Physics 601 (Fall 2012)
Homework Assignment 6: Solutions

Due: Friday October 12, 2012

Canonical Transformations From the Hamiltonian $H = \frac{1}{2}(q^{-2} + p^2 q^4)$ we find the equations of motion

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = pq^4, \\ \dot{p} &= -\frac{\partial H}{\partial q} = q^{-3} - 2p^2 q^3. \end{aligned}$$

Using the first equation of motion we find $p = \dot{q}q^{-4}$ and therefore $\dot{p} = \ddot{q}q^{-4} - 4\dot{q}^2 q^{-5}$. We can substitute this in the second equation of motion, and we get $\ddot{q}q^{-4} - 4\dot{q}^2 q^{-5} = q^{-3} - 2\dot{q}^2 q^{-5}$ or $\ddot{q} - 2\dot{q}^2 q^{-1} - q = 0$.

We would like to write the Hamiltonian as $H(P, Q) = \frac{1}{2}(P^2 + Q^2)$. Let us take for example $Q = \frac{1}{q}$ and determine the generating function $S(p, Q)$ and the expression for P. We integrate $q = \frac{\partial S}{\partial p} = \frac{1}{Q}$ and find $S(p, Q) = \frac{p}{Q} + g(Q)$ with an arbitrary function $g(Q)$. The other transformation is then $P = \frac{\partial S}{\partial Q} = -\frac{p}{Q^2} + g'(Q)$. The new Hamiltonian $H(P, Q) = \frac{1}{2}(P^2 + Q^2) = \frac{1}{2}(q^{-2} + p^2 q^4 - 2pq^2 g'(Q) + g'(Q)^2)$. This will be equal to the original Hamiltonian for the choice $g(Q) \equiv 0$, or $P = -pq^2$.

The solution for the transformed system is $Q = A\cos(t + \phi)$ and $P = A\sin(t + \phi)$ with the constants A and ϕ determined by the initial conditions. Using the transformation $q = \frac{1}{Q}$, the solution for q is

$$q = \frac{1}{A}\frac{1}{\cos(t + \phi)}.$$

If we plug this in the equation of motion for q found earlier, we find

$$\begin{aligned} \dot{q} &= \frac{1}{A}\frac{\sin(t + \phi)}{\cos^2(t + \phi)}, \\ \ddot{q} &= 2\frac{1}{A}\frac{\sin^2(t + \phi)}{\cos^3(t + \phi)} + \frac{1}{A}\frac{1}{\cos(t + \phi)}, \\ -2\dot{q}^2 q^{-1} &= -2\frac{1}{A}\frac{\sin^2(t + \phi)}{\cos^3(t + \phi)}. \end{aligned}$$

From these expressions it is clear that indeed $\ddot{q} - 2\dot{q}^2 q^{-1} - q = 0$.

Fetter & Walecka, Problem 6.10 a) The Lagrangian of the system is given by equation (20.26) $L = \frac{1}{2}ma^2[\omega^2 + (\omega + \dot{\theta})^2 + 2\omega(\omega + \dot{\theta})\cos\theta]$, and the generalized momentum by equation (20.27) $p = ma^2[\omega(1 + \cos\theta) + \dot{\theta}]$, which we can invert to $\dot{\theta} = \frac{p}{ma^2} - \omega(1 + \cos\theta)$. The Hamiltonian in equation (20.28) then becomes

$$\begin{aligned} H &= ma^2\left[\frac{1}{2}\dot{\theta}^2 - \omega^2(1+\cos\theta)\right] \\ &= ma^2\left[\frac{1}{2}\left(\frac{p}{ma^2} - \omega(1+\cos\theta)\right)^2 - \omega^2(1+\cos\theta)\right] \\ &= \frac{p^2}{2ma^2} - p\omega(1+\cos\theta) + \frac{1}{2}ma^2\omega^2\left[(1+\cos\theta)^2 - 2(1+\cos\theta)\right] \\ &= \frac{p^2}{2ma^2} - p\omega(1+\cos\theta) - \frac{1}{2}ma^2\omega^2\sin^2\theta. \end{aligned}$$

Hamilton's equations are now

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial \theta} = -p\omega\sin\theta + ma^2\omega^2\sin\theta\cos\theta, \\ \dot{\theta} &= \frac{\partial H}{\partial p} = \frac{p}{ma^2} - \omega(1+\cos\theta). \end{aligned}$$

From the second equation, which we already obtained earlier, we find $\dot{p} = ma^2(\ddot{\theta} - \omega\dot{\theta}\sin\theta)$, and we can substitute this into the first equation:

$$ma^2(\ddot{\theta} - \omega\dot{\theta}\sin\theta) = -ma^2\left(\dot{\theta} + \omega(1+\cos\theta)\right)\omega\sin\theta + ma^2\omega^2\sin\theta\cos\theta.$$

This simplifies to $\ddot{\theta} + \omega^2\sin\theta = 0$, the equation for the simple pendulum.

b) The Hamilton-Jacobi equation is, with $S(\theta, \alpha, t) = W(\theta, \alpha) - \alpha t$ and using the expression for H before expansion,

$$ma^2\left[\frac{1}{2}\left(\frac{1}{ma^2}\frac{dW}{d\theta} - \omega(1+\cos\theta)\right)^2 - \omega^2(1+\cos\theta)\right] = \alpha.$$

We can solve for $\frac{dW}{d\theta}$ and find

$$\frac{dW}{d\theta} = ma^2\left[\omega(1+\cos\theta) \pm \sqrt{\frac{2}{ma^2}\left(\alpha + ma^2\omega^2(1+\cos\theta)\right)}\right].$$

Therefore, when we integrate, we obtain

$$W(\theta, \alpha) = \int d\theta ma^2\omega(1+\cos\theta) \pm \int d\theta\sqrt{2ma^2\left(\alpha + ma^2\omega^2\left(1+\cos\theta\right)\right)}.$$

For the other integration constant β we find an elliptic integral

$$\begin{aligned} \beta + t &= \frac{dW}{d\alpha} = \pm\int d\theta \frac{\sqrt{2ma^2}}{2\sqrt{2ma^2\left(\alpha + ma^2\omega^2\left(1+\cos\theta\right)\right)}}, \\ &= \pm\int d\theta \frac{1}{2\sqrt{\alpha + ma^2\omega^2 + ma^2\omega^2\cos\theta}}. \end{aligned}$$

Fetter & Walecka, Problem 6.4 a) We use the relativistic gamma factor $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$, which still depends on $\dot{x}_i$, to write $L = -\frac{1}{\gamma}mc^2 - V(\vec{r})$. Using

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_i} &= \frac{m\dot{x}_i}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m \dot{x}_i = p_i, \\ \frac{\partial L}{\partial x_i} &= -\frac{\partial V}{\partial x_i} = F_i. \end{aligned}$$

and with the relativistic momentum $\vec{p} = \gamma m \vec{v}$ and conservative force $\vec{F}$ the Lagrange equation then becomes simply $\dot{\vec{p}} = \vec{F}$.

b) The Hamiltonian is, using $p^2 = \gamma^2 m^2 v^2$ or $v^2 = p^2 \left(m^2 + \frac{p^2}{c^2}\right)^{-1}$ and therefore $\gamma = \sqrt{1 + \frac{p^2}{m^2 c^2}}$,

$$H = \vec{p} \cdot \vec{v} - L = \gamma m v^2 + \frac{1}{\gamma} mc^2 + V(\vec{r}) = \gamma mc^2 + V(\vec{r}) = \sqrt{m^2c^4 + p^2c^2} + V(\vec{r}).$$

Since this does not depend explicitly on t, the Hamiltonian will be a conserved quantity.

c) Central forces result in planar motion. In polar coordinates the momentum is $\vec{p} = p_r \hat{e}_r + \frac{1}{r} p_\phi \hat{e}_\phi$ and $p^2 = p_r^2 + \frac{p_\phi^2}{r^2}$. The Hamiltonian is then

$$H = c\sqrt{m^2c^2 + p_r^2 + \frac{p_\phi^2}{r^2}} + V(r).$$

This is cyclic in ϕ, and the momentum p_ϕ will therefore be constant. Using $\vec{L} = \vec{r} \times \vec{p} = \hat{e}_r \times p_\phi \hat{e}_\phi = p_\phi \hat{e}_\theta$, we can rewrite the Hamiltonian as $H = c\sqrt{m^2c^2 + p_r^2 + \frac{L^2}{r^2}} + V(r)$.

Fetter & Walecka, Problem 6.14 a) Using the Hamiltonian $H = c\sqrt{m^2c^2 + p_r^2 + \frac{p_\phi^2}{r^2}} + V(r)$ we can write the Hamilton-Jacobi equation, with $S = W_r + W_\phi - Et$, as

$$c\sqrt{m^2c^2 + \left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W_\phi}{\partial \phi}\right)} + V(r) = E.$$

We immediately have $W_\phi = \phi p_\phi$ for the cyclic variable ϕ, and find for W_r

$$\begin{aligned} \frac{\partial W_r}{\partial r} &= \pm\frac{1}{c}\sqrt{(E - V(r))^2 - p_\phi^2 c^2 r^{-2} - m^2c^4}, \\ W_r &= \pm\frac{1}{c}\int dr \sqrt{(E - V(r))^2 - p_\phi^2 c^2 r^{-2} - m^2c^4}. \end{aligned}$$

After integration and differentiation we find two expressions

$$\begin{aligned}
\beta &= \frac{\partial S}{\partial E} = \frac{\partial W_r}{\partial E} - t \\
&= \pm\frac{1}{c}\int dr \frac{E - V(r)}{\sqrt{(E - V(r))^2 - p_\phi^2 c^2 r^{-2} - m^2 c^4}} - t, \\
\beta_\phi &= \frac{\partial S}{\partial p_\phi} = \frac{\partial W_r}{\partial p_\phi} + \phi \\
&= \pm\frac{1}{c}\int dr \frac{-p_\phi c^2 r^{-2}}{\sqrt{(E - V(r))^2 - p_\phi^2 c^2 r^{-2} - m^2 c^4}} + \phi.
\end{aligned}$$

After integration, the first expression will give $r(t)$ while the second expression will result in $r(\phi)$.

b) For the gravitational potential, we have

$$\begin{aligned}
\beta_\phi - \phi &= \pm\frac{1}{c}\int dr \frac{-p_\phi c^2 r^{-2}}{\sqrt{\left(E + GMm\frac{1}{r}\right)^2 - p_\phi^2 c^2 \frac{1}{r^2} - m^2 c^4}} \\
&= \pm\frac{1}{c}\int dr \frac{-p_\phi c^2 r^{-2}}{\sqrt{-\left(p_\phi^2 c^2 - (GMm)^2\right)\frac{1}{r^2} + 2EGMm\frac{1}{r} - (m^2 c^4 - E^2)}}.
\end{aligned}$$

We define now the positive constants

$$\begin{aligned}
A &= \left(p_\phi^2 c^2 - (GMm)^2\right), \\
B &= EGMm, \\
C &= \left(m^2 c^4 - E^2\right),
\end{aligned}$$

such that the discriminant

$$\begin{aligned}
B^2 - AC &= E^2 (GMm)^2 - (p_\phi^2 c^2 - (GMm)^2)(m^2 c^4 - E^2) \\
&= E^2 (GMm)^2 - p_\phi^2 c^2 (m^2 c^4 - E^2) + (GMm)^2 (m^2 c^4 - E^2) \\
&= (GMm)^2 m^2 c^4 - p_\phi^2 c^2 (m^2 c^4 - E^2)
\end{aligned}$$

is positive for $p_\phi c > GMm$ and $mc^2 > E$. We rewrite the integral as

$$\begin{aligned}
\beta_\phi - \phi &= \pm\frac{1}{c}\int \frac{-p_\phi c^2 r^{-2} dr}{\sqrt{-\frac{A}{r^2} + 2\frac{B}{r} - C}} \\
&= \pm\frac{1}{c}\int \frac{-p_\phi c^2 r^{-2} dr}{\sqrt{\left(\frac{B^2}{A} - C\right) - \left(\frac{B}{\sqrt{A}} - \frac{\sqrt{A}}{r}\right)^2}} \\
&= \pm\frac{1}{c}\int \frac{-\frac{p_\phi c^2}{\sqrt{A}} d\left(\frac{B}{\sqrt{A}} - \frac{\sqrt{A}}{r}\right)}{\sqrt{\left(\frac{B^2}{A} - C\right) - \left(\frac{B}{\sqrt{A}} - \frac{\sqrt{A}}{r}\right)^2}} \\
\phi - \beta_\phi &= \pm\frac{p_\phi c}{\sqrt{A}} \cos^{-1} \frac{\frac{B}{\sqrt{A}} - \frac{\sqrt{A}}{r}}{\sqrt{\frac{B^2}{A} - C}} = \pm\frac{p_\phi c}{\sqrt{A}} \cos^{-1} \frac{1 - \frac{A}{B}\frac{1}{r}}{\sqrt{1 - \frac{AC}{B^2}}},
\end{aligned}$$

where we used $\int \frac{du}{\sqrt{k^2-u^2}} = -\cos^{-1}\frac{u}{k}$. We can solve this for r as

$$r(\phi) = \frac{A}{B}\frac{1}{1-\sqrt{1-\frac{AC}{B^2}}\cos\left(\pm\frac{\sqrt{A}}{p_\phi c}(\phi-\beta_\phi)\right)}.$$

The general solution for a conic section with semimajor axis a, and eccentricity e, is given by equation (3.20b),

$$r = \frac{a(1-e^2)}{1-e\cos\phi}.$$

so we can identify for an ellipse with $e < 1$

$$\begin{aligned} e &= \sqrt{1-\frac{AC}{B^2}} \approx 1-\frac{AC}{2B^2} < 1, \\ a &= \frac{A}{B}\frac{1}{(1-e^2)} = \frac{B}{C}. \end{aligned}$$

The only difference is the scale factor $\frac{\sqrt{A}}{p_\phi c}$ in the argument of the cosine. This will ensure that a change over 2π in ϕ starting at the perihelium will not bring r back to the perihelium value. Instead, ϕ will have to change over $2\pi\frac{p_\phi c}{\sqrt{A}}$ before r is back at the perihelium, and the perihelium will precess. The perihelium precession per rotation is given by

$$\begin{aligned} \delta\phi &= 2\pi\left(1-\frac{p_\phi c}{\sqrt{A}}\right) \\ &= 2\pi\left(1-\frac{p_\phi c}{\sqrt{p_\phi^2 c^2 - G^2M^2m^2}}\right) \\ &\approx 2\pi\left(1-\left(1-\frac{1}{2}\frac{G^2M^2m^2}{p_\phi^2 c^2}\right)\right) \\ &= \pi\left(\frac{GMm}{p_\phi c}\right)^2. \end{aligned}$$

For Mercury, with

$$\begin{aligned} \frac{GM}{c^2} &= 1.475\,\mathrm{km}, \\ m &= 3.30\cdot 10^{23}\,\mathrm{kg}, \\ p_\phi &= 9.1\cdot 10^{38}\,\mathrm{kg\ m^2\,s^{-1}}, \\ \tau &= 87.97\,\mathrm{days}, \end{aligned}$$

this evaluates to $\delta\phi = 8.1\cdot 10^{-8}$ radians, or for 100 years $\Delta\phi = 3.4\cdot 10^{-5}$ radians, or 6.9 arc seconds.

c) When $p_\phi < \frac{GMm}{c}$ but still $E < mc^2$ the constant A is not positive anymore, and we can now define instead $A = \left(G^2M^2m^2 - p_\phi^2c^2\right)$ without changing B and C. The integral becomes now

$$\begin{aligned}\beta_\phi - \phi &= \pm\frac{1}{c}\int \frac{-p_\phi c^2 r^{-2} dr}{\sqrt{\frac{A}{r^2} + 2\frac{B}{r} - C}} \\ &= \pm\frac{1}{c}\int \frac{-p_\phi c^2 r^{-2} dr}{\sqrt{\left(\frac{\sqrt{A}}{r} + \frac{B}{\sqrt{A}}\right)^2 - \left(\frac{B^2}{A} + C\right)}},\end{aligned}$$

which will lead to an eccentricity

$$e = \sqrt{1 + \frac{AC}{B^2}} \approx 1 + \frac{AC}{2B^2} > 1$$

for a hyperbola.

Fetter & Walecka, Problem 6.17 a) From the generating function $S(\vec{q}, \vec{P}, t) = \vec{q}\cdot\vec{P} + \epsilon G(\vec{q}, \vec{P}, t)$ we can immediately write the transformations and expand

$$\begin{aligned}p_i &= \frac{\partial S}{\partial q_i} = P_i + \epsilon\frac{\partial G}{\partial q_i}(\vec{q}, \vec{P}, t), \\ P_i &= p_i - \epsilon\frac{\partial G}{\partial q_i}(\vec{q}, \vec{P}, t) \\ &= p_i - \epsilon\frac{\partial G}{\partial q_i}(\vec{q}, \vec{p} - \epsilon\frac{\partial G}{\partial q_i}(\vec{q}, \vec{P}, t), t) \\ &= p_i - \epsilon\frac{\partial G}{\partial q_i}(\vec{q}, \vec{p}, t) + \mathcal{O}(\epsilon^2), \\ Q_i &= \frac{\partial S}{\partial P_i} = q_i + \epsilon\frac{\partial G}{\partial P_i}(\vec{q}, \vec{P}, t) \\ &= q_i + \epsilon\frac{\partial G}{\partial P_i}(\vec{q}, \vec{p} - \epsilon\frac{\partial G}{\partial q_i}(\vec{q}, \vec{P}, t), t) \\ &= q_i + \epsilon\frac{\partial G}{\partial p_i}(\vec{q}, \vec{p}, t) + \mathcal{O}(\epsilon^2).\end{aligned}$$

b) Under this canonical transformation we can write the change in F as

$$\begin{aligned}dF &= \sum_i \frac{\partial F}{\partial q_i} dq_i + \sum_i \frac{\partial F}{\partial p_i} dp_i \\ &= \epsilon\sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \epsilon\sum_i \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \\ &= \epsilon[F, G].\end{aligned}$$

c) The change in the Hamiltonian under these transformations is $dH = \epsilon[H, G]$. If G is a constant of motion then, by equation (37.5), we must have $[H, G] = 0$. In a system of particles with two-body potentials, we have

$$H = \sum_i \frac{|\vec{p}_i|^2}{2m_i} + \sum_{i\neq j}\frac{1}{2}V(\vec{r}_i, \vec{r}_j).$$

The total linear momentum is $G\hat{e}_G = \sum_i \vec{p}_i$, or $G = \sum_i \vec{p}_i \cdot \hat{e}_G$. The change dH in the Hamiltonian under the corresponding transformation

$$\begin{aligned} \vec{P}_i &= \vec{p}_i, \\ \vec{R}_i &= \vec{r}_i + \epsilon \hat{e}_G \cdot \sum_j \delta_{ij} = \vec{r}_i + \epsilon \hat{e}_G, \end{aligned}$$

is then

$$\begin{aligned} dH = H(P,R) - H(p,r) &= \sum_{i \neq j} \frac{1}{2} \left(V(\vec{R}_i, \vec{R}_j) - V(\vec{r}_i, \vec{r}_j) \right) \\ &= \sum_{i \neq j} \frac{1}{2} \left(V(\vec{r}_i + \epsilon \hat{e}_G, \vec{r}_j + \epsilon \hat{e}_G) - V(\vec{r}_i, \vec{r}_j) \right) \\ &= \sum_{i \neq j} \frac{1}{2} \epsilon \left(\frac{\partial V}{\partial \vec{r}_i} + \frac{\partial V}{\partial \vec{r}_j} \right) \cdot \hat{e}_G + \mathcal{O}(\epsilon^2). \end{aligned}$$

This will be zero if $V(\vec{r}_i, \vec{r}_j) = V(\vec{r}_i - \vec{r}_j)$, or the two-body potential only depends on the distance between the two particles.

The total angular momentum is $G\hat{e}_G = \sum_i \vec{r}_i \times \vec{p}_i$, or $G = \sum_i (\vec{r}_i \times \vec{p}_i) \cdot \hat{e}_G = \sum_i (\hat{e}_G \times \vec{r}_i) \cdot \vec{p}_i = -\sum_i (\hat{e}_G \times \vec{p}_i) \cdot \vec{r}_i$. The change dH in the Hamiltonian under the corresponding transformation

$$\begin{aligned} \vec{P}_i &= \vec{p}_i - \epsilon \hat{e}_G \times \vec{p}_i, \\ \vec{R}_i &= \vec{r}_i + \epsilon \hat{e}_G \times \vec{r}_i, \end{aligned}$$

is then

$$\begin{aligned} dH = H(P,R) - H(p,r) &= \sum_{i \neq j} \frac{1}{2} \left(V(\vec{R}_i, \vec{R}_j) - V(\vec{r}_i, \vec{r}_j) \right) \\ &= \sum_{i \neq j} \frac{1}{2} \left(V(\vec{r}_i + \epsilon \hat{e}_G \times \vec{r}_i, \vec{r}_j + \epsilon \hat{e}_G \times \vec{r}_j) - V(\vec{r}_i, \vec{r}_j) \right) \\ &= \sum_{i \neq j} \frac{1}{2} \epsilon \left(\frac{\partial V}{\partial \vec{r}_i} \cdot (\hat{e}_G \times \vec{r}_i) + \frac{\partial V}{\partial \vec{r}_j} \cdot (\hat{e}_G \times \vec{r}_j) \right) + \mathcal{O}(\epsilon^2) \\ &= \sum_{i \neq j} \frac{1}{2} \epsilon \left(\frac{\partial V}{\partial \vec{r}_i} \times \vec{r}_i + \frac{\partial V}{\partial \vec{r}_j} \times \vec{r}_j \right) \cdot \hat{e}_G + \mathcal{O}(\epsilon^2) \\ &= \sum_{i \neq j} \frac{1}{2} \epsilon \frac{\partial V}{\partial \vec{r}_i} \times (\vec{r}_i - \vec{r}_j) \cdot \hat{e}_G + \mathcal{O}(\epsilon^2), \end{aligned}$$

where we used $\vec{P}_i \cdot \vec{P}_i = \vec{p}_i \cdot \vec{p}_i + \mathcal{O}(\epsilon^2)$ and the result found for the linear momentum. This will be zero if the force $\vec{F} = -\frac{\partial V}{\partial \vec{r}_i}$ is parallel to $\vec{r}_i - \vec{r}_j$, the line connecting the two particles.