds = \sqrt{\left(\frac{dX}{d\sigma}\right)^2 + \left(\frac{dY}{d\sigma}\right)^2 + \left(\frac{dZ}{d\sigma}\right)^2}\, d\sigma$$

The *Design Trajectory* is the path specified in terms of the path length in the Earth-fixed reference frame. For a relativistic accelerator where the particles move at the velocity of light, $L_{tot}=ct_{tot}$.

$$s:[0,L_{tot}]\rightarrow \mathrm{R}^3$$

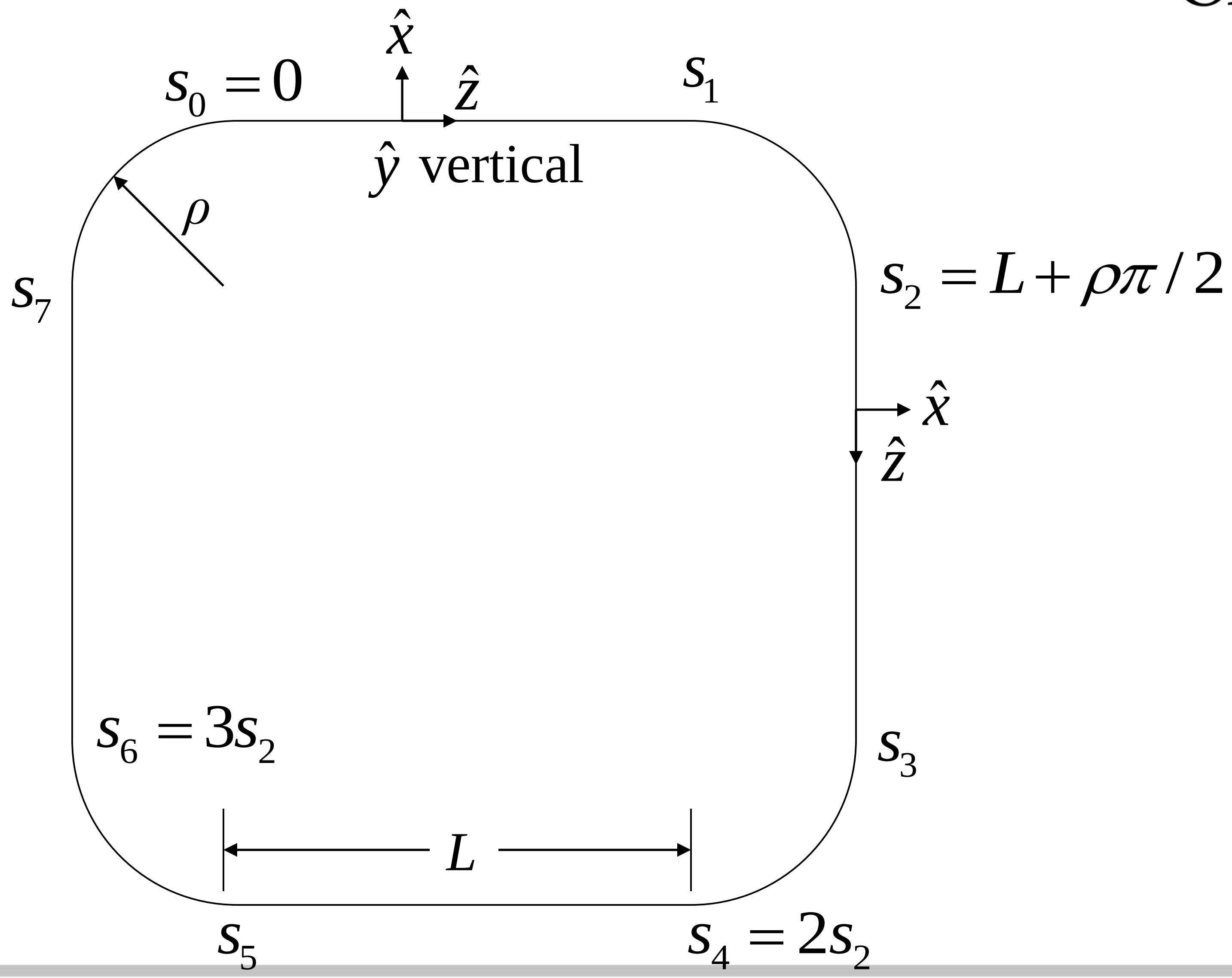
$$s \rightarrow \vec{X}(s)=\big(X(s),Y(s),Z(s)\big)$$

The first step in designing any accelerator, is to specify bending magnet locations that are consistent with the arc portions of the Design Trajectory.

Comment on Design Trajectory

The notion of specifying curves in terms of their path length is standard in courses on the vector analysis of curves. A good discussion in a Calculus book is Thomas, *Calculus and Analytic Geometry*, 4th Edition, Articles 14.3-14.5. Most vector analysis books have a similar, and more advanced discussion under the subject of "Frenet-Serret Equations". Because all of our design trajectories involve only arcs of circles and straight lines (dipole magnets and the drift regions between them define the orbit), we can concentrate on a simplified set of equations that "only" involve the radius of curvature of the design orbit. It may be worthwhile giving a simple example.

4-Fold Symmetric Synchrotron



Its Design Trajectory

$(0,0,s)$	$0 < s < L = s_1$
$(0,0,L) + \rho\left(\cos\left((s-s_1)/\rho\right) - 1, 0, \sin\left((s-s_1)/\rho\right)\right)$	$s_1 < s < s_2$
$(-\rho, 0, L+\rho) + (s-s_2)(-1,0,0)$	$s_2 < s < s_3$
$(-L-\rho, 0, L+\rho) + \rho\left(-\sin\left((s-s_3)/\rho\right), 0, \cos\left((s-s_3)/\rho\right) - 1\right)$	$s_3 < s < s_4$
$(-L-2\rho, 0, L) + (s-s_4)(0,0,-1)$	$s_4 < s < s_5$
$(-L-2\rho, 0, 0) + \rho\left(1-\cos\left((s-s_5)/\rho\right), 0, -\sin\left((s-s_5)/\rho\right)\right)$	$s_5 < s < s_6$
$(-L-\rho, 0, -\rho) + (s-s_6)(1,0,0)$	$s_6 < s < s_7$
$(-\rho, 0, -\rho) + \rho\left(\sin\left((s-s_7)/\rho\right), 0, 1-\cos\left((s-s_7)/\rho\right)\right)$	$s_7 < s < 4s_2$

Betatron Design Trajectory

$$s:[0,2\pi R]\to \mathrm{R}^3$$

$$s \;\to\; \vec{X}\left(s\right)=\left(R\cos\left(s/R\right),R\sin\left(s/R\right),0\right)$$

Use path length *s* as independent variable instead of *t* in the dynamical equations.

$$\frac{d}{ds}=\frac{1}{\Omega_c R}\frac{d}{dt}$$

Betatron Motion in *s*

$$\frac{d^2\delta r}{dt^2}+(1-n)\Omega_c^2\delta r=\Omega_c^2 R\frac{\Delta p}{p}$$

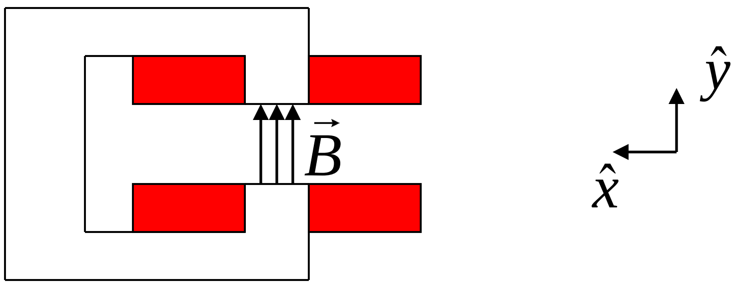
$$\frac{d^2\delta z}{dt^2}+n\Omega_c^2\delta z=0$$

$$\Downarrow$$

$$\frac{d^2\delta r}{ds^2}+\frac{(1-n)}{R^2}\delta r=\frac{1}{R}\frac{\Delta p}{p}$$

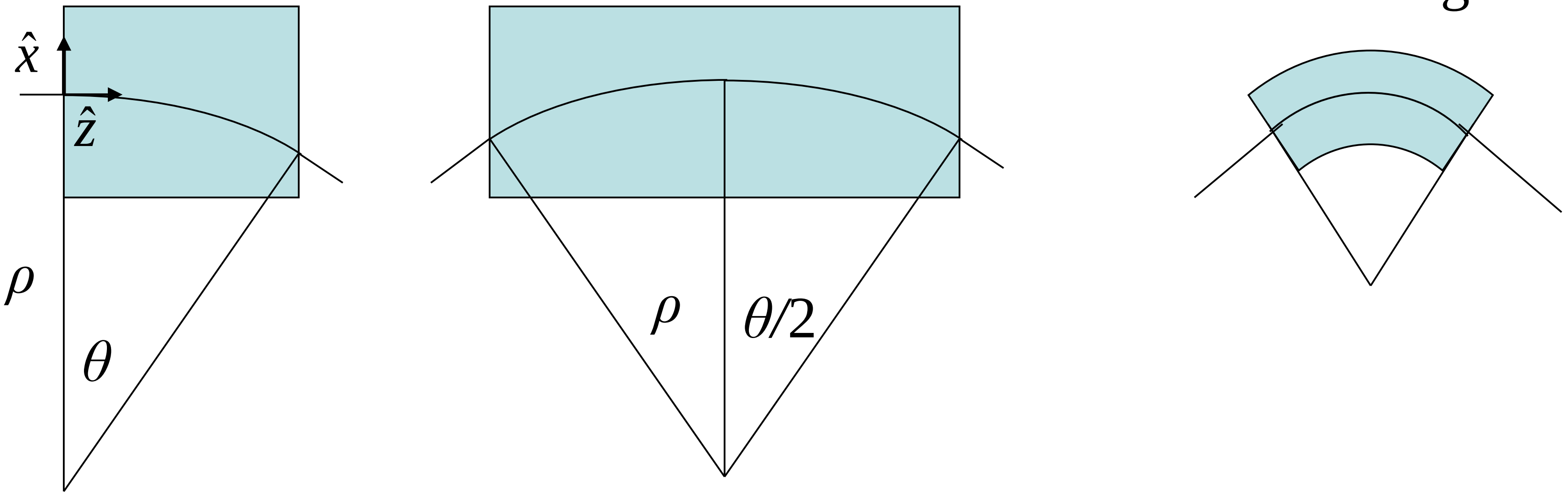
$$\frac{d^2\delta z}{ds^2}+\frac{n}{R^2}\delta z=0$$

Bend Magnet Geometry



Rectangular Magnet of Length L

Sector Magnet



Bend Magnet Trajectory

For a uniform magnetic field

$$\frac{d(\gamma m\vec{V})}{dt} = \left[\vec{E} + \vec{V}\times\vec{B}\right]$$

$$\frac{d(\gamma m V_x)}{dt} = -qV_z B_y$$

$$\frac{d(\gamma m V_z)}{dt} = qV_x B_y$$

$$\frac{d^2 V_x}{dt^2} + \Omega_c^2 V_x = 0 \qquad \frac{d^2 V_z}{dt^2} + \Omega_c^2 V_z = 0$$

For the solution satisfying boundary conditions: $\vec{X}(0) = 0 \qquad \vec{V}(0) = V_{0z}\hat{z}$

$$X(t) = \frac{p}{qB_y}\left(\cos(\Omega_c t) - 1\right) = \rho\left(\cos(\Omega_c t) - 1\right) \qquad \Omega_c = qB_y / \gamma m$$

$$Z(t) = \frac{p}{qB_y}\sin(\Omega_c t) = \rho\sin(\Omega_c t)$$

Magnetic Rigidity

The magnetic rigidity is:

$$B\rho = \left|B_y \rho\right| = \frac{p}{|q|}$$

It depends only on the particle momentum and charge, and is a convenient way to characterize the magnetic field. Given magnetic rigidity and the required bend radius, the required bend field is a simple ratio. Note particles of momentum 100 MeV/*c* have a rigidity of 0.334 T m.

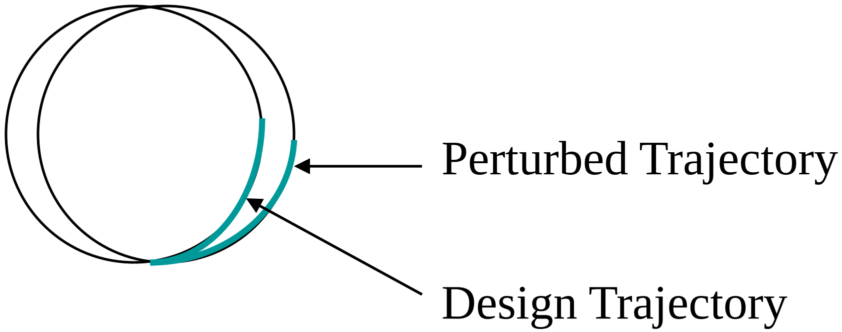
Long Dipole Magnet

$$BL = B\rho\left(2\sin\left(\theta/2\right)\right)$$

Normal Incidence (or exit) Dipole Magnet

$$BL = B\rho\sin\left(\theta\right)$$

Natural Focusing in Bend Plane



Can show that for either a displacement perturbation or angular perturbation from the design trajectory

$$\frac{d^2x}{ds^2} = -\frac{x}{\rho_x^2(s)} \qquad \frac{d^2y}{ds^2} = -\frac{y}{\rho_y^2(s)}$$

Quadrupole Focusing

$$\vec{B}(x,y) = B'(s)(x\hat{y} + y\hat{x})$$

$$\gamma m \frac{d\mathrm{v}_{\mathrm{x}}}{ds} = -qB'(s)\,x \qquad \gamma m \frac{d\mathrm{v}_{\mathrm{y}}}{ds} = qB'(s)\,y$$

$$\frac{d^2x}{ds^2} + \frac{B'(s)}{B\rho}x = 0 \qquad \frac{d^2y}{ds^2} - \frac{B'(s)}{B\rho}y = 0$$

Combining with the previous slide

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho}\right]x = 0 \qquad \frac{d^2y}{ds^2} + \left[\frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}\right]y = 0$$

Hill's Equation

Define focusing strengths (with units of m^{-2})

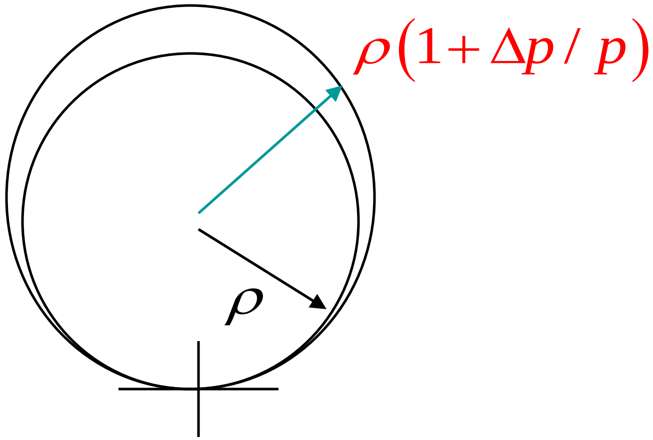
$$k_x(s) = \frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho} \qquad k_y = \frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}$$

$$\frac{d^2x}{ds^2} + k_x(s)x = 0 \qquad \frac{d^2y}{ds^2} + k_y(s)y = 0$$

Note that this is like the harmonic oscillator, or exponential for constant *K*, but more general in that the focusing strength, and hence oscillation frequency depends on *s*

Energy Effects

$$\Delta x(s) = \frac{p}{eB_y}\frac{\Delta p}{p}\left(1-\cos(s/\rho)\right)$$



This solution is not a solution to Hill's equation directly, but *is* a solution to the inhomogeneous Hill's Equations

$$\frac{d^2x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]x=\frac{1}{\rho_x(s)}\frac{\Delta p}{p}$$

$$\frac{d^2y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]y=\frac{1}{\rho_y(s)}\frac{\Delta p}{p}$$

Inhomogeneous Hill's Equations

Fundamental transverse equations of motion in particle accelerators for small deviations from design trajectory

$$\frac{d^2x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]x=\frac{1}{\rho_x(s)}\frac{\Delta p}{p}$$

$$\frac{d^2y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]y=\frac{1}{\rho_y(s)}\frac{\Delta p}{p}$$

ρ radius of curvature for bends, B' transverse field gradient for magnets that focus (positive corresponds to horizontal focusing), $\Delta p/p$ momentum deviation from design momentum. Homogeneous equation is 2$^{\text{nd}}$ order *linear* ordinary differential equation.

Dispersion

From theory of linear ordinary differential equations, the general solution to the inhomogeneous equation is the sum of **any** solution to the inhomogeneous equation, called the particular integral, plus two linearly independent solutions to the homogeneous equation, whose amplitudes may be adjusted to account for boundary conditions on the problem.

$$x(s)=x_p(s)+A_x x_1(s)+B_x x_2(s) \qquad y(s)=y_p(s)+A_y y_1(s)+B_y y_2(s)$$

Because the inhomogeneous terms are proportional to $\Delta p/p$, the particular solution can generally be written as

$$x_p(s)=D_x(s)\frac{\Delta p}{p} \qquad y_p(s)=D_y(s)\frac{\Delta p}{p}$$

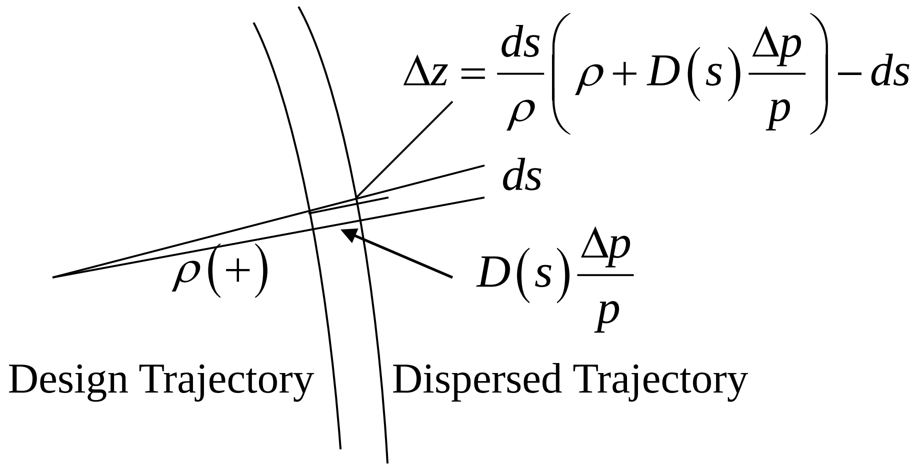
where the dispersion functions satisfy

$$\frac{d^2 D_x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]D_x=\frac{1}{\rho_x(s)} \qquad \frac{d^2 D_y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]D_y=\frac{1}{\rho_y(s)}$$

M_{56}

In addition to the transverse effects of the dispersion, there are important effects of the dispersion along the direction of motion. The primary effect is to change the time-of-arrival of the off-momentum particle compared to the on-momentum particle which traverses the design trajectory.

$$d(\Delta z) = D(s)\frac{\Delta p}{p}\frac{ds}{\rho(s)}$$



$$M_{56} = \int_{s_1}^{s_2}\left\{\frac{D_x(s)}{\rho_x(s)} + \frac{D_y(s)}{\rho_y(s)}\right\}ds$$

Solutions Homogeneous Eqn.

Dipole

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\left(s-s_i\right)/\rho\right) & \rho\sin\left(\left(s-s_i\right)/\rho\right) \\ -\sin\left(\left(s-s_i\right)/\rho\right)/\rho & \cos\left(\left(s-s_i\right)/\rho\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the focusing direction $k = B' / B\rho$

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k}\,(s-s_i)\right) & \sin\left(\sqrt{k}\,(s-s_i)\right)/\sqrt{k} \\ -\sqrt{k}\sin\left(\sqrt{k}\,(s-s_i)\right) & \cos\left(\sqrt{k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \dfrac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the defocusing direction $k = B' / B\rho$

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}\,(s-s_i)\right) & \sinh\left(\sqrt{-k}\,(s-s_i)\right)/\sqrt{-k} \\ \sqrt{-k}\sinh\left(\sqrt{-k}\,(s-s_i)\right) & \cosh\left(\sqrt{-k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \dfrac{dx}{ds}(s_i) \end{pmatrix}$$

Transfer Matrices

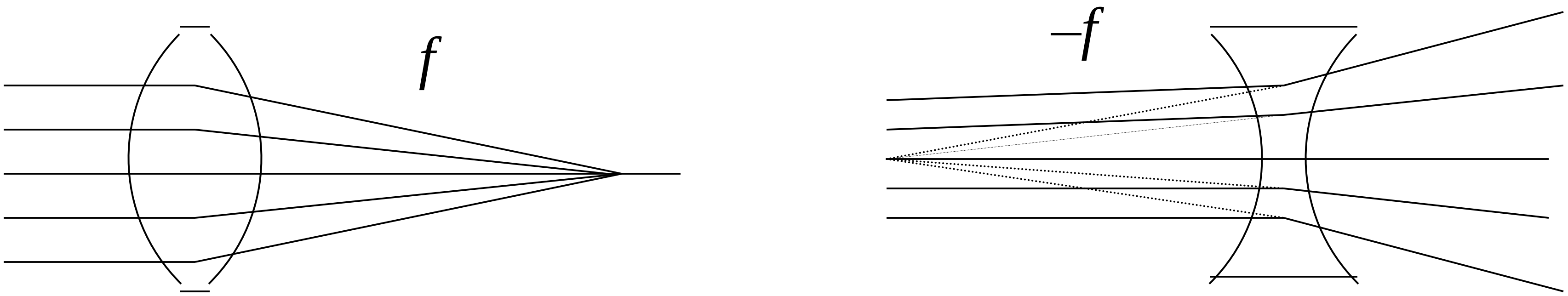
Dipole with bend Θ (put coordinate of final position in solution)

$$\begin{pmatrix} x(s_{after}) \\ \dfrac{dx}{ds}(s_{after}) \end{pmatrix} = \begin{pmatrix} \cos(\Theta) & \rho\sin(\Theta) \\ -\sin(\Theta)/\rho & \cos(\Theta) \end{pmatrix} \begin{pmatrix} x(s_{before}) \\ \dfrac{dx}{ds}(s_{before}) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x(s_{after}) \\ \dfrac{dx}{ds}(s_{after}) \end{pmatrix} = \begin{pmatrix} 1 & L_{drift} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(s_{before}) \\ \dfrac{dx}{ds}(s_{before}) \end{pmatrix}$$

Thin Lenses



Thin Focusing Lens (limiting case when argument goes to zero!)

$$\begin{pmatrix} x\left(s_{lens}+\varepsilon\right) \\ \dfrac{dx}{ds}\left(s_{lens}+\varepsilon\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x\left(s_{lens}-\varepsilon\right) \\ \dfrac{dx}{ds}\left(s_{lens}-\varepsilon\right) \end{pmatrix}$$

Thin Defocusing Lens: change sign of f

Composition Rule: Matrix Multiplication!

Element 1	Element 2

s_0 s_1 s_2

$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} \qquad \begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

More generally

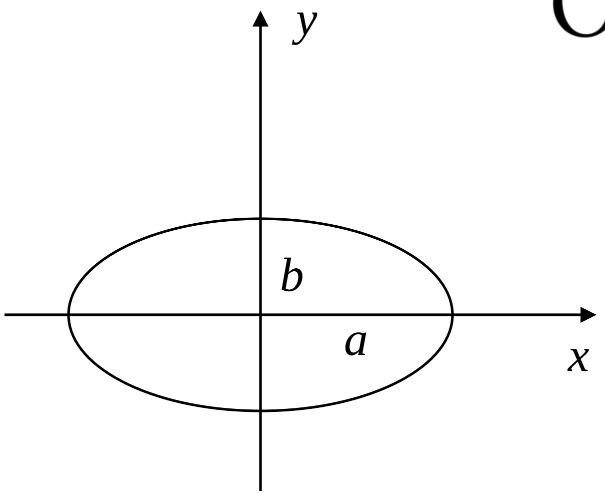
$$M_{tot} = M_N M_{N-1} \ldots M_2 M_1$$

Remember: First element farthest RIGHT

Some Geometry of Ellipses

Equation for an upright ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



In beam optics, the equations for ellipses are normalized (by multiplication of the ellipse equation by *ab*) so that the area of the ellipse divided by π appears on the RHS of the defining equation. For a general ellipse

$$Ax^2 + 2Bxy + Cy^2 = D$$

The area is easily computed to be

$$\frac{\text{Area}}{\pi} \equiv \varepsilon = \frac{D}{\sqrt{AC - B^2}} \qquad \text{Eqn. (1)}$$

So the equation is equivalently

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon$$

$$\gamma = \frac{A}{\sqrt{AC - B^2}}, \quad \alpha = \frac{B}{\sqrt{AC - B^2}}, \quad \text{and} \quad \beta = \frac{C}{\sqrt{AC - B^2}}$$

When normalized in this manner, the equation coefficients clearly satisfy

$$\beta\gamma - \alpha^2 = 1$$

Example: the defining equation for the upright ellipse may be rewritten in following suggestive way

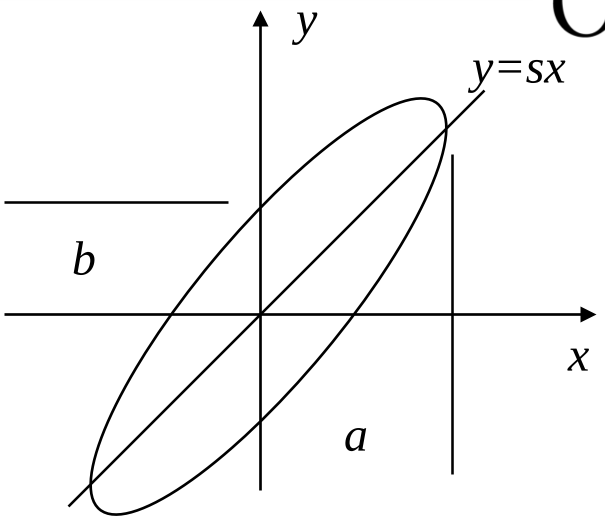
$$\frac{b}{a}x^2 + \frac{a}{b}y^2 = ab = \varepsilon$$

$\beta = a/b$ and $\gamma = b/a$, note $x_{\max} = a = \sqrt{\beta\varepsilon}, \quad y_{\max} = b = \sqrt{\gamma\varepsilon}$

General Tilted Ellipse

Needs 3 parameters for a complete description. One way

$$\frac{b}{a}x^2 + \frac{a}{b}(y - sx)^2 = ab = \varepsilon$$



where s is a slope parameter, a is the maximum extent in the x-direction, and the y-intercept occurs at $\pm b$, and again ε is the area of the ellipse divided by π

$$\frac{b}{a}\left(1 + s^2\frac{a^2}{b^2}\right)x^2 - 2s\frac{a}{b}xy + \frac{a}{b}y^2 = ab = \varepsilon$$

Identify

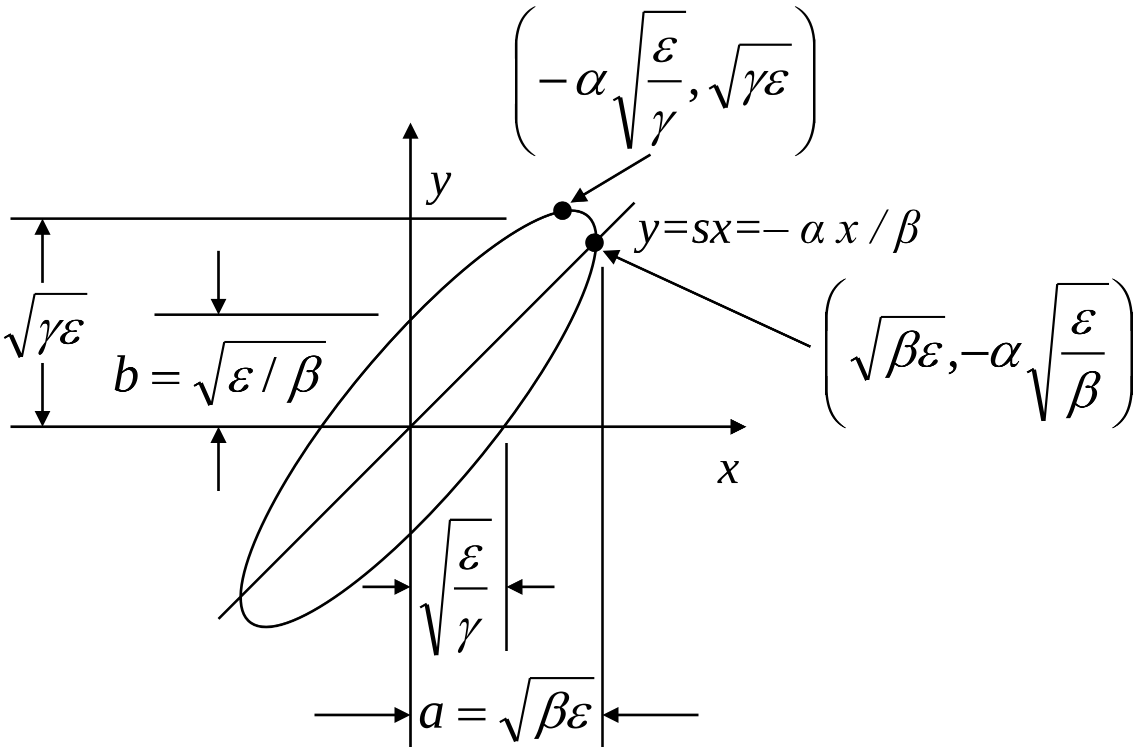
$$\gamma = \frac{b}{a}\left(1+s^2\frac{a^2}{b^2}\right), \quad \alpha = -\frac{a}{b}s, \quad \beta = \frac{a}{b}$$

Note that $\beta\gamma - \alpha^2 = 1$ automatically, and that the equation for ellipse becomes

$$x^2 + \left(\beta y + \alpha x\right)^2 = \beta\varepsilon$$

by eliminating the (redundant!) parameter γ

Ellipse in the β-function Description



As for the upright ellipse $x_{\max} = \sqrt{\beta\varepsilon}, \quad y_{\max} = \sqrt{\gamma\varepsilon}$