Accelerator Physics
Particle Acceleration

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Lecture 9

K-V Distribution

- Single value for the transverse Hamiltonian

$$\frac{1}{\varepsilon_x}\left(\frac{x^2+\left(\alpha_x x+\beta_x x'\right)^2}{\beta_x}\right)+\frac{1}{\varepsilon_y}\left(\frac{y^2+\left(\alpha_y y+\beta_y y'\right)^2}{\beta_y}\right)=C$$

$$\psi\left(x,x',y,y'\right)\propto\delta\left(C-1\right)$$

$$\rho\left(z\right)=\frac{I}{\pi\beta cXY}$$

$$K=\frac{I}{I_0}\frac{2}{\beta^3\gamma^3} \qquad I_0=\frac{4\pi\varepsilon_0 mc^3}{q}$$

$$E_x=\frac{I}{\pi\varepsilon_0\beta c}\frac{x}{Y\left(X+Y\right)}$$

K-V Envelope Equation

$$E_x = \frac{I}{\pi \varepsilon_0 \beta c} \frac{x}{X\left(X+Y\right)}$$

$$E_y = \frac{I}{\pi \varepsilon_0 \beta c} \frac{y}{Y\left(X+Y\right)}$$

$$x'' + k_y x - \frac{2K}{X\left(X+Y\right)} x = 0$$

$$y'' + k_y x - \frac{2K}{Y\left(X+Y\right)} x = 0$$

Envelope Equation

$$X'' + k_x X - \frac{2K}{\left(X+Y\right)} - \frac{\varepsilon_x^2}{X^3} = 0$$

$$Y'' + k_y Y - \frac{2K}{\left(X+Y\right)} - \frac{\varepsilon_y^2}{Y^3} = 0$$

Waterbag Distribution

- Lemons and Thode were first to point out SC field is solved as Bessel Functions for a certain equation of state. Later, others, including my advisor and I showed the equation of state was exact for the waterbag distribution.

$$H_T = \frac{p_z^2}{2m}\left(x'^2 + y'^2\right) + \frac{m\omega_0^2\left(x^2+y^2\right)}{2} + e\phi_{SC}$$

$$\psi = A\Theta\left(H_0 - H_T\right)$$

$$n(r) = \iint \psi\, dx'dy' = n_0\left(1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right) \qquad \phi_0 = H_0/e$$

$$\sigma_v^2 = \frac{\iint \psi\, p_z^2\left(x'^2+y'^2\right)dx'dy'}{m^2\iint \psi\, dx'dy'} = \frac{H_0}{m}\left(1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right)$$

Self-consistent potential solves

$$\nabla^2\phi_{SC} + \frac{\phi_{SC}}{\lambda_D^2} = \frac{en_0}{\varepsilon_0}\left[\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - 1\right]$$

$$\lambda_D = \frac{\sigma_v}{\omega_p} = \sqrt{\frac{\varepsilon_0 m}{e^2 n_0}\frac{H_0}{m}} = \sqrt{\frac{\varepsilon_0 H_0}{e^2 n_0}}$$ Debye Length

Analytic solutions in terms of Modified Bessel Functions

$$e\phi(r) = \frac{m\omega_0^2\left(x^2+y^2\right)}{2} + A\left(I_0\left(r/\lambda_D\right) - 1\right) + BK_0\left(r/\lambda_D\right)$$

$B = 0$ by boundary condition

A chosen so that solution without I_0 solution to inhomogeneous eqn.

Equation for Beam Radius

Now

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\right]\frac{r^2}{2} = 2$$

$$\therefore A = m\lambda_D^2\left(2\omega_0^2 - \omega_p^2\right)$$

At $r = r_b$ the density vanishes

$$H_0 = m\lambda_D^2\left(2\omega_0^2 - \omega_p^2\right)\left(1 - I_0\left(r_b / \lambda_D\right)\right)$$

$$1 + \frac{\omega_p^2}{2\omega_0^2 - \omega_p^2} = I_0\left(r_b / \lambda_D\right)$$

$$n_b\left(r\right) = \hat{n}_b \frac{I_0\left(r_b / \lambda_D\right) - I_0\left(r / \lambda_D\right)}{I_0\left(r_b / \lambda_D\right) - 1}$$