Theoretical Mechanics
Mid-Term Exam
October 18, 2018

1. A simple pendulum (mass M and length L) is suspended from a cart (mass m) that oscillates on the end of a spring of force constant k (Fig. 1)
   a. Assuming the angle ϕ remains small, write down the system's Lagrangian, mass matrix $\underline{M}$, potential matrix $\underline{K}$, and the equations of motion of this system .
   b. Assume that $m=M=L=g=1$ and $k=2$ (all in appropriate units) find the normal frequencies.
   c. Determine the corresponding normal modes. (Hint: construct your matrices then apply the small angle approximation.)

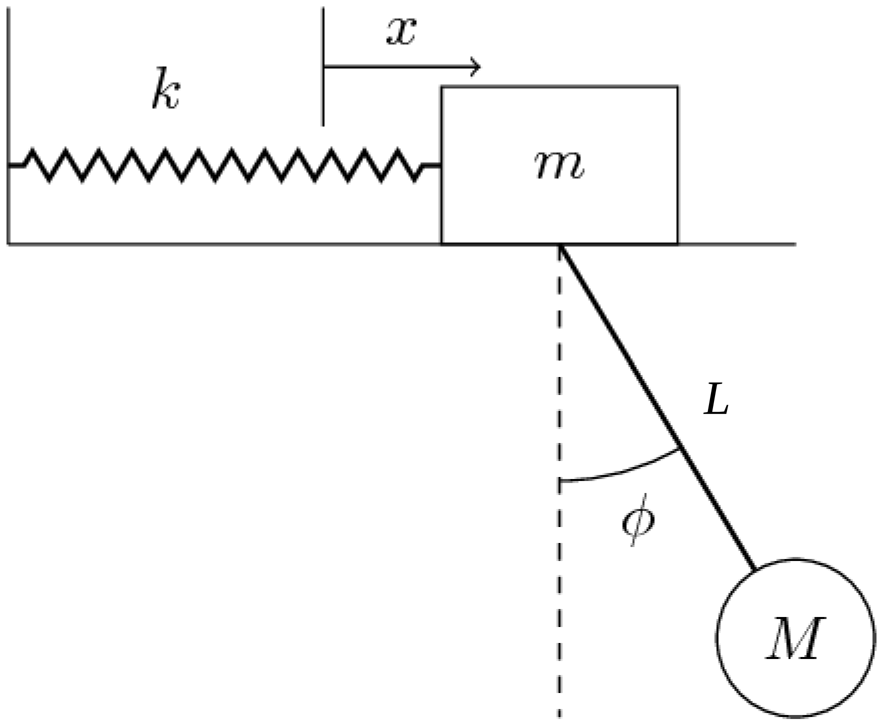

Fig. 1

2. As mentioned several times in lectures, a good way to understand the magnetic field is in terms of the magnetic field form

$$\omega_{\vec{B}}^2 = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy,$$

   where $\vec{B}$ is the usual magnetic field "vector".
   a. How is the Maxwell Equation $\nabla \cdot \vec{B} = 0$ expressed in terms of the exterior derivative?
   b. Suppose γ is closed curve and σ_1 and σ_2, two non–intersecting surfaces with $\partial\sigma_1 = \partial\sigma_2 = \gamma$. Show, using generalized Stoke's Theorem

$$\int_{\sigma_1} \omega_{\vec{B}}^2 = \int_{\sigma_2} \omega_{\vec{B}}^2.$$

   In other words, the magnetic flux through a closed loop is independent of the surface used to compute it.

c. If a magnetic vector potential $\vec{A}$ is found such that $\vec{B} = \nabla \times \vec{A}$, what is $\int_\gamma \omega^1_{\vec{A}}$?

3. The Lagrangian for the one dimensional motion of a particle in a uniform gravitational is

$$L = \frac{m}{2} v_y^2 - mgy,$$

where y is a vertical coordinate and $v_y = dy/dt$.

a. Show the Hamiltonian is

$$H(\vec{q}, \vec{p}) = \frac{p_y^2}{2m} + mgy,$$

b. Show the Hamiltonian equations of motion give the usual Newtonian equation for a uniformly accelerating motion.

c. Is the Hamiltonian explicitly dependent on time? What is the Hamilton-Jacobi equation for this problem?

d. Solve the Hamilton-Jacobi equation for the action function $S(y, \alpha, t)$. Let $\beta = \partial S / \partial \alpha$. Solve for $y = y(\alpha, \beta)$.

e. Show the initial conditions applied to the solution in d. yield the constants of the motion

$$\alpha = \frac{m}{2} v_0^2 + mgy_0$$
$$\beta = -\frac{v_0}{g}.$$

and the usual equation for uniform acceleration.