Homework Set 5
Physics 319
Classical Mechanics

Problem 9.11

In the rotating frame the Lagrangian is

$$\mathcal{L}\left(\dot{\vec{x}}',\vec{x}'\right)=T-U=\frac{m}{2}\left(\dot{\vec{x}}'+\vec{\Omega}\times\vec{x}'\right)^2-U\left(\vec{x}'\right)$$

The six Euler-Lagrange derivatives for each direction are

$$\mathcal{L}\left(\dot{\vec{x}}',\vec{x}'\right)=\frac{m}{2}\left(\left(\dot{\vec{x}}'\right)^2+2\dot{\vec{x}}'\cdot\left(\vec{\Omega}\times\vec{x}'\right)+\left(\vec{\Omega}\times\vec{x}'\right)\cdot\left(\vec{\Omega}\times\vec{x}'\right)\right)-U\left(\vec{x}'\right)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{x}_i'}=m\left(\dot{x}_i'+\left(\vec{\Omega}\times\vec{x}'\right)_i\right)$$

$$\frac{\partial\mathcal{L}}{\partial x_i'}=m\left(\left(\dot{\vec{x}}'\times\vec{\Omega}\right)_i-\left(\vec{\Omega}\times\left(\vec{\Omega}\times\vec{x}'\right)\right)_i/2+\left(\left(\vec{\Omega}\times\vec{x}'\right)\times\vec{\Omega}\right)_i/2\right)-\frac{\partial U}{\partial x_i'}$$

Therefore, when $\dot{\vec{\Omega}}=0$

$$m\ddot{\vec{x}}'=2m\left(\dot{\vec{x}}'\times\vec{\Omega}\right)+\left(\vec{\Omega}\times\vec{x}'\right)\times\vec{\Omega}-\frac{\partial U}{\partial\vec{x}'}$$

$$m\vec{a}'=\vec{F}'_{coriolis}+\vec{F}'_{centrifugal}+\vec{F}'$$

Problem 9.14

In the frame fixed in the bucket after the water settles down (then the relative velocity and therefore the Coriolis force vanishes) the potential including the gravity and the centrifugal force is

$$U\left(r',z'\right)=-m\frac{\Omega^2r'^2}{2}+mgz'+C$$

If the origin is at the surface of the water at the rotation axis, the constant evaluates to zero if the potential vanishes there. The other points with zero potential (the surface of the water) are at

$$z'=\frac{\Omega^2r'^2}{2g}.$$

The water surface follows a parabola.

Problem 9.23

In the non-rotating frame the force is $\vec{F}=-k\vec{r}$. In a primed frame rotating with angular velocity $\Omega\hat{z}$

$$m\vec{a}'=2m\vec{v}'\times\Omega\hat{z}+m\Omega^2\vec{r}'-k\vec{r}'.$$

When $\Omega=\sqrt{k/m}$, also the oscillation frequency of the harmonic potential, the equation of motion reduces to

$$m\vec{a}'=2m\vec{v}'\times\Omega\hat{z},$$

the same as the 2-D Lorentz force created by a magnetic field of magnitude $2m\Omega / e$. The orbit is a circle whose radius vector precesses with angular frequency 2Ω.

$$\eta'(t) = x'(t) + iy'(t) = \eta_0 + ae^{-2i\Omega t}$$

To go back into the non-rotating frame multiply by $e^{i\Omega t}$ and the result is

$$\eta(t) = x(t) + iy(t) = \eta_0 e^{i\Omega t} + ae^{-i\Omega t}$$

$$x(t) = \cos\Omega t \operatorname{Re}\eta_0 - \sin\Omega t \operatorname{Im}\eta_0 + \cos\Omega t \operatorname{Re} a + \sin\Omega t \operatorname{Im} a$$

$$y(t) = \sin\Omega t \operatorname{Re}\eta_0 + \cos\Omega t \operatorname{Im}\eta_0 - \sin\Omega t \operatorname{Re} a + \cos\Omega t \operatorname{Im} a$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \operatorname{Re}\eta_0 + \operatorname{Re} a & -\operatorname{Im}\eta_0 + \operatorname{Im} a \\ \operatorname{Im}\eta_0 + \operatorname{Im} a & \operatorname{Re}\eta_0 - \operatorname{Re} a \end{pmatrix} \begin{pmatrix} \cos\Omega t \\ \sin\Omega t \end{pmatrix}$$

$$\begin{pmatrix} \cos\Omega t \\ \sin\Omega t \end{pmatrix} = \frac{1}{\eta_0\eta_0^* - aa^*} \begin{pmatrix} \operatorname{Re}\eta_0 - \operatorname{Re} a & \operatorname{Im}\eta_0 - \operatorname{Im} a \\ -\operatorname{Im}\eta_0 - \operatorname{Im} a & \operatorname{Re}\eta_0 + \operatorname{Re} a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\cos^2\Omega t + \sin^2\Omega t = 1 \to Ax^2 + 2Bxy + Cy^2 = \left(\eta_0\eta_0^* - aa^*\right)^2$$

$$A = \left(\operatorname{Re}\eta_0 - \operatorname{Re} a\right)^2 + \left(\operatorname{Im}\eta_0 - \operatorname{Im} a\right)^2 = \eta_0\eta_0^* + aa^* - 2\operatorname{Re}\eta_0 \operatorname{Re} a - 2\operatorname{Im}\eta_0 \operatorname{Im} a$$

$$B = \left(\operatorname{Re}\eta_0 - \operatorname{Re} a\right)\left(\operatorname{Im}\eta_0 - \operatorname{Im} a\right) + \left(-\operatorname{Im}\eta_0 - \operatorname{Im} a\right)\left(\operatorname{Re}\eta_0 + \operatorname{Re} a\right)$$

$$= -2\operatorname{Re}\eta_0 \operatorname{Im} a - 2\operatorname{Im}\eta_0 \operatorname{Re} a$$

$$C = \left(-\operatorname{Im}\eta_0 - \operatorname{Im} a\right)^2 + \left(\operatorname{Re}\eta_0 + \operatorname{Re} a\right)^2 = \eta_0\eta_0^* + aa^* + 2\operatorname{Re}\eta_0 \operatorname{Re} a + 2\operatorname{Im}\eta_0 \operatorname{Im} a$$

Now $A^2 = \left(\eta_0 - a\right)\left(\eta_0^* - a^*\right)$ and $C^2 = \left(\eta_0 + a\right)\left(\eta_0^* + a^*\right)$, so the right hand sides of the expressions for A and C being the lengths of specific complex vectors, must be positive. A straightforward computation shows $AC - B^2 = \left(\eta_0\eta_0^* - aa^*\right)^2 > 0$. By the result in Problem 8.11 quoted, the orbit is an ellipse.

Problem 9.28

a) For the conditions of the problem ($\hat{x}$ is local east and $\hat{y}$ is local north)

$$x = v_0 t\cos\alpha - \Omega v_0 \sin\alpha \sin\theta t^2 + \frac{1}{3}\Omega g t^3 \sin\theta$$

$$y = -\Omega v_0 \cos\alpha \cos\theta t^2$$

$$z = v_0 t \sin\alpha - \frac{1}{2}gt^2 + \Omega v_0 \cos\alpha \sin\theta t^2$$

As the Coriolis acceleration in the z equation is much smaller than the gravitational acceleration, the time for the shell to drop to $z = 0$ is

$$t \doteq \frac{2v_0 \sin\alpha}{g}$$

For the parameters given, $t = 35$ sec and $R = 16.4$ km.

b) Using first approximation

$$y \doteq -\Omega v_0 \cos\alpha \cos\theta \frac{4v_0^2 \sin^2\alpha}{g^2}$$

For a shot due east at $\theta = 40^{\circ}$ the deflection is south 32 m as the shell passes the range point. For a shot due east at $\theta = 140^{\circ}$, the sign reverses and the deflection is north.

Problem 9.33

It is easy enough to do the exact solution. The general solution is

$$\eta(t) = x(t) + iy(t) = e^{-i\Omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right)$$

$$\dot{\eta}(t) = \dot{x}(t) + i\dot{y}(t) = -i\Omega_z e^{-i\Omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right) + e^{-i\Omega_z t}\left(i\omega_0 C_1 e^{i\omega_0 t} - i\omega_0 C_2 e^{-i\omega_0 t}\right)$$

Placing the boundary conditions on the problem yields

$$\eta(t=0) = x(t=0) + iy(t=0) = A = C_1 + C_2$$

$$\dot{\eta}(t=0) = 0 = -i\Omega_z (C_1 + C_2) + i\omega_0 (C_1 - C_2)$$

$$\frac{\Omega_z}{\omega_0} A = (C_1 - C_2)$$

$$C_1 = \left(1 + \frac{\Omega_z}{\omega_0}\right)\frac{A}{2} \qquad C_2 = \left(1 - \frac{\Omega_z}{\omega_0}\right)\frac{A}{2}$$

The solution is therefore

$$\eta(t) = Ae^{-i\Omega_z t}\left(\cos\omega_0 t + i(\Omega_z/\omega_0)\sin\omega_0 t\right)$$

$$x(t) = A\cos\omega_0 t\cos\Omega_z t + A(\Omega_z/\omega_0)\sin\omega_0 t\sin\Omega_z t$$

$$y(t) = -A\cos\omega_0 t\sin\Omega_z t + A(\Omega_z/\omega_0)\sin\omega_0 t\cos\Omega_z t$$

For $\Omega_z \ll \omega_0$ the correct answer is obtained (Equation 9.67).

Problem 10.15

a) Put the cube edge along the positive z-axis with the origin at a corner. Then the integrals to be performed are

$$I = \frac{m}{a^3}\left[\int_0^a x^2 dx \int_0^a dy \int_0^a dz + \int_0^a dx \int_0^a y^2 dy \int_0^a dz\right]$$

$$= \frac{m}{a^3}\left[\frac{a^5}{3} + \frac{a^5}{3}\right] = \frac{2}{3}ma^2$$

b) When the cube is resting on the edge the height of the center of mass is $a/\sqrt{2}$. When the cube lies flat the center of mass is at $a/2$. The potential difference is $mg\left(a/\sqrt{2} - a/2\right) = mga\left(\sqrt{2} - 1\right)/2$. By conservation of energy this energy is transferred into kinetic energy of rotation.

$$\frac{1}{2}\omega^2 \frac{2}{3} ma^2 = mga\left(\sqrt{2}-1\right)/2$$
$$\omega = \sqrt{\frac{3g}{2a}\left(\sqrt{2}-1\right)}$$

Problem 10.24

a) The expressions for the diagonal matrix elements can be understood as follows. For example

$$\begin{aligned}
I_{xx} &= \sum_{m_\alpha} m_\alpha \left(y_\alpha^2 + z_\alpha^2\right) \\
&= \sum_{m_\alpha} m_\alpha \left(\left(y_\alpha - y_{cm} + y_{cm}\right)^2 + \left(z_\alpha - z_{cm} + z_{cm}\right)^2\right) \\
&= \sum_{m_\alpha} m_\alpha \begin{pmatrix} \left(y_\alpha - y_{cm}\right)^2 + 2\left(y_\alpha - y_{cm}\right) y_{cm} + y_{cm}^2 \\ +\left(z_\alpha - z_{cm}\right)^2 + 2\left(z_\alpha - z_{cm}\right) z_{cm} + z_{cm}^2 \end{pmatrix} \\
&= \sum_{m_\alpha} m_\alpha \left(\left(y_\alpha - y_{cm}\right)^2 + \left(z_\alpha - z_{cm}\right)^2\right) + 0 + 0 + My_{cm}^2 + Mz_{cm}^2 \\
&= I_{cm,xx} + My_{cm}^2 + Mz_{cm}^2
\end{aligned}$$

because

$$\sum_{m_\alpha} m_\alpha y_\alpha = My_{cm} \quad \sum_{m_\alpha} m_\alpha z_\alpha = Mz_{cm}$$

An identical argument works for the other moments of inertia. For the products of inertia the same type of calculation applies.

$$\begin{aligned}
I_{xy} &= -\sum_{m_\alpha} m_\alpha x_\alpha y_\alpha \\
&= -\sum_{m_\alpha} m_\alpha \left(x_\alpha - x_{cm} + x_{cm}\right)\left(y_\alpha - y_{cm} + y_{cm}\right) \\
&= -\sum_{m_\alpha} m_\alpha \left(x_\alpha - x_{cm}\right)\left(y_\alpha - y_{cm}\right) - \sum_{m_\alpha} m_\alpha \left(x_\alpha - x_{cm}\right) y_{cm} \\
&\quad -\sum_{m_\alpha} m_\alpha x_{cm}\left(y_\alpha - y_{cm}\right) - Mx_{cm}y_{cm} \\
&= I_{cm,xy} - Mx_{cm}y_{cm}
\end{aligned}$$

b) To displace to the corner of the cube needs $\Delta\vec{x} = \left(-a/2, -a/2, -a/2\right)$. Because the products of inertia for a cube about the center of mass vanish, they must be $-ma^2/4$ when calculated from the corner. Likewise the (identical) moments of inertia are adjusted up to $I_{xx} = I_{yy} = I_{zz} = \frac{m}{6}a^2 + \frac{m}{4}a^2 + \frac{m}{4}a^2 = \frac{8}{12}ma^2$ by displacing to the corner. Both these results are in the matrix in Equation 10.49.

Problem 10.25

a)

$$I_{xx} = \frac{m}{8abc}\left[\int_{-a}^{a} dx \int_{-b}^{b} y^2 dy \int_{-c}^{c} dz + \int_{-a}^{a} dx \int_{-b}^{b} dy \int_{-c}^{c} z^2 dz\right]$$

$$= \frac{m}{2abc}\left[\frac{a2b^3c}{3} + \frac{ab2c^3}{3}\right] = \frac{m}{3}\left(b^2 + c^2\right)$$

By symmetry and permuting various variable combinations

$$I_{yy} = \frac{m}{3}\left(a^2 + c^2\right)$$

$$I_{zz} = \frac{m}{3}\left(a^2 + b^2\right)$$

The products of inertia all vanish because an odd function (*x*,*y*,or *z*) is being integrated over an even distribution when the origin is in the center of the cuboid.

b) Displacing to the corner gives

$$I_{xx} = \frac{m}{3}\left(b^2 + c^2\right) + mb^2 + mc^2 = \frac{4m}{3}\left(b^2 + c^2\right)$$

$$I_{yy} = \frac{4m}{3}\left(a^2 + c^2\right) \qquad I_{zz} = \frac{4m}{3}\left(a^2 + b^2\right)$$

$$I_{xy} = I_{yx} = -mab \quad I_{xz} = I_{zx} = -mac \quad I_{yz} = I_{zy} = -mbc$$

c) Using the inertia tensor to obtain the angular momentum

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \omega \begin{pmatrix} 4m\left(b^2 + c^2\right)/3 \\ -mab \\ -mac \end{pmatrix}$$

Problem 10.35

a) For m at $\left(a, 0, 0\right)$, $2m$ at $\left(0, a, a\right)$, and $3m$ at $\left(0, a, -a\right)$, the inertia tensor is

$$I_{xx} = m(0^2+0^2)+2m(a^2+a^2)+3m(a^2+a^2) = 10ma^2$$
$$I_{yy} = m(a^2+0^2)+2m(0^2+a^2)+3m(0^2+a^2) = 6ma^2$$
$$I_{zz} = m(a^2+0^2)+2m(0^2+a^2)+3m(0^2+a^2) = 6ma^2$$
$$I_{xy} = I_{yx} = -m\cdot a\cdot 0 - 2\cdot 0\cdot a - 3m\cdot 0\cdot(-a) = 0$$
$$I_{xz} = I_{zx} = -m\cdot a\cdot 0 - 2\cdot 0\cdot a - 3m\cdot 0\cdot(-a) = 0$$
$$I_{yz} = I_{zy} = -m\cdot 0\cdot 0 - 2\cdot a\cdot a - 3m\cdot a\cdot(-a) = ma^2$$
$$I = ma^2\begin{pmatrix}10 & 0 & 0\\ 0 & 6 & 1\\ 0 & 1 & 6\end{pmatrix}$$

b) The eigenvalue problem is

$$\det\begin{pmatrix}10-\lambda & 0 & 0\\ 0 & 6-\lambda & 1\\ 0 & 1 & 6-\lambda\end{pmatrix} = 0$$
$$(10-\lambda)\left((6-\lambda)^2-1\right) = 0$$
$$\lambda = 10, 7, 5$$

So the principal moments are $10ma^2$, $7ma^2$, and $5ma^2$. The principal axes are

$$\hat{e}_{10} = \hat{x}, \hat{e}_7 = (\hat{y}+\hat{z})/\sqrt{2}, \hat{e}_5 = (\hat{y}-\hat{z})/\sqrt{2}.$$

Problem 10.48

a) The angular velocity in the "unholy mixture" is $\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3$

$$\hat{e}_2' = -\sin\theta\sin\phi\hat{x} + \sin\theta\cos\phi\hat{y}$$
$$e_3 = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$$
$$\vec{\omega} = \sin\theta\left[\left(\dot{\psi}\cos\phi - \dot{\theta}\sin\phi\right)\hat{x} + \left(\dot{\psi}\sin\phi - \dot{\theta}\cos\phi\right)\hat{y}\right] + \left(\dot{\psi}\cos\theta + \dot{\phi}\right)\hat{z}$$

b) By equation 10.98 $\hat{z} = \cos\theta\hat{e}_3 - \sin\theta\hat{e}_1'$. By the definition of the third Euler angle

$$\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\end{pmatrix} = \begin{pmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}\hat{e}_1'\\ \hat{e}_2'\end{pmatrix}.$$

So

$$\begin{pmatrix}\hat{e}_1'\\ \hat{e}_2'\end{pmatrix} = \begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\end{pmatrix},$$

and

$$\vec{\omega} = -\dot{\phi}\sin\theta\hat{e}_1' + \dot{\theta}\hat{e}_2' + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\hat{e}_3$$
$$= \left[\left(-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi\right)\hat{e}_1 + \left(-\dot{\phi}\sin\theta\sin\psi - \dot{\theta}\sin\psi\right)\hat{e}_2\right] + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\hat{e}_3.$$

It should be noted, as Taylor does in the text discussion, that because $\hat{e}_1'^2 + \hat{e}_2'^2 = \hat{e}_1^2 + \hat{e}_2^2$, the Lagrangian evaluates identically in terms of the Euler angles, independent of whether $\hat{e}_1'$ and $\hat{e}_2'$ or $\hat{e}_1$ and $\hat{e}_2$ are used. They are both orthogonal axis sets.