Homework Set 4
Physics 319
Classical Mechanics

Problem 7.8

a) The Lagrangian is

$$\mathcal{L}(\dot{x}_1, \dot{x}_2, x_1, x_2) = T - U = \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_2^2 - \frac{k}{2}(x_1 - x_2 - l)^2$$

b) In terms of the center of mass coordinates and *x*

$$x_{cm} = \frac{mx_1 + mx_2}{2m} = \frac{x_1 + x_2}{2}$$

$$x_1 = x_{cm} + \frac{x_1 - x_2}{2} \quad x_2 = x_{cm} - \frac{x_1 - x_2}{2}$$

$$\dot{x}_1 = \dot{x}_{cm} + \frac{\dot{x}_1 - \dot{x}_2}{2} \quad \dot{x}_2 = \dot{x}_{cm} - \frac{\dot{x}_1 - \dot{x}_2}{2}$$

$$\mathcal{L} = m\dot{x}_{cm}^2 + m\frac{(\dot{x})^2}{4} - \frac{k}{2}(x - l)^2$$

$$\frac{d}{dt}(2m\dot{x}_{cm}) = 0$$

$$\frac{d}{dt}(m\dot{x}/2) = k(x - l)$$

c) The center of mass moves with uniform velocity. The equation of motion for *x* is

$$\ddot{x} = \frac{2k}{m}(x - l)$$

$$x(t) = l + A\cos(\omega t + \delta)$$

$$\omega = \sqrt{\frac{2k}{m}}$$

Problem 7.23

If one defines $x_c = X + x$, as the inside cart's location in space, the Lagrangian is either

$$\mathcal{L} = T - U = \frac{m}{2}\dot{x}_c^2 - \frac{k(X - x_c)^2}{2}$$

$$\therefore \frac{d}{dt}(m\dot{x}_c) = k(X - x_c)$$

or

$$\mathcal{L} = T - U = \frac{m}{2}\left(\dot{x} - A\omega\sin\omega t\right)^2 - \frac{k\left(x\right)^2}{2}$$

$$\therefore \frac{d}{dt}\left(m\dot{x} - mA\omega\sin\omega t\right) = kx$$

In either case the equation of motion is

$$\ddot{x} + \omega_0^2 x = \omega^2 A\cos\omega t\ .$$

where $\omega_0^2 = k/m$. There is no energy integral for this problem.

Problem 7.29

It is clear that the point *P* is at the location

$$\vec{x}_P\left(t\right) = R\hat{x}\cos\omega t + R\hat{y}\sin\omega t.$$

In terms of the given angle the mass is at the point

$$\vec{x}\left(t\right) = \vec{x}_P\left(t\right) + l\sin\phi\hat{x} - l\cos\phi\hat{y}$$

$$\dot{x}\left(t\right) = -R\omega\sin\omega t + l\dot{\phi}\cos\phi$$

$$\dot{y}\left(t\right) = R\omega\cos\omega t + l\dot{\phi}\sin\phi$$

$$U = mgy = R\sin\omega t - l\cos\phi$$

The Lagrangian is

$$\mathcal{L} = T - U = \frac{m}{2}\left(R^2\omega^2 + 2R\omega l\dot{\phi}\sin\left(\phi - \omega t\right) + l^2\dot{\phi}^2\right) - mg\left(R\sin\omega t - l\cos\phi\right)$$

The equation of motion is

$$\frac{d}{dt}\left(ml^2\dot{\phi} + R\omega l\sin\left(\phi - \omega t\right)\right) = mR\omega l\dot{\phi}\cos\left(\phi - \omega t\right) - mgl\sin\phi$$

$$l\ddot{\phi} = \left(R\omega^2 - R\omega\dot{\phi} + R\omega\dot{\phi}\right)\cos\left(\phi - \omega t\right) - g\sin\phi$$

$$l\ddot{\phi} = R\omega^2\cos\left(\phi - \omega t\right) - g\sin\phi$$

Problem 7.34

Consider a uniform expansion over the spring's length. On the fixed side of the spring the coil does not move and the free side it moves $x\left(t\right)$. If *s* measures the distance along the unextended and uncompressed spring with *s*=0 at the fixed end, then the coil at location *s* moves by

$$\Delta x_{coil}\left(s,t\right) = \frac{s}{L}x\left(t\right)$$

when the expansion in the spring is $x\left(t\right)$. Similarly, the velocity of the coil at location *s* is

$$v_{coil} = \frac{\partial}{\partial t}\Delta x_{coil}\left(s,t\right) = \frac{s}{L}\dot{x}\left(t\right)$$

The total kinetic energy of the coil is

$$T_{coil} = \int dm \frac{v^2}{2} = \frac{M}{L} \int_0^L \frac{s^2 \dot{x}^2}{2L^2} ds = \frac{M}{6} \dot{x}^2,$$

Which needs to be added to the kinetic energy of the car, $m\dot{x}^2/2$. The total Lagrangian is

$$\mathcal{L} = \left[\frac{m}{2} + \frac{M}{6}\right] \dot{x}^2 - \frac{k}{2} x^2 .$$

The usual calculation of the equation of motion leads to sinusoidal oscillations with (angular) frequency $\omega = \sqrt{k/\left(m + M/3\right)}$.

Problem 7.35

In the frame that rotates with the wire the position is

$$\vec{x}_r(t) = \frac{AB}{2}(1+\cos\phi)\hat{x}_r + \frac{AB}{2}\sin\phi\,\hat{y}_r .$$

But the unit vectors rotate as

$$\hat{x}_r(t) = \cos\omega t\hat{x} + \sin\omega t\hat{y}$$
$$\hat{y}_r(t) = -\sin\omega t\hat{x} + \cos\omega t\hat{y}$$

And therefore

$$x(t) = \frac{AB}{2}(1+\cos\phi)\cos\omega t - \frac{AB}{2}\sin\phi\sin\omega t = \frac{AB}{2}\cos\omega t + \frac{AB}{2}\cos(\omega t+\phi)$$
$$y(t) = \frac{AB}{2}(1+\cos\phi)\sin\omega t + \frac{AB}{2}\sin\phi\cos\omega t = \frac{AB}{2}\sin\omega t + \frac{AB}{2}\sin(\omega t+\phi)$$
$$\dot{x}(t) = -\frac{AB}{2}\omega\sin\omega t - \frac{AB}{2}\left(\omega+\dot{\phi}\right)\sin(\omega t+\phi)$$
$$\dot{y}(t) = \frac{AB}{2}\omega\cos\omega t + \frac{AB}{2}\left(\omega+\dot{\phi}\right)\cos(\omega t+\phi)$$
$$\mathcal{L} = T - U = \frac{m}{2}\left[\frac{AB^2}{4}\left(\omega+\dot{\phi}\right)^2 + \frac{AB^2}{2}\omega\left(\omega+\dot{\phi}\right)\cos\phi + \frac{AB^2}{4}\omega^2\right] - 0$$
$$\frac{d}{dt}\left[m\frac{AB^2}{4}\left(\omega+\dot{\phi}+\omega\cos\phi\right)\right] = -m\frac{AB^2}{4}\omega\left(\omega+\dot{\phi}\right)\sin\phi$$
$$\ddot{\phi} - \omega\dot{\phi}\sin\phi = -\omega\left(\omega+\dot{\phi}\right)\sin\phi \rightarrow \ddot{\phi} = -\omega^2\sin\phi$$

This result is “obviously” correct as the centrifugal force from the rotation gives an effective potential energy of

$m\omega^2\left(AB^2 - r^2\right)/2 = m\omega^2\left(AB\right)^2\left(1-\cos\phi\right)/4 \approx m\omega^2\left(AB/2\right)^2\phi^2/2$, and the kinetic energy in the rotating frame is $m\left(AB/2\right)^2\dot{\phi}^2/2$. The Euler-Lagrange equations in this rotating frame give $\ddot{\phi} = -\omega^2 \sin\phi$ exactly, as they should. The (angular) frequency of small oscillations is ω.

Problem 8.12

a) The first part is straight from the lectures where this radius was defined to be r_{m}.

$$r_{\mathrm{m}} = \frac{l^2}{Gm_1 m_2 \mu}$$

b) For small deviations from r_{m}, the effective potential is, to second order in the deviation,

$$U_{eff} = E_{\min} + \frac{d^2 U_{eff}}{dr^2}(r = r_{\mathrm{m}})\frac{(\Delta r)^2}{2} + \cdots$$

$$\frac{d^2 U_{eff}}{dr^2}(r = r_{\mathrm{m}}) = -2\frac{Gm_1 m_2}{r_{\mathrm{m}}^3} + 3\frac{l^2}{\mu r_{\mathrm{m}}^4} = \frac{l^2}{\mu r_{\mathrm{m}}^4}$$

and the motion is a simple harmonic motion with (angular) frequency $\omega = \sqrt{k/\mu} = l/\mu r_{\mathrm{m}}^2$. Because for small deviations $a \approx b \approx r_{\mathrm{m}}$, this is a restatement of Kepler's second law $T = 2\pi ab\mu / l$.

Problem 8.19

For the ellipse $a = 3300\ \mathrm{km}/2 + 6400\ \mathrm{km} = 8050\ \mathrm{km}$. Therefore $f = 8050\ \mathrm{km} - 6700\ \mathrm{km} = 1350\ \mathrm{km}$. The eccentricity is $1350/8050 = 0.1677$. Because $r_{\mathrm{m}} = (1-\varepsilon^2)a = .9719 \times 8050 = 7824\ \mathrm{km}$, the height at 90 degrees is 1424 km.

Problem 8.29

From the lectures we know that on a circular orbit of radius r_{m}, the total energy and kinetic energy and potential energy are $-\frac{(Gm_e m_s)^2 \mu}{2l^2}$, $\frac{(Gm_e m_s)^2 \mu}{2l^2}$, and $-\frac{(Gm_e m_s)^2 \mu}{l^2}$, respectively (consistent with the virial theorem!). When the sun's mass is halved the angular momentum, and hence the kinetic energy is conserved. At this same instant, the potential energy becomes $-\frac{Gm_e m_s}{2r_{\mathrm{m}}} = -\frac{(Gm_e m_s)^2 \mu}{2l^2}$ and so the total energy vanishes. Without further ado we know that the orbit is a parabola out to $r = \infty$ with minimum radius r_{m}. The parabola is tangent to the original orbit at the instant when the mass changed.

Problem 8.30

The equation for the orbit is

$$r(\theta)=\frac{r_{\rm m}}{1+\varepsilon\cos\theta}$$

Because $x=r(\theta)\cos\theta$ and $r(\theta)=r_{\rm m}-\varepsilon x$,

$$x^2+y^2=(r_{\rm m}-\varepsilon x)^2=r_{\rm m}^2-2r_{\rm m}\varepsilon x+\varepsilon^2x^2$$
$$x^2(1-\varepsilon^2)+2r_{\rm m}\varepsilon x+y^2=r_{\rm m}^2$$

For $\varepsilon=1$ the orbit is the parabola

$$y=\pm\sqrt{r_{\rm m}^2-2r_{\rm m}x}\qquad x<r_{\rm m}/2$$

This is Eq. 8.60 recalling $r_{\rm m}$ is Taylor's c.

For $\varepsilon>1$, by completing the square the orbit is the hyperbola

$$\left(x+\frac{r_{\rm m}\varepsilon}{1-\varepsilon^2}\right)^2(1-\varepsilon^2)+y^2=r_{\rm m}^2+\frac{r_{\rm m}^2\varepsilon^2}{1-\varepsilon^2}$$
$$\left(x-\frac{r_{\rm m}\varepsilon}{\varepsilon^2-1}\right)^2\frac{(\varepsilon^2-1)^2}{r_{\rm m}^2}-\frac{\varepsilon^2-1}{r_{\rm m}^2}y^2=1$$

so $\alpha=c/(\varepsilon^2-1)$, $\beta=c/\sqrt{\varepsilon^2-1}$, and $\delta=c\varepsilon/(\varepsilon^2-1)$ in the terms defined by 8.61. Also, one may solve for y

$$y^2=\left(x-\frac{r_{\rm m}\varepsilon}{\varepsilon^2-1}\right)^2(\varepsilon^2-1)-\frac{r_{\rm m}^2}{\varepsilon^2-1}$$
$$y=\pm\sqrt{\left(x-\frac{r_{\rm m}\varepsilon}{\varepsilon^2-1}\right)^2(\varepsilon^2-1)-\frac{r_{\rm m}^2}{\varepsilon^2-1}}\qquad x<\frac{r_{\rm m}(\varepsilon-1)}{\varepsilon^2-1}$$

This result explains the diagram and equation for the hyperbola in Lecture 15.

Problem 8.33

After the satellite is placed into the transfer orbit the relations

$$R_1=\frac{c_2}{1+\varepsilon_2}\qquad R_3=\frac{c_2}{1-\varepsilon_2}$$

apply. Eliminating c_2 provides the eccentricity as $\varepsilon_2=(R_3-R_1)/(R_3+R_1)$ To achieve a circular orbit at P', $c_3=R_3$ must be obtained. Therefore

$$\frac{c_3}{c_2}=\lambda'^2=\frac{1}{1-\varepsilon_2}$$
$$=\frac{R_3+R_1}{R_3+R_1-R_3+R_1}=\frac{R_3+R_1}{2R_1}$$
$$\lambda'=\sqrt{\frac{R_3+R_1}{2R_1}}$$

This is Taylor's Eqn. 8.73.