Physics 601 (Fall 2012)
Homework Assignment 5: Solutions

Due: Thursday October 5, 2012

Fetter & Walecka, Problem 6.2 Using the Hamiltonian for a free particle in spherical coordinates (previous homework assignment), but with r removed as degree of freedom, the Hamiltonian here is $H = \frac{1}{2m\ell^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - mg\ell\cos\theta$. This leads to the following Hamilton's equations:

$$\begin{aligned}
\dot{\phi} &= \frac{1}{m\ell^2}\frac{p_\phi}{\sin^2\theta}, \\
\dot{p}_\phi &= 0, \\
\dot{\theta} &= \frac{1}{m\ell^2}p_\theta \\
\dot{p}_\theta &= \frac{p_\phi^2}{2m\ell^2}\frac{\cos\theta}{\sin^3\theta} - mg\ell\sin\theta.
\end{aligned}$$

We can eliminate p_θ from the last two equations and obtain

$$m\ell^2\ddot{\theta} = \frac{p_\phi^2}{2m\ell^2}\frac{\cos\theta}{\sin^3\theta} - mg\ell\sin\theta.$$

For uniform circular motion $\dot{\phi} = \omega$, $\theta = \theta_0$ and $\ddot{\theta} = \dot{\theta} = 0$, so

$$\begin{aligned}
\omega_0 &= \frac{1}{m\ell^2}\frac{p_\phi}{\sin^2\theta}, \\
p_\phi^2 &= \left(m\ell^2\sin^2\theta_0\right)^2\frac{g}{\ell\cos\theta_0}.
\end{aligned}$$

We introduce small deviation from this circular motion, $\theta = \theta_0 + \Delta\theta$ and $\dot{\theta} = \Delta\dot{\theta}$, and use

$$\begin{aligned}
\frac{1}{\sin^2\theta} &= (\sin\theta_0\cos\Delta\theta + \cos\theta_0\sin\Delta\theta)^{-2} \\
&= \left(\sin\theta_0\left(1 - \frac{\Delta\theta^2}{2}\right) + \cos\theta_0\Delta\theta\right)^{-2} + \mathcal{O}(\Delta\theta^3) \\
&= \frac{1}{\sin^2\theta_0}\left(1 + \frac{\cos\theta_0}{\sin\theta_0}\Delta\theta - \frac{\Delta\theta^2}{2}\right)^{-2} + \mathcal{O}(\Delta\theta^3) \\
&= \frac{1}{\sin^2\theta_0}\left(1 - 2\left(\frac{\cos\theta_0}{\sin\theta_0}\Delta\theta - \frac{\Delta\theta^2}{2}\right) + 3\left(\frac{\cos\theta_0}{\sin\theta_0}\Delta\theta\right)^2\right) + \mathcal{O}(\Delta\theta^3) \\
&= \frac{1}{\sin^2\theta_0}\left(1 - 2\frac{\cos\theta_0}{\sin\theta_0}\Delta\theta + \left(1 + 3\frac{\cos^2\theta_0}{\sin^2\theta_0}\right)\Delta\theta^2\right) + \mathcal{O}(\Delta\theta^3),
\end{aligned}$$

and

$$\begin{aligned}
\cos\theta &= \cos\theta_0\cos\Delta\theta - \sin\theta_0\sin\Delta\theta \\
&= \cos\theta_0\left(\left(1-\frac{\Delta\theta^2}{2}\right) - \frac{\sin\theta_0}{\cos\theta_0}\Delta\theta\right) + \mathcal{O}(\Delta\theta^3) \\
&= \cos\theta_0\left(1 - \frac{\sin\theta_0}{\cos\theta_0}\Delta\theta - \frac{\Delta\theta^2}{2}\right) + \mathcal{O}(\Delta\theta^3).
\end{aligned}$$

The Hamiltonian becomes now

$$\begin{aligned}
H &= \frac{1}{2m\ell^2}\left(p_{\Delta\theta}^2 + \frac{p_\phi^2}{\sin^2\theta_0}\left(1 - 2\frac{\cos\theta_0}{\sin\theta_0}\Delta\theta + \left(1 + 3\frac{\cos^2\theta_0}{\sin^2\theta_0}\right)\Delta\theta^2\right)\right) \\
&\quad - mg\ell\cos\theta_0\left(1 - \frac{\sin\theta_0}{\cos\theta_0}\Delta\theta - \frac{\Delta\theta^2}{2}\right) + \mathcal{O}(\Delta\theta^3) \\
&= \frac{1}{2m\ell^2}p_{\Delta\theta}^2 \\
&\quad + \frac{1}{2m\ell^2}\left(m\ell^2\sin^2\theta_0\right)^2\frac{g}{\ell\cos\theta_0}\left(1 - 2\frac{\cos\theta_0}{\sin\theta_0}\Delta\theta + \left(1 + 3\frac{\cos^2\theta_0}{\sin^2\theta_0}\right)\Delta\theta^2\right) \\
&\quad - mg\ell\cos\theta_0\left(1 - \frac{\sin\theta_0}{\cos\theta_0}\Delta\theta - \frac{\Delta\theta^2}{2}\right) + \mathcal{O}(\Delta\theta^3) + constant \\
&= \frac{1}{2m\ell^2}p_{\Delta\theta}^2 + \frac{1}{2}m\ell^2\frac{g}{\ell\cos\theta_0}\left(1 + 3\cos^2\theta_0\right)\Delta\theta^2 + \mathcal{O}(\Delta\theta^3) + constant.
\end{aligned}$$

This is the Hamiltonian for a harmonic oscillator with $\omega^2 = \frac{g}{\ell\cos\theta_0}\left(1 + 3\cos^2\theta_0\right)$.

Fetter & Walecka, Problem 6.6 The kinetic energy $T = \frac{1}{2}m\dot{q}^2$ and potential energy $V = \frac{1}{2}kq^2$, so the Lagrangian is $L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$. The generalized moment is $p = m\dot{q}$, so the Hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$.

The transformation is canonical if it maintains the Poisson bracket in

$$\begin{aligned}
[P, Q] &= C^2[p,p] + 2im\omega C^2[p,q] + m^2\omega^2[q,q] \\
&= 2im\omega C^2[p,q],
\end{aligned}$$

so $C^2 = \frac{1}{2im\omega}$.

The generating function $S(q, P)$ satisfies two differential expressions:

$$\begin{aligned}
p &= \frac{\partial S}{\partial q} = \frac{P}{C} + im\omega q \\
Q &= \frac{\partial S}{\partial P} = P + 2Cim\omega q,
\end{aligned}$$

which can be integrated:

$$\begin{aligned}
S(q, P) &= \frac{qP}{C} + \frac{1}{2}im\omega q^2 + g(P) \\
S(q, P) &= \frac{1}{2}P^2 + 2Cim\omega qP + h(q),
\end{aligned}$$

with $g(P)$ and $h(q)$ arbitrary functions. Using the expression for C we find that the qP term is indeed identical, and thus $S(q,P) = \frac{1}{2}P^2 + \frac{qP}{C} + \frac{1}{2}im\omega q^2$.

The new Hamiltonian is $H(P,Q) = \frac{1}{2mC^2}QP = i\omega QP$. Hamilton's equations are now decoupled:

$$\begin{aligned}\dot{Q} &= i\omega Q \\ \dot{P} &= -i\omega P,\end{aligned}$$

and easily solved to $Q(t) = \exp i\omega t$ and $P(t) = \exp -i\omega t$. The original coordinates are then $p(t) = \frac{Q+P}{2C} \propto \cos\omega t$ and $q(t) = \frac{Q-P}{2im\omega} \propto \sin\omega t$.

Fetter & Walecka, Problem 6.9 We immediately write the Lagrangian $L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)(q_1^2 + q_2^2) - (q_1^2 + q_2^2)^{-1}$. The generalized momentum is now $p_i = \frac{\partial L}{\partial q_i} = \dot{q}_i(q_1^2 + q_2^2)$, or $\dot{q}_i = \frac{p_i}{q_1^2+q_2^2}$. The Hamiltonian then becomes

$$H = \frac{1}{2}\frac{p_1^2 + p_2^2}{q_1^2 + q_2^2} + \frac{1}{q_1^2 + q_2^2}.$$

We separate the principal function $S(q_1, q_2, P_1, P_2, t)$ into $W_1(q_1, P_1) + W_2(q_2, P_2) - \alpha t$. The Hamilton-Jacobi equation becomes now

$$\frac{1}{2}\frac{1}{q_1^2 + q_2^2}\left[\left(\frac{\partial W_1}{\partial q_1}\right)^2 + \left(\frac{\partial W_2}{\partial q_2}\right)^2 + 2\right] = \alpha,$$

or separately

$$\left(\frac{\partial W_i}{\partial q_i}\right)^2 - 2\alpha q_i^2 = \alpha_i,$$

with $\alpha_1 + \alpha_2 = -2$. The expressions for $W_i(q_i, \alpha_i)$ are then

$$\begin{aligned}W_1(q_1, \alpha, \alpha_1) &= \pm\int\sqrt{\alpha_1 + 2\alpha q_1^2}dq_1, \\ W_1(q_2, \alpha, \alpha_2) &= \pm\int\sqrt{\alpha_2 + 2\alpha q_2^2}dq_2.\end{aligned}$$

The principal function is now

$$S(q_1, q_2, \alpha, \alpha_1) = \pm\int\sqrt{\alpha_1 + 2\alpha q_1^2}dq_1 \pm \int\sqrt{-2 - \alpha_1 + 2\alpha q_2^2}dq_2 - \alpha t$$

which gives for the constants β and β_1 the expressions

$$\begin{aligned}\beta &= \pm\int\frac{q_1^2 dq_1}{\sqrt{\alpha_1 + 2\alpha q_1^2}} \pm \int\frac{q_2^2 dq_2}{\sqrt{-2 - \alpha_1 + 2\alpha q_2^2}} - t, \\ \beta_1 &= \pm\int\frac{dq_1}{2\sqrt{\alpha_1 + 2\alpha q_1^2}} \pm \int\frac{dq_2}{2\sqrt{-2 - \alpha_1 + 2\alpha q_2^2}}dq_2.\end{aligned}$$

These expressions can now (theoretically) be solved and inverted to get q_1 and q_2 as a function of time t and the integration constants α, α_1, β and β_1.

Jacobi's identity for Poisson brackets With $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial^i = \frac{\partial}{\partial p_i}$ we write

$$\begin{aligned}
[A,[B,C]] &= \sum_i [A, \partial_i B \partial^i C - \partial^i B \partial_i C] \\
&= \sum_i [A, \partial_i B \partial^i C] - \sum_i [A, \partial^i B \partial_i C] \\
&= \sum_{i,j} \partial_j A \partial_i^j B \partial^i C + \sum_{i,j} \partial_j A \partial_i B \partial^{ij} C - \sum_{i,j} \partial^j A \partial_{ij} B \partial^i C - \sum_{i,j} \partial^j A \partial_i B \partial_j^i C \\
&\quad - \sum_{i,j} \partial_j A \partial^{ij} B \partial_i C - \sum_{i,j} \partial_j A \partial^i B \partial_i^j C + \sum_{i,j} \partial^j A \partial_j^i B \partial_i C + \sum_{i,j} \partial^j A \partial^i B \partial_{ij} C.
\end{aligned}$$

We find similar expressions for $[B,[C,A]]$ and $[C,[A,B]]$. Inspection shows that all terms will cancel out.

Fetter & Walecka, Problem 6.18 Using the Levi-Civita symbol e_{ijk}, and with $[r_i, p_k] = \delta_{ik}$, we find

$$\begin{aligned}
[L_m, L_n] &= \left[\sum_{ij} e_{ijm} r_i p_j, \sum_{kl} e_{kln} r_k p_l \right] = \sum_{ij} e_{ijm} \sum_{kl} e_{kln} [r_i p_j, r_k p_l] \\
&= \sum_{ij} e_{ijm} \sum_{kl} e_{kln} \left(r_k r_i [p_j, p_l] + r_k [r_i, p_l] p_j + r_i [p_j, r_k] p_l + [r_i, r_k] p_j p_l \right) \\
&= \sum_{ij} e_{ijm} \sum_{kl} e_{kln} \left(r_k \delta_{il} p_j - r_i \delta_{jk} p_l \right) \\
&= \sum_j \sum_{kl} e_{ljm} e_{kln} r_k p_j - \sum_i \sum_{kl} e_{ikm} e_{kln} r_i p_l \\
&= \sum_{jk} (\delta_{jn}\delta_{km} - \delta_{jk}\delta_{nm}) r_k p_j - \sum_{il} (\delta_{lm}\delta_{in} - \delta_{li}\delta_{nm}) r_i p_l \\
&= \sum_{ji} (\delta_{jn}\delta_{im} - \delta_{ji}\delta_{nm}) r_i p_j - \sum_{ij} (\delta_{jm}\delta_{in} - \delta_{ji}\delta_{nm}) r_i p_j \\
&= \sum_{ij} (\delta_{jn}\delta_{im} - \delta_{jm}\delta_{in}) r_i p_j \\
&= \sum_{ij} \sum_k e_{mnk} e_{ijk} r_i p_j \\
&= \sum_k e_{mnk} \sum_{ij} e_{ijk} r_i p_j \\
&= \sum_k e_{mnk} L_k,
\end{aligned}$$

or you could of course just do this for $m = 1$ and $n = 2$ and find $k = 3$ explicitly. For the magnitude we find

$$[L^2, L_n] = [\sum_m L_m L_m, L_n] = 2 \sum_m L_m [L_m, L_n] = 2 \sum_m e_{mnk} L_m L_k = 0.$$

Because the Poisson brackets of the set L_i are different than those of p_i we cannot form a canonical transformation from p to L.